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FIXED POINT THEOREMS FOR BERINDE MAPPINGS

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Abstract

The constants concerning the Banach contraction principle are a little complicated, which was proved in a paper [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 136 (2008), 1861–1869]. In this paper, we prove fixed point theorems for generalized Berinde mappings with constants. The constants are quite simple.

1. Introduction and preliminaries

Recently, we proved the following theorem, which is a generalization of the Banach contraction principle [1] and does characterize the metric completeness.

Theorem 1 ([11]). *Let (X, d) be a complete metric space and let T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \quad \text{implies} \quad d(Tx, Ty) \le rd(x, y)$$

for $x, y \in X$. Then T has a unique fixed point.

In [11], we proved that for each $r \in [0,1)$, $\theta(r)$ is the best constant. Before proving Theorem 1, the author had guessed $(1+r)^{-1}$ is best from his intuition. However, his guess was false. And he was surprised because θ is not simple. So, it is a natural question of whether there is a condition whose constant is $(1+r)^{-1}$. We have not found it; see [6, 8–13].

In this paper, we shall show that the best constant for Berinde mappings is $(1+r)^{-1}$ for each $r \in [0,1)$. Therefore the author believes that the concept has some mathematical beauty.

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DEFINITION 1 (Berinde [3]). Let T be a mapping on a metric space (X, d). Then T is called a *Berinde mapping* if there exist $r \in [0, 1)$ and $B \in [0, \infty)$ such that

$$d(Tx, Ty) \leq rd(x, y) + Bd(Tx, y) \tag{1}$$

for all $x, y \in X$.

DEFINITION 2. Let T be a mapping on a metric space (X, d). Then T is called a *generalized Berinde mapping* if there exist $r \in [0, 1)$ and a function b from X into $[0, \infty)$ such that

$$d(Tx, Ty) \leq rd(x, y) + b(y)d(Tx, y) \tag{2}$$

for all $x, y \in X$.

Contractions and Kannan mappings are Berinde mappings. See [2–5].

2. Main results

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers.

We first prove the following fixed point theorem.

THEOREM 2. *Let (X, d) be a complete metric space and let T be a mapping on X. Let b be a function from X into $[0, \infty)$. Assume that there exists $r \in [0, 1)$ such that*

$$(1 + r)^{-1} d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, y) + b(y)d(Tx, y) \tag{3}$$

for all $x, y \in X$. Then for every $x \in X$, $\{T^n x\}$ converges to a fixed point of T.

We use the following lemma.

LEMMA 1 ([9, 11]). *Let (X, d) be a metric space and let T be a mapping on X. Let $x \in X$ satisfy $d(Tx, T^2 x) \leq rd(x, Tx)$ for some $r \in [0, 1)$. Then for $y \in X$, either*

$$(1 + r)^{-1} d(x, Tx) \leq d(x, y) \qquad \textit{or} \qquad (1 + r)^{-1} d(Tx, T^2 x) \leq d(Tx, y)$$

holds.

PROOF OF THEOREM 2. Since $(1 + r)^{-1} d(x, Tx) \leq d(x, Tx)$, we have

$$d(Tx, T^2 x) \leq rd(x, Tx) + b(Tx)d(Tx, Tx) = rd(x, Tx) \tag{4}$$

for all $x \in X$. Fix $u \in X$. Then from (4), we have $d(T^n u, T^{n+1} u) \leq r^n d(u, Tu)$ and hence $\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) < \infty$. Thus, $\{T^n u\}$ is a Cauchy sequence. Since X is complete, $\{T^n u\}$ converges to some point $z \in X$. By Lemma 1 and (4), we can find a subsequence $\{f(n)\}$ of $\{n\}$ such that

$$(1 + r)^{-1} d(T^{f(n)} u, T^{f(n)+1} u) \leq d(T^{f(n)} u, z).$$

By (3), we have

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} d(T^{f(n)+1}u, Tz) \\ &\le \lim_{n\to\infty} (rd(T^{f(n)}u, z) + b(z)d(T^{f(n)+1}u, z)) \\ &= rd(z, z) + b(z)d(z, z) = 0. \end{aligned}$$

Therefore z is a fixed point of T. □

As a direct consequence, we obtain the following theorem, which is a very slight generalization of Berinde's theorem [5].

