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State-Dependent Scaling Characterization for Interconnected Nonlinear Systems–Part II: Small Gain Theorems for iISS and ISS Properties[¶§]

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Abstract: This paper is devoted to the problem of stability analysis for interconnected integral input-to-state stable(iISS) systems and input-to-state stable(ISS) systems. In the first part of this two-part paper, the state-dependent scaling problem has been proposed as a general mathematical formulation whose solutions explicitly provide Lyapunov functions proving stability properties of feedback and cascade connection of dissipative nonlinear systems. A main purpose of this second part is to demonstrate that the generalization is surely beyond formal applicability, and it effectively deals with systems having diverse and strong nonlinearities which are not covered by classical and existing advanced stability criteria. In particular, this paper derives small-gain-type theorems for interconnected systems involving iISS systems smoothly from the state-dependent scaling formulation as special cases. This paper also provides explicit solutions to the state-dependent scaling problems. The new framework enables us not only to characterize stability of interconnected iISS and ISS systems in a unified manner, but also to extend the ISS small-gain theorem to iISS supply rates seamlessly. The results are considered as some theoretical evidences that the state-dependence of the scaling is crucial to effective treatment of essential nonlinearities.

Keywords: Nonlinear interconnected system, Global asymptotic stability, Lyapunov function, State-dependent scaling, Small gain condition, Integral input-to-state stability, Input-to-state stability

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1 Introduction

The problem of establishing stability properties of nonlinear interconnected systems has been investigated extensively for decades. However, control practice still demands a great deal of development. It is reasonable that the diversity of nonlinearities has been defying the treatment in the way of linear systems theory where universality and effectiveness often come together automatically. For nonlinear systems, it is fundamental to be aware of the total difference between universal applicability and effectiveness. Indeed, we experience conflicts between the two issues in many cases. This is a reason why there are two directions of the research. One direction pursues problem-specific techniques focusing on particularity of individual nonlinearities. People in other field sometimes consider them too heuristic and impractical even when the specialized tricks are effective. The other direction seeks general techniques that are applicable to many cases in a unified way. The generality sometimes not only excludes some strong nonlinearities of great importance, but also renders the essential effectiveness obscure so that the applicability is only formal. It is typical of general 'nonlinear' problems to have no guarantee of the existence of solutions. We often do not know how to solve them even if solutions exist. Naturally, this situation has brought out a quest for a successful fusion of the two directions. From this viewpoint, it is remarkable that the ISS small-gain theorem (also referred to as the nonlinear small-gain theorem) proposed in [1, 2] achieves a balance between the universal applicability and the effectiveness partially[3, 4, 5].

The first part[6] of this two-part paper has presented a unified way to formulate problems of analyzing stability and dissipative properties of nonlinear interconnected systems via state-dependent scaling. Systems to which the state-dependent scaling framework is applicable are not limited to classical systems in standard textbooks, such as finite $\mathcal{L}_2$-gain systems, passive systems and sector nonlinearities. The state-dependent scaling not only enables us to assess stability, but also gives us Lyapunov functions establishing the stability properties of interconnected systems explicitly. Classical stability criteria for systems with mild nonlinearities such as finite $\mathcal{L}_p$-gain systems, passive systems and Lur'e systems can be extracted exactly from the state-dependent scaling characterization as special cases[6]. More importantly, it can be shown that the coverage includes the ISS small-gain theorem for interconnected input-to-state stable(ISS) systems. This paper is devoted to further discussions on the fundamental capability of the state-dependent scaling characterization.

A major advantage of the state-dependent scaling approach over the existing stability criteria is that it is applicable to nonlinear systems disagreeing with classically standard nonlinearities. The purposes of this paper are to demonstrate that the effectiveness is much more than formal applicability, and to provide theoretical illustrations for advanced types of nonlinearities. For this end, this paper concentrates on the interconnected system composed of integral input-to-state stable(iISS) and ISS systems. The existence of solutions to the corresponding state-dependent scaling problems are investigated rigorously, and explicit formulas of the solutions are shown. This paper also derive new theorems of the small-gain-type for them from the state-dependent scaling characterization smoothly. To the best of author's knowledge, the result of small-gain-type theorems involving iISS systems is the first of its kind. The class of ISS systems has been extensively investigated and has been playing an important role in the recent literature of nonlinear control theory[7, 8, 3]. For instance, the fact that cascades of ISS systems are ISS is widely used in stabilization. The ISS small-gain theorem is also a popular tool to establish stability of feedback interconnection of ISS systems. In contrast, the concept of iISS has not yet been fully exploited in analysis and design although the property of iISS by itself has been investigated deeply[9]. The iISS property covers nonlinearities much broader than the ISS property. Indeed, the iISS captures important characteristics essentially nonlinear systems often have[9], and there are many practical systems which are iISS, but not ISS. There are still few tools of making full use of the iISS property in systems analysis and design. For instance, stability criteria similar to the ISS small-gain theorem have not been developed for interconnection involving iISS systems so far. Extension of the ISS small-gain condition to more general systems has been anticipated.

This paper is organized as follows. Section 2 contains a brief review of the general idea presented in [6]. We begin with the introduction of the state-dependent scaling problems to which this paper concentrates on deriving explicit solutions. The section presents a general configuration of nonlinear interconnected systems. It is explained that stability properties of the interconnected system can be established if solutions to the state-dependent scaling problems are found. The main body of this paper begins in Section 3, which is devoted to the issues of when the solutions exist and how they can be found. This paper focuses on the interconnected system consisting of iISS systems and ISS systems. The settings are considered as some special cases of the general settings covered by the state-dependent scaling formulation proposed in [6]. In Section 3, it is demonstrated that we are actually able to obtain solutions to the state-dependent scaling problems, and the solutions are shown explicitly for establishing iISS and ISS properties of the interconnected system. For the feedback interconnection, we derive small-gain-like conditions which are sufficient conditions for the existence of the solutions. It is proved that the conditions become identical to the ISS small-gain theorem in the case of interconnected ISS systems. The solution to the corresponding state-dependent scaling problem provides us with a Lyapunov function for the feedback loop explicitly. More importantly, Section 3 presents results of small-gain-like conditions for the interconnection involving iISS systems. To the best of the author's knowledge, it is the first of its kind. The state-dependent scaling approach allows us to develop the iISS small-gain theorem and the iISS-ISS small-gain theorem in a unified manner. It is shown that there is a reasonable relationship between them and the ISS small-gain theorem. Stability theorems for cascade iISS and ISS systems are also derived as solutions to the state-dependent scaling problems. While Section 3 of this paper deals with supply rates of advanced types such as ISS and iISS properties, discussions in [6] have been concentrated on supply rates which are popular in classical stability analysis such as the $\mathcal{L}_2$ small-gain theorem, the passivity theorems, and the circle and Popov criteria. Using the results obtained in [6] and Section 3 in this paper, the author demonstrates that the classical stability theorems, advanced stability theorems and even new ones can be extracted as special cases of the state-dependent scaling formulation. Stability conditions provided by classical, advanced and new stability theorems are viewed as sufficient conditions for guaranteeing the existence of solutions to the state-dependent scaling problems. In Section 4, the effectiveness of the approach is illustrated through several examples. Finally, concluding remarks are given in Section 5.

This paper uses the following notations. The interval $[0,\infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. Euclidean norm of a vector in $\mathbb{R}^n$ of dimension n is denoted by $|\cdot|$. A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be class $\mathcal{K}$ and written as $\gamma\in\mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0)=0$. A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be class $\mathcal{K}_\infty$ and written as $\gamma\in\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty}\gamma(r)=\infty$. We write $\gamma\in\mathcal{P}$ for a function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ if it is a continuous function satisfying $\gamma(0)=0$ and $\gamma(s)>0$ for all $s\in\mathbb{R}_+\setminus\{0\}$.

2 State-dependent scaling formulation

This section presents a mathematical problem which plays a central role in this paper. Another problem which relaxes the main problem is also presented. This paper refers to those two problems as the state-dependent scaling problems[6]. This section puts system theoretic interpretations on the problems from the viewpoint of stability properties of nonlinear interconnected systems and construction of Lyapunov functions. This section thereby reviews a minimum of necessary preliminaries described in the previous paper[6].

The following is the main mathematical problem to be considered in this paper.

Problem 1 *Given continuously differentiable functions $V_i : (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ and continuous functions $\rho_i : (x_i, x_j, r_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \times \mathbb{R}^{m_i} \to \mathbb{R}$ for $i = 1, 2$ and $j = \{1, 2\} \setminus \{i\}$, find continuous functions $\lambda_i : s \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\lambda_i(s) > 0 \quad \forall s \in (0, \infty) \tag{1}$$

$$\lim_{s\to 0^+} \lambda_i(s) < \infty \tag{2}$$

$$\int_1^\infty \lambda_i(s)ds = \infty \tag{3}$$

for $i = 1, 2$ such that

$$\lambda_1(V_1(t, x_1))\rho_1(x_1, x_2, r_1) + \lambda_2(V_2(t, x_2))\rho_2(x_2, x_1, r_2) \le \rho_e(x_1, x_2, r_1, r_2),$$
$$\forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \tag{4}$$

holds for some continuous function $\rho_e : (x_1, x_2, r_1, r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\rho_e(x_1, x_2, 0, 0) < 0 \quad , \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \{(0, 0)\} \tag{5}$$

A variant of Problem 1 is given by the following which is milder than Problem 1.

Problem 2 *Given a continuously differentiable function $V_2 : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and continuous functions $\rho_1 : (z_1, x_2, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}$ and $\rho_2 : (x_2, z_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$, find continuous functions $\lambda_1 : (t, z_1, x_2, r_1, r_2) \in \mathbb{R}_+ \times \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}_+$, $\lambda_2 : s \in \mathbb{R}_+ \to \mathbb{R}_+$, an increasing continuous function $\xi_1 : s \in [0, N] \to \mathbb{R}_+$ and a continuous function $\varphi_1 : (z_1, x_2, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}_+$ satisfying*

$$\lambda_2(s) > 0 \quad \forall s \in (0, \infty) \tag{6}$$

$$\lim_{s\to 0^+} \lambda_2(s) < \infty \tag{7}$$

$$\int_1^\infty \lambda_2(s)ds = \infty \tag{8}$$

$$\xi_1(s) \ge 0 \quad \forall s \in [0, N] \tag{9}$$

$$\varphi_1(z_1, x_2, r_1) \ge 0, \quad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1} \tag{10}$$

such that

$$\lambda_1(t, z_1, x_2, r_1, r_2)\left[-\xi_1(\varphi_1(z_1, x_2, r_1)) + \xi_1(\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1))\right] +$$
$$\lambda_2(V_2(t, x_2))\rho_2(x_2, z_1, r_2) \le \rho_e(x_2, r_1, r_2),$$
$$\forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \tag{11}$$

holds for some continuous function $\rho_e : (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\rho_e(x_2, 0, 0) < 0 \quad , \forall x \in \mathbb{R}^{n_2} \setminus \{0\} \tag{12}$$

where $N \in [0, \infty]$ is defined by

$$N = \sup_{(z_1, x_2, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1}} [\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1))] \tag{13}$$

The functions λ_i and ξ_i are referred to as state-dependent scaling functions in this paper. It may be worth mentioning that (2) and (7) are redundant mathematically since each λ_i is supposed to be continuous on $\mathbb{R}_+ = [0, \infty)$. The explicit statement of (2) and (7) may be helpful to direct the readers' attention to it.

The inequalities (4) and (11) are central inequalities that need to be solved. This paper calls a pair of λ_1 and λ_2 a solution to Problem 1 if the pair fulfills all requirements stated in Problem 1. In a similar manner, a quartet of λ_1, λ_2, ξ_1 and φ_1 fulfilling all requirements in Problem 2 is called a solution to Problem 2. When the function $\xi_1(s)$ is affine in s, the inequality (11) becomes

$$\lambda_1(t,z_1,x_2,r_1,r_2)\xi_1(\rho_1(z_1,x_2,r_1)) + \lambda_2(V_2(t,x_2))\rho_2(x_2,z_1,r_2) \leq \rho_e(x_2,r_1,r_2),$$
$$\forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \tag{14}$$

Thus, the function φ_1 disappears from (11). In the case of affine $\xi_1(s)$, a solution to Problem 2 becomes the triplet of λ_1, λ_2 and ξ_1. According to Lemma 1 in [6], Problem 1 has a solution only if so does Problem 2 in reasonable settings.

Next, consider the nonlinear interconnected system Σ shown in Fig.1. Suppose that subsystems Σ_1 and Σ_2 are described by

$$\Sigma_1: \ \dot{x}_1 = f_1(t,x_1,u_1,r_1) \tag{15}$$
$$\Sigma_2: \ \dot{x}_2 = f_2(t,x_2,u_2,r_2) \tag{16}$$

These two dynamic systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. If Σ_1 is static, we suppose that Σ_1 is described by

$$\Sigma_1: \ z_1 = h_1(t,u_1,r_1) \tag{17}$$

Then, $u_2 = x_1$ is replaced by $u_2 = z_1$. Assume that $f_1(t,0,0,0) = 0$, $f_2(t,0,0,0) = 0$ and $h_1(t,0,0,0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \geq 0$. The functions f_1, f_2 and h_1 are supposed to be piecewise continuous in t, and locally Lipschitz in the other arguments. The exogenous inputs $r_1 \in \mathbb{R}^{m_1}$ and $r_2 \in \mathbb{R}^{m_2}$ are packed into a single vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. The state vector of the interconnected system Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of Σ_i.

The following theorem demonstrates that stability properties of the nonlinear interconnected system are strongly related to the solutions of the state-dependent scaling problems.

Theorem 1 *Suppose that Σ_1 and Σ_2 are dynamic systems fulfilling the following.*

(i) The system Σ_1 admits the existence of a $\mathbf{C}^1$ function $V_1 : (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+$ such that it satisfies

$$\underline{\alpha}_1(|x_1|) \leq V_1(t,x_1) \leq \bar{\alpha}_1(|x_1|) \tag{18}$$
$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1,u_1,r_1) \leq \rho_1(x_1,u_1,r_1) \tag{19}$$

for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$, where $\underline{\alpha}_1$ and $\bar{\alpha}_1$ are class $\mathcal{K}_\infty$ functions, and $\rho_1 : (x_1,u_1,r_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}$ is a continuous function satisfying $\rho_1(0,0,0) = 0$.

(ii) The system Σ_2 admits the existence of a $\mathbf{C}^1$ function $V_2 : (t,x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ such that it satisfies

$$\underline{\alpha}_2(|x_2|) \leq V_2(t,x_2) \leq \bar{\alpha}_2(|x_2|) \tag{20}$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2,u_2,r_2) \leq \rho_2(x_2,u_2,r_2) \tag{21}$$

for all $x_2 \in \mathbb{R}^{n_2}$, $u_2 \in \mathbb{R}^{n_1}$, $r_2 \in \mathbb{R}^{m_2}$ *and* $t \in \mathbb{R}_+$, *where* $\underline{\alpha}_2$ *and* $\bar{\alpha}_2$ *are class* $\mathcal{K}_\infty$ *functions, and* $\rho_2 : (x_2, u_2, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ *is a continuous function satisfying* $\rho_2(0,0,0) = 0$.

If there is a solution $\{\lambda_1, \lambda_2\}$ *to Problem 1, the equilibrium* $x = [x_1^T, x_2^T]^T = 0$ *of the interconnected system* Σ *is globally uniformly asymptotically stable for* $r \equiv 0$. *Furthermore, there exist a* $\mathbf{C}^1$ *function* $V_{cl} : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ *and class* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ *such that*

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|), \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{22}$$

is satisfied and

$$\frac{dV_{cl}}{dt} \leq \rho_e(x, r), \quad \forall x \in \mathbb{R}^n, r \in \mathbb{R}^m, t \in \mathbb{R}_+ \tag{23}$$

holds along the trajectories of the system Σ.

The previous paper[6] has shown that the properties in Theorem 1 are established by a Lyapunov function in the form of

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{24}$$

Thus, the solutions to the inequality of the sum of scaled supply rates, which is (4), directly lead us to Lyapunov functions.

If a system Σ_i in Fig.1 is static, Problem 1 can be replaced by a weaker Problem 2 where one can employ other flexibilities of functions ξ_i and φ_i.

Theorem 2 *Suppose that* Σ_1 *is a static system, and* Σ_2 *is a dynamic system fulfilling the following.*

(i) The system Σ_1 *satisfies*

$$\rho_1(z_1, u_1, r_1) \geq 0 \tag{25}$$

for all $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ *and* $t \in \mathbb{R}_+$, *where* $\rho_1 : (z_1, u_1, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$ *is a continuous function satisfying* $\rho_1(0,0,0) = 0$.

(ii) The system Σ_2 *satisfies (ii) of Theorem 1.*

If there is a solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ *to Problem 2, the equilibrium* $x = x_2 = 0$ *of the interconnected system* Σ *is globally uniformly asymptotically stable for* $r \equiv 0$. *Furthermore, there exist a* $\mathbf{C}^1$ *function* $V_{cl} : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ *and class* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ *such that*

$$\underline{\alpha}_{cl}(|x_2|) \leq V_{cl}(t,x_2) \leq \bar{\alpha}_{cl}(|x_2|), \quad \forall x_2 \in \mathbb{R}^{n_2}, t \in \mathbb{R}_+ \tag{26}$$

is satisfied and

$$\frac{dV_{cl}}{dt} \leq \rho_e(x_2, r), \quad \forall x_2 \in \mathbb{R}^{n_2}, r \in \mathbb{R}^m, t \in \mathbb{R}_+ \tag{27}$$

holds along the trajectories of the system Σ.

According to [6], a Lyapunov function proving this theorem is given by

$$V_{cl}(t,x_2) = \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{28}$$

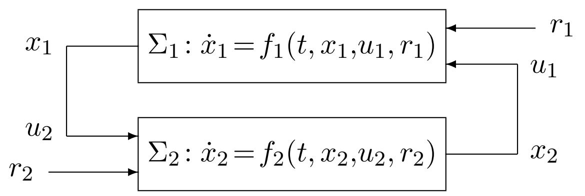

Figure 1: Feedback interconnected system Σ

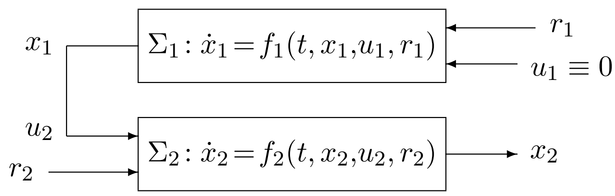

Figure 2: Cascade system Σ_c

In [10, 11, 12], a system Σ_1 satisfying (19) is said to be dissipative. Then, the function ρ_1 is referred to as the supply rate. Following the terminology, in the rest of this paper, a system Σ_i is said to accept a supply rate ρ_i if there exists a $\mathbf{C}^1$ function $V_i(t, x_i)$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_i$, $\bar{\alpha}_i$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \bar{\alpha}_i(|x_i|) \tag{29}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i, u_i, r_i) \leq \rho_i(x_i, u_i, r_i) \tag{30}$$

hold for all x_i, u_i, r_i and t. If Σ_i is a static system, we replace the pair of (29) and (30) by the following single inequality.

$$\rho_i(z_i, u_i, r_i) \geq 0 \tag{31}$$

For convenience, we call the function ρ_i for the static system the supply rate although energy is never stored by any static system.

The central inequalities (4) and (11) of Problem 1 and Problem 2 are not in the form of linear combinations of supply rates. Functional coefficients λ_1, λ_2 and ξ_1 are introduced into the combinations. The use of the functionals λ_1 λ_2 and ξ_1 is contrasted with the early works on Lyapunov stability criteria for interconnected dissipative systems such as [10, 11, 12] where linear combinations of supply rates were employed[6], i.e., constants λ_1, λ_2 and an identity function $\xi_1(s) = s$. The pair of Problem 1 and Problem 2 can be regarded as a general formulation of the state-dependent scaling technique [13, 14, 15].

Cascade systems are special cases of the materials in this section. In other words, the solutions to the state-dependent scaling problems establish stability properties of cascade connection of systems. Indeed, if one of feedback paths $u_1 = x_2$ and $u_2 = x_1$ is disconnected in Fig.1, the interconnected system becomes a cascade connection. When the path of u_i is disconnected, the supply rate $\rho_i(x_i, u_i, r_i)$ simply becomes $\rho_i(x_i, r_i)$. By the cascade system Σ_c, the paper means that the path of $u_1 = x_2$ is cut, i.e., $u_1(t) \equiv 0$, which is depicted in Fig.2.

Problem 1 and Problem 2 are jointly affine in the scaling functions λ_1 and λ_2. It is expected that this affine property is helpful in calculating the solutions, which is the main issue investigated in this paper.

3 Small-gain theorems for iISS and ISS systems

In the previous section, it is shown that the state-dependent scaling problems are directly related to construction of Lyapunov functions, and they provide a unified approach to stability properties of interconnected systems accepting supply rates in a general form. Clearly, solutions to the state-dependent scaling problems exist only if the interconnected system actually possesses the stability property required. It, however, has not been mentioned how easy or difficult it is to find the solutions when the solutions should exist. The purposes of this paper are to address the question of how we are able to obtain solutions to the state-dependent scaling problems, and to give explicit solutions. In the first part of this two-part paper[6], the answers have been given for classically standard supply rates which stability criteria in textbooks of nonlinear systems control deal with. It has been shown in [6] that those classical stability criteria are some of the easiest cases of Problem 1 and Problem 2. Thus, this paper considers several advanced types of supply rate, and seeks explicit solutions for the supply rates.

This section focuses on interconnection of iISS and ISS systems as essential nonlinearities beyond classical ones addressed in [11, 16]. iISS systems and ISS systems are classes of dissipative systems introduced by Sontag[7, 9]. In this section, solutions to the state-dependent scaling problems are derived explicitly for the iISS and the ISS types of supply rate, and the solutions are related to ISS and iISS properties of the feedback loop shown in Fig.1. Small-gain rules are obtained as conditions guaranteeing the existence of the solutions for iISS systems as well as ISS systems. It is the first formulation of its type to address stability of nonlinear interconnections involving iISS systems. For the interconnection of ISS systems, the formulated problem reduces to the ISS small-gain condition which has become popular recently in the area of nonlinear systems control. The formulation of the state-dependent scaling problems enables us to treat iISS systems and ISS systems in a unified manner.

Consider the interconnected system illustrated by Fig.1. It is assumed that each system Σ_i accepts the supply rate in the form of

$$\rho_i(x_i, u_i, r_i) = -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{32}$$

More precisely, we assume that, for each Σ_i, $i = 1, 2$, there exists a $\mathbf{C}^1$ function $V_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(t, x_i) \le \bar{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, t \in \mathbb{R}_+ \tag{33}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i, u_i, r_i) \le -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|)$$
$$, \ \forall x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{m_i}, t \in \mathbb{R}_+ \tag{34}$$

are satisfied for some $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathcal{K}_\infty$ and some α_i, σ_i, $\sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ which are continuous functions satisfying $\alpha_i(0) = \sigma_i(0) = \sigma_{ri}(0) = 0$. The system Σ_i is said to be iISS with respect to input (u_i, r_i) and state x_i if (34) is satisfied for a positive definite function α_i, class $\mathcal{K}$ functions σ_i and σ_{ri}. In the single input case, the second input r_i is null, and the function σ_{ri} vanishes. The function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ iISS Lyapunov function[9]. If α_i is additionally a class $\mathcal{K}_\infty$ function, the system Σ_i is said to be ISS with respect to input (u_i, r_i) and state x_i, and the function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ ISS Lyapunov function[17]. The trajectory-based definition of ISS and iISS may be seen more often than the Lyapunov-based definition this paper adopts. The Lyapunov-based definition is more suitable for the state-space version of stability analysis. The two types of definition is equivalent in the sense that the existence of ISS (iISS) Lyapunov functions is necessary and sufficient for ISS (iISS, respectively)[17, 9]. It is clear from the definition that ISS implies iISS. The converse is not true. Therefore, we can expect that stability of interconnection of iISS systems should requires more restrictive conditions than that of ISS systems.