Corollary 1. *Let (X, d) be a complete metric space and let T be a generalized Berinde mapping on X. Then for every $x \in X$, $\{T^n x\}$ converges to a fixed point of T.*

We also obtain the following.

Corollary 2. *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there exist $r \in [0, 1)$ and $B \in [0, \infty)$ such that*

$$(1 + r)^{-1} d(x, Tx) \le d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \le rd(x, y) + Bd(Tx, y)$$

for all $x, y \in X$. Then for every $x \in X$, $\{T^n x\}$ converges to a fixed point of T.

We next prove that for every $r \in [0, 1)$, $(1 + r)^{-1}$ is the best constant in Theorem 2 and Corollary 2.

Theorem 3. *For each $r \in [0, 1)$, there exist a complete metric space (X, d), a mapping T on X and $B \in [0, \infty)$ such that T does not have any fixed points and*

$$(1 + r)^{-1} d(x, Tx) < d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \le rd(x, y) + Bd(Tx, y) \tag{5}$$

for all $x, y \in X$.

Proof. In the case of $r = 0$, we have shown the existence of X and T in [11]. So we assume $r > 0$. Define real sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ as follows:

- $u_0 = -1$
- $u_n = 1 + 2(-r)^n/(1 + 2r)$ for $n \in \mathbf{N}$
- $v_n = -u_n$ for $n \in \mathbf{N} \cup \{0\}$

Put

$$X = \{u_n : n \in \mathbf{N} \cup \{0\}\} \cup \{v_n : n \in \mathbf{N} \cup \{0\}\}$$

and let d be the usual metric. Define a mapping T on X by $Tu_n = u_{n+1}$ and $Tv_n = v_{n+1}$ for $n \in \mathbf{N} \cup \{0\}$. It is obvious that X is complete and T does not have any fixed points. We note $\lim_n u_n = v_0$,

$$v_2 < v_4 < \cdots < u_0 < \cdots < v_3 < v_1 < u_1 < u_3 < \cdots < v_0 < \cdots < u_4 < u_2,$$

$$d(u_1, v_1) = \frac{2}{1+2r} > \frac{2}{1+2r}(r - r^3) = d(u_1, u_3)$$

and

$$d(u_n, u_{n+1}) = r^n \frac{2+2r}{1+2r}$$

for all $n \in \mathbf{N} \cup \{0\}$. Put

$$\delta := \sup\{d(x, y) : x, y \in X\} = d(u_2, v_2)$$

and

$$B := \delta / d(u_1, u_3) > \delta / \inf\{d(u_i, v_j) : i, j \in \mathbf{N}\}.$$

Dividing the following nine cases, we shall show (5).

(i) $y = x$
(ii) $y = Tx$
(iii) $y = v_0$ and $x \in \{u_0\} \cup \{v_n : n \in \mathbf{N}\}$
(iv) $y = v_0$ and $x \in \{u_n : n \in \mathbf{N}\}$
(v) $y = u_0$
(vi) $y, x \in \{u_n : n \in \mathbf{N}\}$
(vii) $y = u_1$ and $x \in \{v_0\} \cup \{v_n : n \in \mathbf{N}\}$
(viii) $y \in \{u_n : n \in \mathbf{N}, n \geq 2\}$ and $x \in \{u_0, v_0\} \cup \{v_n : n \in \mathbf{N}\}$
(ix) $y \in \{v_n : n \in \mathbf{N}\}$

$y \backslash x$	u_0	u_n $(n \geq 1)$	v_0	v_n $(n \geq 1)$
v_0	(iii)	(iv)	(i)	(iii)
u_1	(ii)	(vi)	(vii)	(vii)
u_n $(n \geq 2)$	(viii)	(vi)	(viii)	(viii)