3.1 Interconnection of iISS systems

We first consider the interconnected system composed of two systems described by supply rates ρ_i of the iISS type.

Theorem 3 *Assume that functions $\rho_i(x_i, u_i, r_i)$, $i = 1, 2$ are in the form of (32) consisting of*

$$\alpha_1 \in \mathcal{P}, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{K} \tag{35}$$

$$\alpha_2 \in \mathcal{P}, \quad \sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{K} \tag{36}$$

Suppose that there exist $c_i > 0$, $i = 1, 2$ and $q \geq 1$ such that

$$[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq c_1\alpha_1(\bar{\alpha}_1^{-1}(s)), \quad \forall s \in \mathbb{R}_+ \tag{37}$$

$$c_2\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq [\alpha_2(\bar{\alpha}_2^{-1}(s))]^q, \quad \forall s \in \mathbb{R}_+ \tag{38}$$

$$c_1 < c_2 \tag{39}$$

are satisfied. Then, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \ \alpha_{cl} \in \mathcal{P}, \ \sigma_{cl} \in \mathcal{K} \tag{40}$$

(ii) In the case of $\alpha_2 \in \mathcal{K}$, a solution to Problem 1 with respect to (40) is given by

$$\lambda_1 = \frac{\nu c_1}{\delta^2}, \quad \lambda_2(s) = \nu q[\delta\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^{q-1} \tag{41}$$

where ν is any positive constant, and

$$\delta = \left(\frac{c_1}{c_2}\right)^{\frac{1}{q+2}} \tag{42}$$

(iii) In the case of $\alpha_2 \notin \mathcal{K}$, there exists $\hat{\alpha}_2 \in \mathcal{K}$ such that

$$\hat{\alpha}_2(s) \leq \alpha_2(s), \ c\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq [\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s))]^q, \ \forall s \in \mathbb{R}_+ \tag{43}$$

hold, and a solution to Problem 1 with respect to (40) is the same as (ii) except that α_2 is replaced by $\hat{\alpha}_2$.

Theorem 1 yields the following directly, which shows that the triplet of (37), (38) and (39) is a condition that allows us to establish stability of the feedback interconnected system with iISS supply rates.

Corollary 1 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32), (35) and (36). Suppose that there exist $c_i > 0$, $i = 1, 2$ and $q > 0$ such that (37), (38) and (39) are satisfied. Then, the interconnected system Σ is iISS with respect to input r and state x*

It is worth stressing that Corollary 1 assures the iISS for all $q > 0$, In the case of $q \geq 1$, the claim of Corollary 1 follows directly from Theorem 1. To obtain the case of $0 < q < 1$, we switch Σ_1 and Σ_2, and apply Theorem 3 to the systems whose subscripts 1 and 2 are exchanged each other.

Remark 1 The two conditions in (37) and (38) necessitate $\liminf_{s\to\infty} \alpha_1(s) > 0$ and $\liminf_{s\to\infty} \alpha_2(s) > 0$ since σ_1 and σ_2 are class $\mathcal{K}$ functions. Consequently, Theorem 3 implicitly requires Σ_1 and Σ_2 to accept positive definite functions α_1 and α_2 which are class $\mathcal{K}$ functions.

Remark 2 It is worth mentioning that the set of (37)-(39) implies at least one system Σ_i of Σ_1 and Σ_2 is required to be ISS with respect to input u_i and state x_i under $r_i(t) \equiv 0$. In order to understand this statement precisely. two points should be emphasized. First, that system Σ_i does not have to be ISS in the presence of the external input r_i. Secondly, the pair of α_i and σ_i of that system Σ_i does not necessarily form a supply rate of the ISS type in (37) and (38). In other words, $\alpha_1 \in \mathcal{P} \setminus \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{P} \setminus \mathcal{K}_\infty$ are allowed in (37) and (38) simultaneously. To verify the statement in the beginning of this remark, we consider the supply rate (32) where $\alpha_i \in \mathcal{P}$ and $\sigma_1 \in \mathcal{K}$ hold. In addition, we assume $\alpha_i \in \mathcal{K}$ due to Remark 1. Then, the conditions (37) and (38) yield

$$\left[\frac{\sigma_2(\underline{\alpha}_1^{-1}(s))}{\alpha_2(\bar{\alpha}_2^{-1}(s))}\right]^q \leq \frac{c_1\alpha_1(\bar{\alpha}_1^{-1}(s))}{c_2\sigma_1(\underline{\alpha}_2^{-1}(s))}, \quad \forall s \in \mathbb{R}_+ \setminus \{0\}$$

From this inequality and (39), we obtain

$$\lim_{s\to\infty} \left[\frac{\sigma_2(\underline{\alpha}_1^{-1}(s))}{\alpha_2(\bar{\alpha}_2^{-1}(s))}\right]^q \leq \lim_{s\to\infty} \frac{\alpha_1(\bar{\alpha}_1^{-1}(s))}{\sigma_1(\underline{\alpha}_2^{-1}(s))} \tag{44}$$

Suppose that $\alpha_1 \in \mathcal{K} \setminus \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K} \setminus \mathcal{K}_\infty$ hold. Then, limiting values of σ_1 and σ_2 toward ∞ are guaranteed to be finite by (44) and $q > 0$ since σ_1 and σ_2 are class $\mathcal{K}$ functions. From (44) and $q > 0$ it also follows that

$$\alpha_2(\infty) < \sigma_2(\infty) \Rightarrow \alpha_1(\infty) \geq \sigma_1(\infty) \tag{45}$$

$$\alpha_1(\infty) < \sigma_1(\infty) \Rightarrow \alpha_2(\infty) \geq \sigma_2(\infty) \tag{46}$$

It can be verified that the system Σ_i is ISS if $\alpha_i(\infty) \geq \sigma_i(\infty)$ holds[17]. In other words, the inequality $\alpha_i(\infty) \geq \sigma_i(\infty)$ guarantees the existence of a $\mathbf{C}^1$ ISS Lyapunov function for a supply rate composed of another pair of $\alpha_i \in \mathcal{K}_\infty$ and $\sigma_1 \in \mathcal{K}$. Therefore, the property of (45) and (46) implies that the set of (37)-(39) requires at least one of Σ_1 and Σ_2 to be ISS with respect to input u_i and state x_i under $r_i(t) \equiv 0$. The requirement of (45) and (46) is natural in view of the 'small gain' for the stability of the interconnection, and it can be intuitively explained as follows. Suppose that neither of the iISS systems Σ_1 and Σ_2 is ISS for $r_i(t) \equiv 0$. Then, there are no iISS Lyapunov functions whose supply rates satisfy $\alpha_i(\infty) \geq \sigma_i(\infty)$. Thus, in the absence of r_i, iISS Lyapunov functions $V_1(x_1)$ and $V_2(x_2)$ given 'arbitrarily' satisfy

$$\frac{dV_1(x_1)}{dt} \leq -\alpha_1(\bar{\alpha}_1^{-1}(V_1(x_1)) + \sigma_1(\underline{\alpha}_2^{-1}(V_2(x_2)) \tag{47}$$

$$\frac{dV_2(x_2)}{dt} \leq -\alpha_2(\bar{\alpha}_2^{-1}(V_2(x_2)) + \sigma_2(\underline{\alpha}_1^{-1}(V_1(x_1)) \tag{48}$$

along the trajectories of Σ_i, and

$$\alpha_1(\infty) < \sigma_1(\infty), \quad \alpha_2(\infty) < \sigma_2(\infty) \tag{49}$$

Due to (49), there exist sufficient large $l_1, l_2 > 0$ such that $\alpha_1(\infty) < \sigma_1(\underline{\alpha}_2^{-1}(l_2))$ and $\alpha_2(\infty) < \sigma_2(\underline{\alpha}_1^{-1}(l_1))$ hold. We have $dV_i(x_i)/dt \geq 0$ for $x_i \in \mathbf{U}_i(l_i) = \{x_i \in \mathbb{R}^{n_i} : V_i(x_i) \geq l_i\}$ if we can assume that the pair $\{\alpha_i, \sigma_i\}$ are selected such that the gap in the inequality (47) or (48) is sufficiently small in $\mathbf{U}_i(l_i)$. Hence, the simultaneous property (49) contradicts the global asymptotic stability of $x = 0$.

Problem 1 also enables us to establish the stability of the cascade connection of iISS systems. The following is obtained by letting $\sigma_1 = 0$.

Corollary 2 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32), (35) and (36). Suppose that there exist $c_1 > 0$ and $q > 0$ such that (37) is satisfied. Then, the cascade system Σ_c is iISS with respect to input r and state x.*

3.2 Interconnection of ISS and iISS systems

In this subsection, we consider the interconnection of an iISS system and an ISS system.

Theorem 4 *Assume that functions $\rho_i(x_i, u_i, r_i)$, $i = 1, 2$ are in the form of (32) consisting of*

$$\alpha_1 \in \mathcal{K}_\infty, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{K} \tag{50}$$

$$\alpha_2 \in \mathcal{P}, \quad \sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{K} \tag{51}$$

Suppose that there exist $c_i > 1$, $i = 1, 2$ and $k > 0$ such that

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w)]^k}{c_1\sigma_1(w)} \leq \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2^{-1}(s)]^k}{c_1\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+ \tag{52}$$

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s), \quad \forall s \in \mathbb{R}_+ \tag{53}$$

are satisfied. Then, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \ \alpha_{cl} \in \mathcal{K}, \ \sigma_{cl} \in \mathcal{K} \tag{54}$$

(ii) In the case of $\alpha_2 \in \mathcal{K}$, a solution to Problem 1 with respect to (54) is given by

$$\lambda_1(s) = \max_{w\in[0,s]} \nu c_1 c_2^q \underline{\delta}^{\frac{q}{q+1}} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(w)]^q}{\alpha_1 \circ \bar{\alpha}_1^{-1}(w)} \tag{55}$$

$$\lambda_2(s) = \nu q [\underline{\delta}^{\frac{1}{q+1}} \alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^{q-1} \tag{56}$$

where ν, $\underline{\delta}$ and q are any constants satisfying

$$\nu > 0, \quad 1 > \underline{\delta} > 0 \tag{57}$$

$$c_2^q > [\underline{\delta}(c_1 - 1)]^{-1}, \quad q \geq k, \quad q > 1 \tag{58}$$

(iii) In the case of $\alpha_2 \notin \mathcal{K}$, there exists $\hat{\alpha}_2 \in \mathcal{K}$ such that

$$\hat{\alpha}_2(s) \leq \alpha_2(s) \tag{59}$$

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w)]^k}{c_1\sigma_1(w)} \leq \frac{[\hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2^{-1}(s)]^k}{c_1\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+ \tag{60}$$

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \leq \hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s), \quad \forall s \in \mathbb{R}_+ \tag{61}$$

hold, and a solution to Problem 1 with respect to (54) is the same as (ii) except that α_2 is replaced by $\hat{\alpha}_2$.

Furthermore, the statements (i), (ii) and (iii) are true even in the case of $\alpha_1 \in \mathcal{K}$ fulfilling

$$\lim_{s\to\infty} \alpha_1(s) = \bar{\eta} \lim_{s\to\infty} \{\sigma_1(s) + \sigma_{r1}(s)\} \tag{62}$$

for some $\bar{\eta} > 1$ if the constants c_1, $\underline{\delta}$ and q satisfy

$$\frac{(1-\underline{\delta}^{\frac{1}{q+1}})\bar{\eta}(\bar{\nu}+1)}{(1-\underline{\delta}^{\frac{1}{q+1}})\bar{\eta}(\bar{\nu}+1)-\bar{\nu}} < c_1 \tag{63}$$

$$\frac{\bar{\nu}}{\bar{\nu}+1} < (1-\underline{\delta}^{\frac{1}{q+1}})\bar{\eta} \tag{64}$$

where $\bar{\nu} \geq 0$ is given by

$$\bar{\nu} \lim_{s\to\infty} \sigma_1(s) = \lim_{s\to\infty} \sigma_{r1}(s) \tag{65}$$

It is stressed that there always exist ν, $\underline{\delta}$ and q fulfilling (57) and (58). The function $\lambda_1(s)$ given in (55) fulfills $\lim_{s\to 0^+} \lambda_1(s) < \infty$, which is guaranteed by (52). In fact, the left hand side of (52) is a non-decreasing continuous function due to the maximization. The right hand side of (52) takes finite positive value at all $s \in (0,\infty)$. In this situation, the inequality of (52) implies

$$\lim_{s\to 0+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} < \infty \tag{66}$$

Hence, the function $\lambda_1(s)$ given in (55) is a non-decreasing continuous function and $\lim_{s\to 0^+} \lambda_1(s) < \infty$ is satisfied.