In the first case, we have

$$d(Tx, Ty) = 0 < rd(x, y) + Bd(Tx, y).$$

In the second case, we have

$$d(Tx, Ty) = rd(x, y) = rd(x, y) + Bd(Tx, y).$$

In the third case, we have

$$d(Tx, Ty) < \delta < Bd(u_1, y) \leq Bd(Tx, y) < rd(x, y) + Bd(Tx, y).$$

In the fourth case, since

$$(1+r)^{-1}d(x,Tx) = d(x,y),$$

(5) holds. In the fifth case, we can show (5) as in the third and fourth cases. In the sixth case, we have

$$d(Tx,Ty) = rd(x,y) \le rd(x,y) + Bd(Tx,y).$$

In the seventh case, we have

$$d(Tx,Ty) \le \delta < Bd(v_1,y) \le Bd(Tx,y) < rd(x,y) + Bd(Tx,y).$$

In the eighth case, we have

$$d(Tx,Ty) < \delta = Bd(u_1,u_3) \le Bd(Tx,y) < rd(x,y) + Bd(Tx,y).$$

In the ninth case, we can show (5) as in the sixth, seventh and eighth cases. □

3. Additional result

We finally give an example which informs that there exists a generalized Berinde mapping which is not a Berinde mapping.

Example 1. Put

$$X = \{0\} \cup \{-1/n : n \in \mathbf{N}\} \cup \{+1/n : n \in \mathbf{N}\}$$

and let d be the usual metric. Define a mapping T on X by

$$Tx = \begin{cases} -x & \text{if } x \in \{(-1)^n/n : n \in \mathbf{N}\}, \\ 0 & \text{if } x \in \{0\} \cup \{-(-1)^n/n : n \in \mathbf{N}\}. \end{cases}$$

Then T is a generalized Berinde mapping but is not a Berinde mapping.

Proof. We first show that T is a generalized Berinde mapping. Define a mapping b from X into $[0,\infty)$ by

$$b(y) = \begin{cases} 1 & \text{if } y = 0, \\ 2/\inf\{d(x,y) : x \neq y\} & \text{if } y \neq 0. \end{cases}$$

Dividing the following four cases, we shall show (2) with $r = 1/2$.

(i) $y = 0$ and $x \in \{0\} \cup \{-(-1)^n/n : n \in \mathbf{N}\}$
(ii) $y = 0$ and $x \in \{(-1)^n/n : n \in \mathbf{N}\}$
(iii) $y \neq 0$ and $Tx = y$
(iv) $y \neq 0$ and $Tx \neq y$

In the first case, we have

$$d(Tx,Ty) = 0 \le (1/2)d(x,y) + b(y)d(Tx,y).$$

In the second case, we have

$$d(Tx, Ty) = b(y)d(Tx, y) < (1/2)d(x, y) + b(y)d(Tx, y).$$

In the third case, there exists $n \in \mathbf{N}$ such that $x = (-1)^n/n$ and $y = -(-1)^n/n$. So we have

$$d(Tx, Ty) = (1/2)d(x, y) = (1/2)d(x, y) + b(y)d(Tx, y).$$

In the fourth case, we have

$$d(Tx, Ty) < 2 \le b(y)d(Tx, y) \le (1/2)d(x, y) + b(y)d(Tx, y).$$

Next, let us prove that T is not a Berinde mapping. Arguing by contradiction, we assume that T is a Berinde mapping, that is, there exist $r \in [0, 1)$ and $B \in [0, \infty)$ satisfying (1) for $x, y \in X$. Putting $x_n = (-1)^n/n$ and $y_n = (-1)^{n+1}/(n+1)$, we have

$$\begin{aligned}
2 &= \lim_{n\to\infty} n\left(\frac{1}{n} + \frac{1}{n+1}\right) = \lim_{n\to\infty} nd(Tx_n, Ty_n) \\
&\le \lim_{n\to\infty} n(rd(x_n, y_n) + Bd(Tx_n, y_n)) \\
&= \lim_{n\to\infty} n\left(r\left(\frac{1}{n} + \frac{1}{n+1}\right) + B\left(\frac{1}{n} - \frac{1}{n+1}\right)\right) \\
&= 2r.
\end{aligned}$$

This is a contradiction. Therefore T is not a Berinde mapping. □

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