Remark 3 The readers may be confused with the claim regarding $\alpha_1 \in \mathcal{K}$ in Theorem 4 since $V_1(t,x_1)$ seems to be only an iISS Lyapunov function at a glance. The claim is, however, reasonable. We should be aware that the existence of $\bar{\eta} > 1$ satisfying (62) implies that the system Σ_1 is ISS with respect to input (u_1, r_1) and state x_1 [17]. It is also verified that there is another function $\tilde{V}_1(t,x_1)$ qualified as a $\mathbf{C}^1$ ISS Lyapunov function with $\tilde{\alpha}_1 \in \mathcal{K}_\infty$. Furthermore, it is worth mentioning that if the exogenous signal r_1 is absent, the two cases of $\alpha_1 \in \mathcal{K}$ and $\alpha_1 \in \mathcal{K}_\infty$ can be treated exactly in the same way. Indeed, the inequalities (63) and (64) are automatically satisfied when $\bar{\nu} = 0$ holds.

The following is a direct corollary of Theorem 4, which establishes the iISS property of the mixed interconnection of iISS and ISS systems.

Corollary 3 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32), (50) and (51). Suppose that there exist $c_i > 0$, $i = 1,2$ and $k > 0$ such that (52) and (53) are satisfied. Then, the interconnected system Σ is iISS with respect to input r and state x.*

Remark 4 The assumption (52) can be replaced by a simpler assumption that there exists a constant $k > 0$ achieving at least one of

$$\frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} \text{ is non-decreasing} \tag{67}$$

$$\frac{[\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^k}{\sigma_1 \circ \underline{\alpha}_2^{-1}(s)} \text{ is non-decreasing} \tag{68}$$

It is easily verified that each of (67) and (68) implies (52) under the assumption (53).

Stability of the cascade system can be also obtained from Theorem 4. Since the expression (52) is not ready for the case of $\sigma_1 = 0$, an alternative expression is used.

Corollary 4 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32), (50) and (51). Suppose that there exists $k > 0$ such that*

$$\lim_{s\to 0+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} < \infty \tag{69}$$

holds, Then, the cascade system Σ_c is iISS with respect to input r and state x.

It is known that the cascade of ISS systems are ISS. Corollary 4 shows that the stability of the cascade connection is ensured even if one system driven by the other system is only iISS under an additional condition (69). The following corollary deals with the situation where an iISS is driving an ISS system.

Corollary 5 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32) and*

$$\alpha_1 \in \mathcal{P}, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{K} \tag{70}$$

$$\alpha_2 \in \mathcal{K}_\infty, \quad \sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{K} \tag{71}$$

Then, the cascade system Σ_c is iISS with respect to input r and state x.

This corollary is obtained easily from application of Theorem 4 to the systems whose scripts 1 and 2 are permuted. Corollary 5 which does not pose any additional conditions is a natural extension of a known fact that the cascade of an ISS system and a globally asymptotically stable system is globally asymptotically stable.

3.3 Interconnection of ISS systems

This subsection deals with the interconnection consisting of ISS systems. We are able to obtain a solution to the state-dependent scaling problem for ISS supply rates as follows.

Theorem 5 *Assume that functions $\rho_i(x_i, u_i, r_i)$, $i = 1, 2$ are in the form of (32) consisting of*

$$\alpha_1 \in \mathcal{K}_\infty, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{K} \tag{72}$$

$$\alpha_2 \in \mathcal{K}_\infty, \quad \sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{K} \tag{73}$$

Suppose that there exist $c_i > 1$, $i = 1, 2$ such that

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{74}$$

is satisfied. Then, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \ \alpha_{cl} \in \mathcal{K}_\infty, \ \sigma_{cl} \in \mathcal{K} \tag{75}$$

(ii) In the case of $\sigma_1 \in \mathcal{K}_\infty$, a solution to Problem 1 with respect to (75) is given by

$$\lambda_1(s) = \left[\nu_1 \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \left[\frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right]^m \tag{76}$$

$$\lambda_2(s) = \frac{c_2}{\delta(c_2-1)} \left[\nu_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right] \left[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{m+1} \tag{77}$$

where $\nu_1 : s \in \mathbb{R}_+ \to \mathbb{R}_+$ is any non-decreasing continuous function satisfying

$$\nu_1(s) > 0, \quad \forall s \in (0, \infty) \tag{78}$$

and δ, τ_1 and m are any real numbers satisfying

$$0 \le m, \quad 0 < \delta < 1, \quad 1 < \tau_1 \le c_1 \tag{79}$$

$$\frac{\tau_1}{[\delta^2(\tau_1 - 1)(c_2 - 1)]^{\frac{1}{m+1}}} \le c_1 \tag{80}$$

(iii) In the case of $\sigma_1 \notin \mathcal{K}_\infty$, there exists $\hat{\sigma}_1 \in \mathcal{K}_\infty$ such that

$$\sigma_1(s) \le \hat{\sigma}_1(s), \quad \forall s \in \mathbb{R}_+ \tag{81}$$

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{82}$$

hold, and a solution to Problem 1 with respect to (75) is the same as (ii) except that σ_1 is replaced by $\hat{\sigma}_1$.

It is worth noting that there always exist m, δ, τ_1 such that (79) and (80) hold.

Remark 5 Solutions to Problem 1 are not unique. This point can be seen clearly by looking at one of the easiest cases. Suppose that $\nu\sigma_1(s) = \alpha_2(s)$ holds for some $\nu > 0$. Pick

$$\lambda_1(s) = \delta\nu, \quad \delta = \max\{1/c_1, 1/c_2\} \tag{83}$$

$$\lambda_2(s) = 1 \tag{84}$$

which are not in the form of (76) and (77). The inequality (4) is satisfied with (75) and

$$\alpha_{cl}(s) = \min_{s=|x|} \{(1-\delta)\delta\nu\alpha_1(|x_1|) + (1-\delta)\alpha_2(|x_2|)\}$$

$$\sigma_{cl}(s) = \max_{s=|r|} \{\delta\nu\sigma_{r1}(|r_1|) + \sigma_{r2}(|r_2|)\}$$

if

$$\sigma_2(s) \le \delta^2\nu\alpha_1(s), \quad \delta\nu\sigma_1(s) \le \delta\alpha_2(s), \quad \forall s \in \mathbb{R}_+$$

hold. Due to $\nu\sigma_1(s) = \alpha_2(s)$, the above two inequalities are satisfied if

$$\alpha_1^{-1} \circ \frac{1}{\delta}\sigma_1 \circ \alpha_2^{-1} \circ \frac{1}{\delta}\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+$$

holds. This is guaranteed when $c_1 > 1$ and $c_2 > 1$ satisfy (74). Thus, the pair of (83) and (84) which are different from (76) and (77) solves Problem 1.

The following corollary is obtained directly from Theorem 5.

Corollary 6 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32), (72) and (73). If there exist $c_i > 1$, $i = 1, 2$ such that (74) is satisfied, the interconnected system Σ is ISS with respect to input r and state x.*

The statement of Corollary 6 by itself is essentially the same as the ISS small-gain theorem presented in [1, 2], This paper, however, proposes a new approach to the ISS small-gain theorem through Theorem 5. The combination of Corollary 6 and Theorem 5 forms a state-dependent scaling version of the ISS small-gain theorem. The proof derived from the state-dependent scaling problem gives explicit information about how to construct a Lyapunov function to establish the ISS property of the feedback interconnected system. In fact, the Lyapunov function is given explicitly by (24) where λ_1 and λ_2 are given by (76) and (77). It contrasts sharply with the original ISS small-gain

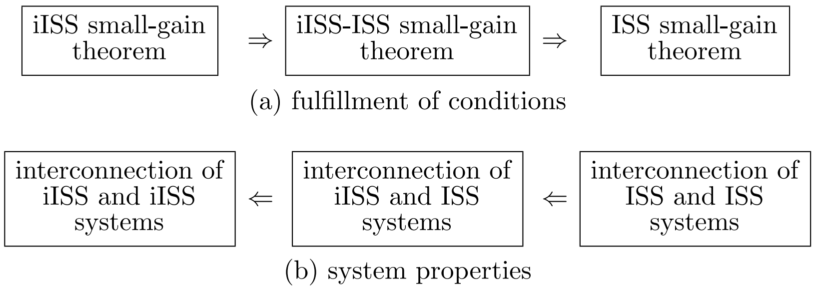

Figure 3: Relationships between small-gain theorems

theorem[1, 2, 3] which are stated and proved by using trajectories of systems. In this sense, the state-dependent scaling approach is constructive in view of Lyapunov functions. The Lyapunov function which leads us to the ISS small-gain theorem is not necessarily unique. Indeed, There is another type proof of the ISS small-gain theorem based on the existence of a different Lyapunov function. In [18], the existence of a smooth Lyapunov function is proved by presenting non-smooth functions which determine a Lyapunov function in an implicit manner. In contrast, this paper demonstrates that the equation (24) defined with state-dependent scaling functions $\{\lambda_1, \lambda_2\}$ given by Theorem 5 provides us with an explicit formula for the Lyapunov function. Another desirable feature of the state-dependent scaling approach is that it allows a smooth transition to stability criteria for more general systems. For instance, this paper explains the ISS small-gain theorem as a special case of the state-dependent scaling problems.

Theorem 5 also covers stability of the cascade connection. Switching Σ_1 and Σ_2 and letting $\sigma_2 = 0$ in Theorem 5, we obtain the following.

Corollary 7 *Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (32), (72) and (73). Then, the cascade system Σ_c is ISS with respect to input r and state x.*

It is known that the cascade connection of ISS systems is ISS, and another Lyapunov-type proof of this fact can be found in [3]. In ISS analysis of open-loop systems and cascade systems, Lyapunov functions have been used successfully by [7, 17, 19, 3]. This paper extends their techniques to feedback systems naturally, and this section has demonstrated that a similar type of Lyapunov function can be tailored for proving the ISS small-gain theorem for feedback interconnected systems. Regardless of the difference between feedback and cascade, the construction of the Lyapunov function falls with in the same single framework of state-dependent scaling problems which can be solved explicitly.

3.4 Relation between existence conditions

The inequality (74) guaranteeing the existence of solutions to Problem 1 for ISS supply rates is identical to a condition derived by [1, 2]. It is called the ISS small-gain condition since it guarantees that the feedback interconnection of ISS systems is ISS. The fact is widely referred to as the ISS small-gain theorem in the literature. Theorem 5 describes the ISS small-gain *theorem* as a special case of the existence of state-dependent scaling functions solving Problem 1. Corollary 3 demonstrates that the ISS small-gain *condition* can lead us to stability of the feedback interconnection *even if one of the systems is only iISS* under an additional condition (52). Corollary 1 deals with the interconnection of systems individually described by iISS supply rates, and the conditions for the existence of a solution are given in terms of gain-like functions. The author calls Corollary 3 the iISS-ISS small-gain theorem. In a similar manner, the author refers to Corollary 1 as the iISS small-gain theorem. There are reasonable relationships between the iISS small-gain theorem, the iISS-ISS small-gain theorem and the ISS small-gain theorem(Corollary 6) as described by the following.

Theorem 6 *Suppose that σ_1 and σ_2 are class $\mathcal{K}$ functions.*

(i) Assume that $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{P}$ hold. If there exist a pair of $c_1 > 0$, $c_2 > 0$ and $q \geq 1$ such that (37)-(39) are satisfied, there exist another pair of $c_1 > 1$, $c_2 > 1$ and $k > 0$ such that (52) and (53) hold.

(ii) Assume that $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K}_\infty$ hold. If there exist a pair of $c_1 > 1$, $c_2 > 1$ and $k > 0$ such that (52) and (53) hold, the inequality (74) is satisfied.

An interpretation is illustrated in Fig.3. The broader the class of systems covered by a theorem is, the more restrictive the condition for the existence is.

Naturally, solutions to state-dependent scaling problems are not unique. For example, the pair $\{\lambda_1, \lambda_2\}$ given in Theorem 3 is a solution to the problem for supply rates considered in Theorem 4 and Theorem 5. In the same manner, the pair $\{\lambda_1, \lambda_2\}$ given in Theorem 4 is also a solution to Theorem 5. Given particular functions of supply rates, we are sometimes able to find an ad hoc solution. An important benefit from Theorem 3, Theorem 4 and Theorem 5 is that we can predict the existence of solutions before solving the problem. The theorems also provide us with analytical solutions in the closed form for iISS and ISS supply rates.

Remark 6 Theorem 3-5 and Corollary 1-7 are valid even when either or both of Σ_1 and Σ_2 do not have the exogenous signals r_1 and r_2. For example, the function ρ_e becomes

$$\rho_e(x, r) = -\alpha_{cl}(|x|) \tag{85}$$

when both of the exogenous signals are absent. If r_i is absent in Σ_i, all terms containing σ_{ri}, μ_r, τ_r, τ_{ri} or θ_{ri} in the proofs disappear.

3.5 Interconnection of iISS and static systems

In this section, we consider interconnection of static and dynamic systems. When a system Σ_i is static, it is supposed that Σ_i accepts a supply rate in the form of

$$\rho_i(z_i, u_i, r_i) = -\alpha_i(|z_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{86}$$

More precisely, we assume that

$$-\alpha_i(|z_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \geq 0, \quad \forall u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{m_i}, t \in \mathbb{R}_+ \tag{87}$$

holds for some continuous functions α_i, σ_i, $\sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy $\alpha_i(0) = \sigma_i(0) = \sigma_{ri}(0) = 0$. In addition, we assume

$$\liminf_{s\to\infty} \alpha_i(s) \geq \liminf_{s\to\infty} \{\sigma_i(s) + \sigma_{ri}(s)\} \tag{88}$$

without loss of generality for static systems. To see this, suppose that the system Σ_i does not admit α_i, σ_i and σ_{ri} satisfying (88). Due to $\liminf_{s\to\infty} \alpha_i(s) < \liminf_{s\to\infty} \{\sigma_i(s) + \sigma_{ri}(s)\}$ and (87), the boundedness of the inputs $u_i(t)$ and $r_i(t)$ does not guarantee the boundedness of the output $z_i(t)$. The size of $u_i(t)$ and $r_i(t)$ needs to be sufficiently small to obtain bounded $z_i(t)$. This fact contradicts the assumption that $h_i(t, u_i, r_i)$ is locally Lipschitz with respect to u_i on $\mathbb{R}^{n_{ui}}$ and r_i on $\mathbb{R}^{m_i}$.

Consider the interconnected system shown in Fig.1. Suppose that Σ_1 is a static system described by (17), while Σ_2 is a dynamic system described by (16). The following theorem provides a solution to the state-dependent scaling problem for such an interconnected system.

Theorem 7 *Assume that functions $\rho_1(z_1, u_1, r_1)$ and $\rho_2(x_2, u_2, r_2)$ are in the form of (86) and (32), respectively, and consist of*

$$\alpha_1 \in \mathcal{K}_\infty, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{K} \tag{89}$$

$$\alpha_2 \in \mathcal{P}, \quad \sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{K} \tag{90}$$

Suppose that there exist $c_i > 1$, $i = 1, 2$ such that

$$c_2\sigma_2 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \leq \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{91}$$

is satisfied. Then, the following hold.

(i) Problem 2 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x_2, r) = -\alpha_{cl}(|x_2|) + \sigma_{cl}(|r|)), \ \alpha_{cl} \in \mathcal{P}, \ \sigma_{cl} \in \mathcal{K} \tag{92}$$

(ii) A solution to Problem 2 with respect to (92) is given by

$$\lambda_1 = \lambda_2 = \nu \tag{93}$$

$$\xi_1(s) = \sigma_2 \circ \alpha_1^{-1}(s) \tag{94}$$

$$\varphi_1(s) = \alpha_1(s) \tag{95}$$

where ν is any positive constant.

Furthermore, the statements (i) and (ii) are true even in the case of $\alpha_1 \in \mathcal{K}$ if the constant c_1 satisfies

$$\frac{\bar{\eta}(\bar{\nu}+1)}{\bar{\eta}(\bar{\nu}+1) - \bar{\nu}} \leq c_1 \tag{96}$$

where $\bar{\eta} \geq 1$ and $\bar{\nu} \geq 0$ denote constants which fulfill

$$\lim_{s\to\infty} \alpha_1(s) = \bar{\eta} \lim_{s\to\infty} \{\sigma_1(s) + \sigma_{r1}(s)\} \tag{97}$$

$$\bar{\nu} \lim_{s\to\infty} \sigma_1(s) = \lim_{s\to\infty} \sigma_{r1}(s) \tag{98}$$

Theorem 7 yields the following due to Theorem 2.

Corollary 8 *Assume that Σ_1 is a static system accepting a supply rate ρ_1 in the form of (86) and (89), and Σ_2 is a dynamic system accepting a supply rate ρ_2 in the form of (32) and (90). If there exist $c_i > 1$, $i = 1, 2$ such that (91) is satisfied, the interconnected system Σ is iISS with respect to input r and state x_2. Furthermore, if α_2 is additionally assumed to be a class $\mathcal{K}_\infty$ function, the interconnected system Σ is ISS with respect to input r and state x_2.*

We come at a similar consequence by using Theorem 4 instead of Theorem 7. It is due to the inclusive relation between Problem 1 and Problem 2. In other words, we can prove the iISS property of the closed loop by using λ_1 and λ_2 given by (55), (56) and $\xi_1(s) = s$. Note that $\underline{\alpha}_1(|z_1|) = V_1(z_1) = \bar{\alpha}_1(|z_1|)$ is used for the static system Σ_1 in Problem 1. We should be aware that, compared with (91), the pair of (52) and (53) is conservative. In the case of $\alpha_2 \in \mathcal{K}_\infty$, i.e., when Σ_2 is ISS, we can also invoke Theorem 5 to obtain the ISS property in Corollary 8.

An important point of Corollary 8 derived from Theorem 7 is that the system Σ_2 is not required to be ISS. The small-gain condition (91) without any additional constraints is sufficient for the stability even when the dynamic system Σ_2 is only iISS. It contrasts with the case where the interconnected system consists of only dynamic systems.

Remark 7 Theorem 7 and Corollary 8 are valid even when either or both of Σ_1 and Σ_2 do not have the exogenous signal r_i. When r_1 is absent, the constant c_1 is required to satisfy only $c_1 \geq 1$ in both the cases of $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K}$ of Corollary 8. Indeed, it is verified with $\bar{\nu} = 0$ in (96) and (98).

Remark 8 When a static system Σ_i satisfies (87) for some $\sigma_i \in \mathcal{K}$ and $\sigma_{ri} \in \mathcal{K}$, we can assume $\alpha_i \in \mathcal{K}_\infty$ without loss of generality. In fact, it can be verified that the inequality (88) guarantees the existence of class $\mathcal{K}$ functions $\hat{\sigma}_i$ and $\hat{\sigma}_{ri}$ satisfying

$$-\hat{\alpha}_i(|z_i|) + \hat{\sigma}_i(|u_i|) + \hat{\sigma}_{ri}(|r_i|) \geq 0, \quad \forall u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{m_i}, t \in \mathbb{R}_+$$

for arbitrarily given $\hat{\alpha}_i \in \mathcal{K}_\infty$ if (87) holds. Therefore, the assumption (87) given with $\sigma_i \in \mathcal{K}$ and $\sigma_{ri} \in \mathcal{K}$, implies that the magnitude of output z_i is nonlinearly bounded by the magnitude of the inputs u_i and r_i.

Consider the cascade system Σ_c shown in Fig.2. Suppose that the system Σ_1 is static, and and the system Σ_2 is dynamic. The following corollary is obtained from Corollary 8 by letting $\sigma_1 = 0$.

Corollary 9 *Assume that Σ_1 is a static system accepting a supply rate ρ_1 in the form of (86) and (89), and Σ_2 is a dynamic system accepting a supply rate ρ_2 in the form of (32) and (90). Then, the cascade system Σ_c is iISS with respect to input r and state x_2. Furthermore, if α_2 is additionally assumed to be a class $\mathcal{K}_\infty$ function, the cascade system Σ_c is ISS with respect to input r and state x_2.*

This fact is natural since the static system is nonlinearly bounded.

4 Examples

This section illustrates the effectiveness and versatility of the state-dependent scaling characterization through several simple examples. It is shown how scaling functions are obtained successfully, and how the state-dependent scaling approach enables us to discover Lyapunov functions establishing stability properties for various classes of nonlinearities. Systems employed in this section are the same as those employed in [6]. The examples are numbered in the same order. This section, however, takes a different approach. In [6], the state-dependent scaling problems are tackled directly without having any guarantees of the existence of solutions a priori. By contrast, this section uses the results of small-gain theorems presented in Section 3 to check if solutions to the state-dependent scaling problems exist in advance. The results of the small-gain theorems enable one to assess stability properties of the interconnection without calculating the solutions. In addition, if one uses the formulas for the solutions derived in Section 3, Lyapunov functions are obtained automatically.

Example 1 The first example of Fig.1 is the interconnected system defined for $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ and $r_2 \in \mathbb{R}_+$ by

$$\Sigma_1 : \ \dot{x}_1 = -\left(\frac{x_1}{x_1+1}\right)^2 + 3\left(\frac{x_2}{x_2+1}\right)^2, \quad x_1(0) \in \mathbb{R}_+ \tag{99}$$

$$\Sigma_2 : \ \dot{x}_2 = -\frac{4x_2}{x_2+1} + \frac{2x_1}{x_1+1} + 6r_2, \quad x_1(0) \in \mathbb{R}_+ \tag{100}$$

Clearly, these two subsystems are iISS with respect to input (u_i, r_i) and state x_i, where $u_1 = x_2$ and $u_2 = x_1$ hold, and r_1 is null. It is verified that neither Σ_1 nor Σ_2 is ISS with respect to input (u_i, r_i) and state x_i. Since $x(0) \in \mathbb{R}_+^2$ and $r_2(t) \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$ imply $x(t) \in \mathbb{R}_+^2, \forall t \in \mathbb{R}_+$, the simplest choices of iISS Lyapunov functions for individual Σ_1 and Σ_2 are $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$. In fact, we obtain

$$\begin{aligned} \frac{dV_1}{dt} &= \rho_1(x_1, x_2) = -\alpha_1(x_1) + \sigma_1(x_2), \\ &\qquad \alpha_1(s) = \left(\frac{s}{s+1}\right)^2, \ \sigma_1(s) = 3\left(\frac{s}{s+1}\right)^2 \end{aligned} \tag{101}$$

$$\begin{aligned} \frac{dV_2}{dt} &= \rho_2(x_2, x_1, r_2) = -\alpha_2(x_2) + \sigma_2(x_1) + \sigma_{r2}(r_2), \\ &\qquad \alpha_2(s) = \frac{4s}{s+1}, \ \sigma_2(s) = \frac{2s}{s+1}, \ \sigma_{r2}(s) = 6s \end{aligned} \tag{102}$$

For these functions of supply rates, the inequalities (37) and (38) are obtained as

$$2^q \left(\frac{s}{s+1}\right)^q \le c_1 \left(\frac{s}{s+1}\right)^2, \quad \forall s \in \mathbb{R}_+ \tag{103}$$

$$3c_2 \left(\frac{s}{s+1}\right)^2 \le 4^q \left(\frac{s}{s+1}\right)^q, \quad \forall s \in \mathbb{R}_+ \tag{104}$$

These two inequalities and $0 < c_1 < c_2$ are achieved by $q = 2$, $c_1 = 4$ and $c_2 \in (4, 16/3]$. Thus, the iISS property of the interconnected system given by (99) and (100) follows directly from Corollary 1. It is worth mentioning that the inequalities (103) and (104) are never achieved for $q \neq 2$. Theorem 3 automatically provides us with a subset of solutions to Problem 1. Using the formula (41), we obtain

$$\lambda_1(s) = 1, \quad \lambda_2(s) = bs/(s+1), \ b \in [1.6119, 2) \tag{105}$$

An iISS Lyapunov function of the interconnected system is calculated directly from (24) as

$$V_{cl}(x_1, x_2) = x_1 + b(x_2 - \log(x_2 + 1)), \ b \in [1.6119, 2)$$

The value of $\lambda_1\rho_1 + \lambda_2\rho_2$ with (105) and $b = 1.7$ is plotted on the state space in Fig. 4. For visual simplicity, the surface is drawn for $r_2 = 0$. It is observed that the surface of $\lambda_1\rho_1 + \lambda_2\rho_2$ is below the horizontal plane of zero. This confirms that Problem 1 is solved by the choice (105) of state-dependent scaling functions, which is consistent with Theorem 3. It is easily verified that Problem 1 cannot be solved by any constant $\lambda_1, \lambda_2 > 0$. This is an example that demonstrates the effectiveness of state-dependence of scaling functions for supply rates describing essential nonlinearities. The state-dependence enables us to establish the stability property of the nonlinear system which has not been covered by previously existing stability criteria.

Example 2 Next, consider

$$\Sigma_1 : \dot{x}_1 = -\frac{2x_1}{x_1+1} + \frac{x_2}{(x_1+1)(x_2+1)}, \ x_1(0) \in \mathbb{R}_+ \tag{106}$$

$$\Sigma_2 : \dot{x}_2 = -\frac{4x_2}{x_2+1} + x_1, \quad x_2(0) \in \mathbb{R}_+ \tag{107}$$

Note that $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ holds for all $t \in \mathbb{R}_+$. One system Σ_1 is ISS, and the other system Σ_2 is only iISS. Indeed, the choice $V_1(x_1) = x_1$ yields

$$\begin{aligned} \frac{dV_1(x_1)}{dt} &= \rho_1(x_1, x_2) \le -\alpha_1(x_1) + \sigma_1(x_2) \\ &\qquad \alpha_1(s) = \frac{2s}{s+1}, \ \sigma_1(s) = \frac{s}{s+1} \end{aligned} \tag{108}$$

Thus, the system Σ_1 is ISS since (62) is satisfied for $\bar{\eta} > 1$ [17]. The system Σ_2 is not ISS since we have $x_2 \to \infty$ as $t \to \infty$ for $x_1(t) \equiv 5$. The system Σ_2 is iISS since the choice $V_2(x_2) = x_2$ yields

$$\frac{dV_2(x_2)}{dt} = \rho_2(x_2, x_1) = -\alpha_2(x_2) + \sigma_2(x_1)$$
$$\alpha_2(s) = \frac{4s}{s+1}, \ \sigma_2(s) = s \tag{109}$$

It is easily seen that if λ_1 and λ_2 are restricted to constants, Problem 1 is not solvable. We need to find appropriate functions for λ_1 and λ_2. From

$$c_2\sigma_2 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) = \frac{c_1c_2s}{(2-c_1)s+2}$$

it follows that the condition (53) is identical with

$$8 - c_1c_2 - 4c_1 \geq 0$$

There exist such c_1, $c_2 > 1$. Thus, the small-gain condition (53) is fulfilled. Since Σ_2 is not ISS, we cannot invoke the ISS small-gain theorem. We, however, have

$$\frac{\sigma_2(s)}{\alpha_1(s)} = \frac{s+1}{2}, \quad \frac{\alpha_2(s)}{\sigma_1(s)} = 4$$

which fulfill (67) and (68) for $k = 1$. The inequalities (63) and (64) are satisfied for $\bar{\nu} = 0$. Hence, Corollary 3 concludes that the origin $x = 0$ is globally asymptotically stable. Using $c_1 = 1.2$, $c_2 = 2.5$, $q = 2$, $\nu = 2c_1^{-1}c_2^{-2/3}$ and $\underline{\delta} = 0.9044$, we obtain scaling functions from the formulas (55) and (56) in Theorem 4 as

$$\lambda_1(x_1) = x_1(x_1+1), \quad \lambda_2(x_2) = \frac{7x_2}{x_2+1} \tag{110}$$

An iISS Lyapunov function of the interconnected system is obtained as

$$V_{cl}(x_1, x_2) = \frac{x_1^3}{3} + \frac{x_1^2}{2} + 7(x_2 - \log(x_2+1))$$

The surface plot of $\lambda_1\rho_1 + \lambda_2\rho_2$ with (110) is shown in Fig. 5. It is observed that the choice (110) solves Problem 1.

Example 3 Finally, we consider the interconnected system described by

$$\Sigma_1 : \ \frac{dx_1}{dt} = -2x_1 + x_2 \tag{111}$$
$$\Sigma_2 : \ \frac{dx_2}{dt} = -2x_2^5 + x_2^3x_1^2 \tag{112}$$

The state vector is $x = [x_1, x_2]^T \in \mathbb{R}^2$. Both the two systems Σ_i are ISS. It is easily verified with $V_1(x_1) = x_1^2$ and $V_2(x_2) = x_2^2$. In fact, their time-derivatives along trajectories of the individual systems lead to the following supply rates of the ISS type.

$$\frac{dV_1}{dt} = \rho_1(x_1, x_2) \leq -\alpha_1(x_1) + \sigma_1(|x_2|)$$
$$\alpha_1(s) = 3s^2, \ \sigma_1(s) = s^2 \tag{113}$$
$$\frac{dV_2}{dt} = \rho_2(x_2, x_1) \leq -\alpha_2(x_2) + \sigma_2(|x_1|)$$
$$\alpha_2(s) = \frac{8}{3}s^6, \ \sigma_2(s) = \frac{2}{3}s^6 \tag{114}$$

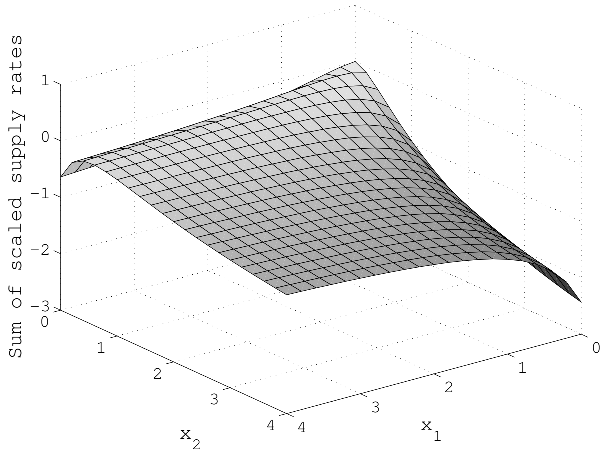

Figure 4: Example 1: State-dependently scaled combination of supply rates with functions λ_1 and λ_2 calculated from the iISS small-gain theorem.

Global asymptotic stability of $x = 0$ is proved if there exist solutions to Problem 1. The ISS small-gain condition (74) is calculated as

$$\left(\frac{c_1}{3}\right)^3 \left(\frac{c_2}{4}\right) \leq 1$$

Obviously, there exist $c_1, c_2 > 1$ fulfilling this condition, so that Theorem 5 guarantees the existence of solutions to Problem 1. Thus, Corollary 6 proves the global asymptotic stability of $x = 0$. The formulas (76) and (77) automatically give us a solution as follows:

$$\lambda_1(s) = s^3, \quad \lambda_2(s) = s \tag{115}$$

Here, $c_1 = 3$, $c_2 = 2$, $\nu_1 = 3/8$, $m = 0$, $\delta = 3/4$ and $\tau_1 = c_1$ are used. Figure 6 shows that the state-dependent scaling functions given in (115) actually solve Problem 1. Note that Problem 1 cannot be solved by any constant $\lambda_1, \lambda_2 > 0$. An ISS Lyapunov function of the overall system is calculated as

$$V_{cl}(x_1, x_2) = \frac{x_1^8}{4} + \frac{x_2^4}{2}$$

The examples presented in this section reveal that the state-dependence of scaling functions, in other words 'nonlinear combination of individual supply rates' or 'nonlinear combination of individual storage functions', is vital for dealing with strong nonlinearities which are not covered by popular classical stability criteria.

5 Conclusions

This paper has discussed the effectiveness of the state-dependent scaling approach to stability analysis of interconnected dissipative systems. The iISS and ISS properties are the focuses of this paper. The idea of the state-dependent scaling problems is formed by an inequality representing the sum of nonlinearly scaled supply rates of dissipative systems. Solving the equality for parameters called scaling functions, we are able to obtain Lyapunov functions of feedback and cascade connected systems explicitly. The effectiveness of the state-dependent scaling approach is not limited to the settings of popular classical stability criteria and the ISS small-gain theorem. Explicit formulas of

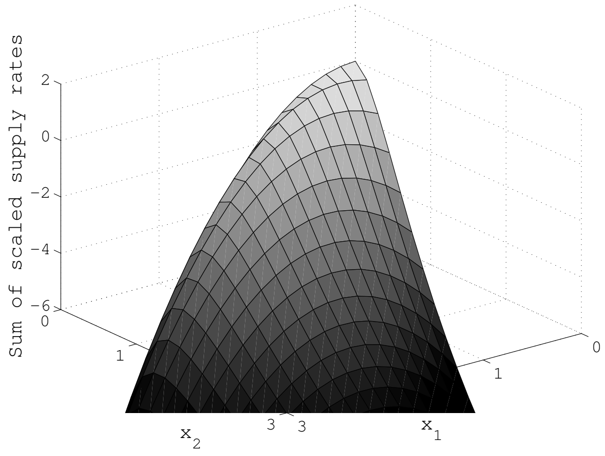

Figure 5: Example 2: State-dependently scaled combination of supply rates with functions λ_1 and λ_2 calculated from the iISS-ISS small-gain theorem.

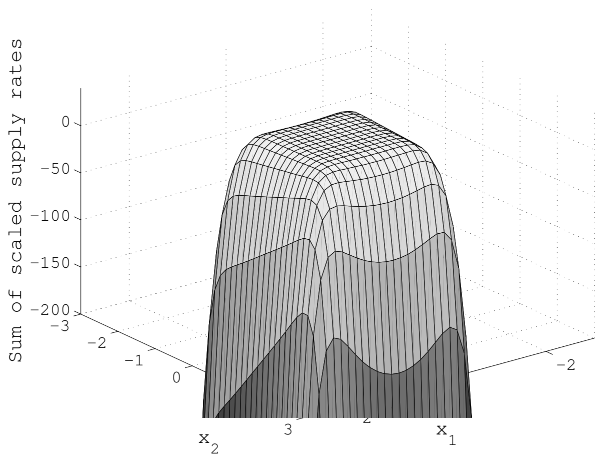

Figure 6: Example 3: State-dependently scaled combination of supply rates with functions λ_1 and λ_2 calculated from the ISS small-gain theorem.

solutions to the state-dependent scaling problems can be obtained for supply rates which are more general than the classical supply rates and the ISS supply rates. In fact, this paper has succeeded in deriving solutions to the problem involving iISS supply rates. Sufficient conditions for the existence of the solutions are obtained as small-gain-like theorems for feedback interconnected systems involving iISS systems. This paper has developed the iISS small-gain theorem and the iISS-ISS small-gain theorem which generalize the ISS small-gain theorem smoothly. It is an interesting direction of future research to seek analytical formulas of solutions to the state-dependent scaling problems for supply rates which are more general than the iISS property.

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Appendix

A Calculating scaling functions

A.1 Proof of Theorem 3

In the case of $q = 1$, the function λ_2 given in (41) becomes $\lambda_2 = \nu > 0$. Using (37) and (38), we obtain

$$\lambda_1\rho_1 + \lambda_2\rho_2 \leq -(\lambda_1 - \lambda_2 c_1)\alpha_1(s) - (\lambda_2 - \lambda_1/c_2)\alpha_2(s)$$

Let $\rho_e(x,r)$ be chosen as

$$\rho_e(x,r) = -(1-\delta)\left[\lambda_1\alpha_1(|x_1|) + \lambda_2\alpha_2(|x_2|)\right] + \lambda_1\sigma_{r1}(|r_1|) + \lambda_2\sigma_{r2}(|r_2|)$$

Then, the inequality (4) is achieved with (41) if

$$c_1 \geq \delta c_1, \quad \delta^3 c_2 \geq c_1 \tag{116}$$

hold. Due to (39) and (42), we have $0 < \delta < 1$ and $\delta^3 c_2 = c_1$. Thus, both the inequalities in (116) are guaranteed. This proves *(i)*, *(ii)* and *(iii)* in the case of $q = 1$. Next, we assume that $q > 1$ and $\alpha_2 \in \mathcal{K}$ hold. Let μ and $\tilde{\mu}$ be any positive constants satisfying

$$\left(\frac{\tilde{\mu}}{\mu}\right)^q = \delta \tag{117}$$

Define $p > 1$ by

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{118}$$

Since (39) and (42) implies $0 < \delta < 1$, we have $0 < \tilde{\mu} < \mu$ which ensures the existence of $\mu_r > 0$ satisfying

$$\frac{1}{\tilde{\mu}^p} \geq \frac{1}{\mu^p} + \frac{1}{\mu_r^p} \tag{119}$$

Using Young's inequality

$$xy \leq \frac{1}{p}\left|\frac{x}{a}\right|^p + \frac{1}{q}|ay|^q \quad ,\forall x, y \in \mathbb{R}$$

which holds for any $a \neq 0$, we obtain

$$\begin{aligned}
&\lambda_2(V_2(t,x_2))\left\{-\alpha_2(|x_2|) + \sigma_2(|x_1|) + \sigma_{r2}(|r_2|)\right\} \\
&\leq -\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + \frac{\nu q}{\tilde{\mu}^q}\left[\frac{1}{p}\left(\frac{\tilde{\mu}^q}{\nu q \mu}\lambda_2(V_2(t,x_2))\right)^p + \right. \\
&\qquad \left. \frac{\mu^q}{q}\sigma_2(|x_1|)^q + \frac{1}{p}\left(\frac{\tilde{\mu}^q}{\nu q \mu_r}\lambda_2(V_2(t,x_2))\right)^p + \frac{\mu_r^q}{q}\sigma_{r2}(|r_2|)^q\right]
\end{aligned} \tag{120}$$

Define $\rho_e(x,r)$ by

$$\rho_e(x,r) = -(1-\delta)\left[\lambda_1\alpha_1(|x_1|) + \lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right] + \lambda_1\sigma_{r1}(|r_1|) + \nu\left(\frac{\mu_r}{\tilde{\mu}}\right)^q \sigma_{r2}(|r_2|)^q$$

Since $0 < \delta < 1$ holds, the function $\rho_e(x,r)$ with $\lambda_1 > 0$ and $\lambda_2 \in \mathcal{K}$ given in (41) satisfies (40). Define λ_1 as in (41). A sufficient condition for (4) is obtained as

$$-\frac{\nu c_1}{\delta^2}\delta\alpha_1(|x_1|) + \nu\left(\frac{\mu}{\tilde{\mu}}\right)^q \sigma_2(|x_1|)^q \leq 0, \quad \forall x_1 \in \mathbb{R}^{n_1} \tag{121}$$

$$\left(\frac{1}{\nu q}\right)^{\frac{1}{q-1}} \frac{q-1}{q}\lambda_2(V_2(t,x_2))^{\frac{q}{q-1}} - \delta\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + \frac{\nu c_1}{\delta^2}\sigma_1(|x_2|) \leq 0, \ \forall x_2 \in \mathbb{R}^{n_2}; \ \forall t \in \mathbb{R}_+ \tag{122}$$

Due to (117), the inequality (121) is identical to

$$[\sigma_2(s)]^q \leq c_1\alpha_1(s), \quad \forall s \in \mathbb{R}_+$$

which is ensured by (37). Since $\alpha_2 \in \mathcal{K}$ is non-decreasing, the inequality (122) holds if

$$\left(\frac{1}{\nu q}\right)^{\frac{1}{q-1}} \frac{q-1}{q} \lambda_2(s)^{\frac{q}{q-1}} - \delta\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s)) + \frac{\nu c_1}{\delta^2}\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq 0, \quad \forall s \in \mathbb{R}_+ \tag{123}$$

is satisfied. When $\lambda_2 \in \mathcal{K}$ is given by (41), the inequality (123) is equivalent to

$$\frac{\nu c_1}{\delta^2}\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq \nu\delta^q[\alpha_2(\bar{\alpha}_2^{-1}(s))]^q \quad \forall s \in \mathbb{R}_+$$

Due to (42), this inequality is identical to (38). Thus, the part of *(ii)* has been proved for $q > 1$. In the case of *(iii)*, the inequality (38) guarantees the existence of a class $\mathcal{K}$ function $\hat{\alpha}_2$ which satisfies (43). Due to $\hat{\alpha}_2(s) \leq \alpha_2(s)$, the inequality (4) holds with α_2 if it holds with $\hat{\alpha}_2$. The rest of the proof is the same as *(ii)*. The claim *(i)* follows directly from *(ii)* and *(iii)*.

Proof of Theorem 4

Define δ and choose $\bar{\delta}$ as

$$\delta = \underline{\delta}^{\frac{1}{q+1}}, \quad \delta < \bar{\delta} < 1 \tag{124}$$

The inequality (58) and $c_2 > 1$ ensure the existence of μ and $\tilde{\mu}$ satisfying $0 < \tilde{\mu} < \mu$ and

$$\left(\frac{c_2\tilde{\mu}}{\mu}\right)^q \geq \frac{1}{\underline{\delta}(c_1 - 1)} \tag{125}$$

Suppose $\tau > 1$. Then, there exists $\tau_r > 1$ such that

$$1 - \frac{1}{\tau} - \frac{1}{\tau_r} \geq \bar{\delta}\left(1 - \frac{1}{\tau}\right) \tag{126}$$

is satisfied. Using these τ and τ_r, define the following class $\mathcal{K}$ functions.

$$\theta_1(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau\sigma_1(s), \quad \theta_{r1}(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_r\sigma_{r1}(s)$$

Combining calculations in individual cases separated by $\alpha_1(|x_1|) \geq \tau\sigma_1(|x_2|)$, $\alpha_1(|x_1|) < \tau\sigma_1(|x_2|)$, $\alpha_1(|x_1|) \geq \tau_r\sigma_{r1}(|r_1|)$ and $\alpha_1(|x_1|) < \tau_r\sigma_{r1}(|r_1|)$, we obtain

$$\begin{aligned}&\lambda_1(V_1(t,x_1))\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\}\\ &\leq \bar{\delta}\left(-1 + \frac{1}{\tau}\right)\lambda_1(V_1(t,x_1))\alpha_1(|x_1|) + \lambda_1(\theta_1(|x_2|))\sigma_1(|x_2|) + \lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)\end{aligned}$$

on the assumption that $\lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Define $p > 1$ by (118), and pick $\mu_r > 0$ satisfying (119). Using Young's inequality, we obtain (120). Define $\rho_e(x, r)$ as

$$\begin{aligned}\rho_e(x,r) = &-(\bar{\delta} - \delta)\frac{\tau - 1}{\tau}\lambda_1(\underline{\alpha}_1(|x_1|))\alpha_1(|x_1|) - (1-\delta)\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\\ &+\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|) + \nu\left(\frac{\mu_r}{\tilde{\mu}}\right)^q\sigma_{r2}(|r_2|)^q\end{aligned}$$

The inequality (4) is achieved if the pair of λ_1 and λ_2 solves

$$-\delta\frac{\tau - 1}{\tau}\lambda_1(s)\alpha_1(\bar{\alpha}_1^{-1}(s)) + \nu\left(\frac{\mu}{\tilde{\mu}}\right)^q[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq 0, \ \forall s \in \mathbb{R}_+ \tag{127}$$

$$\frac{1}{p}\left(\frac{1}{\nu q}\right)^{p-1}\lambda_2(s)^p - \delta\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s)) + \lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq 0, \ \forall s \in \mathbb{R}_+ \tag{128}$$

and if $\lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Here, we assumed $\alpha_2 \in \mathcal{K}$ in obtaining (128). The inequality (127) holds if and only if

$$\frac{\nu\mu^q\tau[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q}{\tilde{\mu}^q\delta(\tau-1)\alpha_1(\bar{\alpha}_1^{-1}(s))} \leq \lambda_1(s), \quad \forall s \in \mathbb{R}_+ \tag{129}$$

is achieved by λ_1. Substitute λ_2 chosen as (56) into (128), we obtain

$$\lambda_1(\theta_1(s))\sigma_1(s) \leq \nu[\delta\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q \ \forall s \in \mathbb{R}_+ \tag{130}$$

Hence, the pair of (127) and (128) holds if the non-decreasing function λ_1 given by (55) satisfies (129) and (130). The choice of λ_1 satisfies (129) with $\tau = c_1$ since

$$\left(\frac{\mu}{\tilde{\mu}}\right)^q \frac{1}{\delta(c_1-1)} \leq c_2^q\delta^{\frac{q}{q+1}}$$

is implied by (124) and (125). The function λ_1 given in (55) satisfies (130) with $\tau = c_1$ if

$$\max_{w\in[0,\theta_1(s)]} \delta^{\frac{q}{q+1}} \frac{[c_2\sigma_2(\underline{\alpha}_1^{-1}(w))]^q}{\alpha_1(\bar{\alpha}_1^{-1}(w))} \leq \frac{[\delta\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q}{c_1\sigma_1(s)} \ \forall s \in \mathbb{R}_+$$

holds. Due to (124), this is equivalent to

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2(\underline{\alpha}_1^{-1}(\theta_1(w)))]^q}{\alpha_1(\bar{\alpha}_1^{-1}(\theta_1(w)))} \leq \frac{[\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q}{c_1\sigma_1(s)} \ \forall s\in\mathbb{R}_+ \tag{131}$$

Note that

$$\max_{w\in[0,s]} c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w) \ \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)$$

is guaranteed by (53). The inequality (52) implies that (52) still holds even if k is replaced by $q > k$. Therefore, the pair of (52) and (53) ensures (131). Hence, the non-decreasing functions λ_1 and λ_2 given in (55) and (56), respectively, achieve (127) and (128) for $\tau = c_1$. Due to the non-decreasing property of λ_1, we arrive at (4). This completes the part of *(ii)*. When α_2 is not class $\mathcal{K}$, it is clear that there always exists a class $\mathcal{K}$ function $\hat{\alpha}_2$ fulfilling (59), (60) and (61) due to (52) and (53). The inequality $\hat{\alpha}_2(s) \leq \alpha_2(s)$ implies that (4) holds with α_2 if it holds with $\hat{\alpha}_2$. Hence, the poof for the part *(iii)* is the same as *(ii)*. The claim *(i)* follows directly from *(ii)* and *(iii)*.
In the case of $\alpha_1 \in \mathcal{K}$, for any constant τ satisfying

$$\frac{(1-\underline{\delta}^{\frac{1}{q+1}})\overline{\eta}(\bar{\nu}+1)}{(1-\underline{\delta}^{\frac{1}{q+1}})\overline{\eta}(\bar{\nu}+1)-\bar{\nu}} < \tau \leq \overline{\eta}\,(\bar{\nu}+1)$$

we have

$$\lim_{s\to\infty} \alpha_1(s) \geq \tau \lim_{s\to\infty} \sigma_1(s)$$

and there exist $\tau_r > 1$ and $\bar{\delta} \in (\delta, 1)$ such that (126) and

$$\lim_{s\to\infty} \alpha_1(s) \geq \tau_r \lim_{s\to\infty} \sigma_{r1}(s)$$

are satisfied under the assumption (64). Using these τ_r and $\bar{\delta}$, we can repeat the proofs of (i), (ii) and (iii) with $\tau = \min\{c_1, \overline{\eta}\,(\bar{\nu}+1)\}$.

Proof of Corollary 4

The proof is the same as the proof of Theorem 4 up to (129) and (130). In the case of $\sigma_1 = 0$, the inequality (130) derived for the choice λ_2 in (56) is satisfied automatically. The condition (69) guarantees the existence of a non-decreasing function λ_1 given by (55), and the choice λ_1 solves (129).

Proof of Theorem 5

Assume that $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$. Let $\bar{\delta}$ be a real number satisfying $0 < \delta < \bar{\delta} < 1$, and set $\tau_2 = c_2$. Since $\tau_1 > 1$ and $\tau_2 > 1$ hold, there exist $\tau_{r1} > 1$ and $\tau_{r2} > 1$ such that

$$1 - \frac{1}{\tau_i} - \frac{1}{\tau_{ri}} \geq \bar{\delta}\left(1 - \frac{1}{\tau_i}\right), \quad i = 1, 2$$

are satisfied. Define the following class $\mathcal{K}$ functions for $i = 1, 2$.

$$\theta_i(s) = \bar{\alpha}_i \circ \alpha_i^{-1} \circ \tau_i \sigma_i(s), \quad \theta_{ri}(s) = \bar{\alpha}_i \circ \alpha_i^{-1} \circ \tau_{ri}\sigma_{ri}(s)$$

Since the functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ $i = 1, 2$ given in (76) and (77) are non-decreasing, we obtain

$$\begin{aligned} &\lambda_1(V_1(t, x_1))\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\} \\ &\quad \leq \bar{\delta}\left(-1 + \frac{1}{\tau_1}\right)\lambda_1(\underline{\alpha}_1(|x_1|))\alpha_1(|x_1|) + \lambda_1(\theta_1(|x_2|))\sigma_1(|x_2|) + \lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|) \quad (132) \\ &\lambda_1(V_2(t, x_2))\{-\alpha_2(|x_2|) + \sigma_2(|x_1|) + \sigma_{r2}(|r_2|)\} \\ &\quad \leq \bar{\delta}\left(-1 + \frac{1}{\tau_2}\right)\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|) + \lambda_2(\theta_2(|x_1|))\sigma_2(|x_1|) + \lambda_2(\theta_{r2}(|r_2|))\sigma_{r2}(|r_2|) \quad (133) \end{aligned}$$

by combining calculations in individual cases separated by $\alpha_i(|x_i|) \geq \tau_i\sigma_i(|x_j|)$, $\alpha_i(|x_i|) < \tau_i\sigma_i(|x_j|)$, $\alpha_i(|x_i|) \geq \tau_{ri}\sigma_{ri}(|r_i|)$ and $\alpha_i(|x_i|) < \tau_{ri}\sigma_{ri}(|r_i|)$. Thus, the inequality (4) is achieved if

$$\lambda_1(\theta_1(s))\sigma_1(s) \leq \delta\frac{\tau_2 - 1}{\tau_2}\lambda_2(\underline{\alpha}_2(s))\alpha_2(s), \ \forall s \in \mathbb{R}_+ \quad (134)$$

$$\lambda_2(\theta_2(s))\sigma_2(s) \leq \delta\frac{\tau_1 - 1}{\tau_1}\lambda_1(\underline{\alpha}_1(s))\alpha_1(s), \ \forall s \in \mathbb{R}_+ \quad (135)$$

are satisfied. In fact, $\alpha_{cl} \in \mathcal{K}_\infty$ and $\sigma_{cl} \in \mathcal{K}$ in (75) are given by

$$\begin{aligned} \alpha_{cl}(s) &= \min_{s=|x|}\left\{(\bar{\delta} - \delta)\frac{\tau_1 - 1}{\tau_1}\lambda_1(\underline{\alpha}_1(|x_1|))\alpha_1(|x_1|) + (\bar{\delta} - \delta)\frac{\tau_2 - 1}{\tau_2}\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right\} \\ \sigma_{cl}(s) &= \max_{s=|r|}\{\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|) + \lambda_2(\theta_{r2}(|r_2|))\sigma_{r2}(|r_2|)\} \end{aligned}$$

Hence, verification of (134) and (135) suffices to prove *(i)* and *(ii)*. It is easily seen that (134) and (135) are fulfilled if λ_1 and λ_2 achieve

$$\sigma_2(s)\sigma_1(\underline{\alpha}_2^{-1}(\theta_2(s)))\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(\theta_2(s)))) \leq \frac{\delta^2(\tau_1 - 1)(\tau_2 - 1)}{\tau_1\tau_2}\alpha_2(\underline{\alpha}_2^{-1}(\theta_2(s)))\alpha_1(s)\lambda_1(\underline{\alpha}_1(s)) \quad (136)$$

$$\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq \delta\frac{\tau_2 - 1}{\tau_2}\lambda_2(s)\alpha_2(\underline{\alpha}_2^{-1}(s)) \quad (137)$$

for all $s \in \mathbb{R}_+$. From $s \leq \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2(s)$ it follows that

$$\tau_2\sigma_2(s) \leq \alpha_2 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \tau_2\sigma_2(s) = \alpha_2(\underline{\alpha}_2^{-1}(\theta_2(s)))$$

Thus, (136) is implied by

$$\sigma_1(\underline{\alpha}_2^{-1}(\theta_2(s)))\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(\theta_2(s)))) \leq \frac{\delta^2(\tau_1-1)(\tau_2-1)}{\tau_1}\alpha_1(s)\lambda_1(\underline{\alpha}_1(s)) \tag{138}$$

Using (76) we have

$$\lambda_1 \circ \theta_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s) = \left[\nu_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right]\left[\alpha_2 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right]\left[\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right]^m$$

Thus, inserting (76) we obtain the left hand side of (138) as

$$\left[\nu_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right]\left[\alpha_2 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right]\left[\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right]^{m+1}$$

Since $\bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s) \leq s$ and

$$\lambda_1 \circ \underline{\alpha}_1(s) = \left[\nu_1 \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\left[\frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]^m$$

hold, the right hand side of (138) is larger than or equal to

$$\delta^2(\tau_1-1)(\tau_2-1)\left[\nu_1 \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\left[\frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]^{m+1}$$

Remember that ν_1 and α_2 are non-decreasing. The inequality (138) holds if

$$\tau_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s) \tag{139}$$

$$\frac{\tau_1}{[\delta^2(\tau_1-1)(\tau_2-1)]^{\frac{1}{m+1}}}\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s) \tag{140}$$

are satisfied for $s \in \mathbb{R}_+$. Since $\tau_1 \leq c_1$ and $\tau_2 \leq c_2$ are satisfied. the inequality (74) guarantees (139). Due to (80), the inequality (74) also implies (140). On the other hand, using

$$\lambda_1 \circ \bar{\alpha}_1 \circ \alpha_1^{-1}(\tau_1 s) = \nu_1(s)\left[\alpha_2 \circ \sigma_1^{-1}(s)\right]s^m$$
$$\theta_1 \circ \underline{\alpha}_2^{-1}(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_1\sigma_1 \circ \underline{\alpha}_2^{-1}(s)$$

we obtain

$$\frac{\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s))}{\alpha_2(\underline{\alpha}_2^{-1}(s))} = \left[\nu_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]\left[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{m+1}$$

Hence, $\lambda_2(s)$ given in (77) solves (137). Therefore, the inequality (4) is achieved by λ_1 and λ_2 given in (76) and (77). If σ_i is not class $\mathcal{K}_\infty$, it is obvious that there are functions $\hat{\sigma}_i \in \mathcal{K}_\infty$ which fulfill

$$\sigma_i(s) \leq \hat{\sigma}_i(s), \quad \forall s \in \mathbb{R}_+, \quad i = 1,2 \tag{141}$$

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\hat{\sigma}_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{142}$$

Then, we can follow the above argument by replacing $\sigma_i \in \mathcal{K}$ with $\hat{\sigma}_i \in \mathcal{K}_\infty$. Note that, due to (141), the characteristic inequality (4) is guaranteed to be achieved with σ_i if it is achieved with $\hat{\sigma}_i$. Hence, all claims of *(i)*, *(ii)* and *(iii)* have been proved.

B Relating small-gain theorems

B.1 Proof of Theorem 6

(i) Suppose that (37)-(39) holds for some $c_1 > 0$, $c_2 > 0$ and $q \geq 1$. The inequality (37) implies

$$[\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tilde{c}_1\sigma_1(s)]^q \leq c_1\tilde{c}_1\sigma_1(s)$$

for arbitrary $\tilde{c}_1 > 0$. Combining this inequality with the inequality (38), we obtain

$$c_2[\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tilde{c}_1\sigma_1(s)]^q \leq c_1\tilde{c}_1[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^q$$

which is identical to

$$\begin{aligned} &\tilde{c}_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tilde{c}_1\sigma_1(s) \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \\ &\tilde{c}_2 = \left(\frac{c_2}{c_1\tilde{c}_1}\right)^{1/q} \end{aligned}$$

Under the assumption (39), there exists $\tilde{c}_1 > 1$ such that $\tilde{c}_2 > 1$ holds. Thus, we arrive at (53). From (37) and (38) it follows that, for arbitrary $\hat{c}_1$, $\hat{c}_2 > 0$,

$$\begin{aligned} &\max_{w\in[0,s]} \frac{[\hat{c}_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \hat{c}_1\sigma_1(w)]^q}{\hat{c}_1\sigma_1(w)} \leq \hat{c}_2^q c_1, \quad \forall s\in\mathbb{R}_+ \\ &\frac{c_2}{\hat{c}_1} \leq \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2^{-1}(s)]^q}{\hat{c}_1\sigma_1(s)}, \quad \forall s\in\mathbb{R}_+ \end{aligned}$$

hold. Taking $\hat{c}_1 = \tilde{c}_1$ and $\hat{c}_2 = \tilde{c}_2$, we obtain (52).
(ii) It is obvious since (53) is identical to (74).

C Calculating scaling functions for static systems

C.1 Proof of Theorem 7

For $\zeta > 0$, we obtain

$$\begin{aligned} \sigma_2(|u_2|) &\leq \sigma_2 \circ \alpha_1^{-1} (\sigma_1(|x_2|) + \sigma_{r1}(|r_1|)) \\ &\leq \sigma_2 \circ \alpha_1^{-1} \circ (1+1/\zeta)\sigma_1(|x_2|) + \sigma_2 \circ \alpha_1^{-1} \circ (\zeta+1)\sigma_{r1}(|r_1|) \end{aligned} \tag{143}$$

from $\sigma_2 \in \mathcal{K}$, (86) and (89). Define

$$\rho_e(x_2, r) = -\nu(1-\delta)\alpha_2(|x_2|) + \nu\sigma_2 \circ \alpha_1^{-1} \circ (\zeta+1)\sigma_{r1}(|r_1|) + \nu\sigma_{r2}(|r_2|)$$

for $0 < \delta < 1$. For the choice of (93), (94) and (95) with $\nu > 0$, the inequality (11) becomes

$$\sigma_2 \circ \alpha_1^{-1} \circ (1+1/\zeta)\sigma_1(|x_2|) \leq \delta\alpha_2(|x_2|), \ x_2 \in \mathbb{R}^{n_2} \tag{144}$$

Hence, there exist constants $0 < \delta < 1$ and $\zeta > 0$ such that (11) is achieved if (91) holds for some $c_1 > 1$ and $c_2 > 1$. In the case of $\alpha_1 \in \mathcal{K}$, if the constant ζ in (144) belongs to

$$\frac{1}{\bar{\eta}(\bar{\nu}+1)-1} \leq \zeta \leq \bar{\eta}\left(1+\frac{1}{\bar{\nu}}\right) - 1 \tag{145}$$

then the following inequalities hold.

$$\lim_{s\to\infty} \alpha_1(s) \geq \left(1+\frac{1}{\zeta}\right) \lim_{s\to\infty} \sigma_1(s), \quad \lim_{s\to\infty} \alpha_1(s) \geq (1+\zeta) \lim_{s\to\infty} \sigma_{r1}(s)$$

These properties guarantee that (143) valid even for $\alpha_1 \in \mathcal{K}$. Note that the assumption (88) ensures the existence of $\bar{\eta} \geq 1$ satisfying (97). The inequality (145) is equivalent to

$$\frac{\bar{\eta}(\bar{\nu}+1)}{\bar{\eta}(\bar{\nu}+1)-\bar{\nu}} \leq 1 + \frac{1}{\eta} \leq \bar{\eta}\left(\bar{\nu}+1\right),$$

There exist constants $0 < \delta < 1$ and $\zeta > 0$ such that (144) and (145) are satisfied if (91) holds for some c_1 and c_2 satisfying (96) and and $c_2 > 1$.