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BLOWUP OF RADIAL SOLUTIONS TO A PARABOLIC-ELLIPTIC SYSTEM RELATED TO CHEMOTAXIS

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Abstract

We study blowup of radial solutions to a parabolic-elliptic system related to a biological model. The model describes chemotactic aggregation of cellular slime molds which move preferentially towards relatively high concentrations of a chemical secreted by the amoebae themselves. We think that blowup of solutions to the system corresponds to chemotactic aggregation of the living things. In the model, the relation between the movement of cells and the chemical concentration is represented by the chemotaxis coefficient. The behavior of solutions is determined by the chemotaxis coefficient and the initial function. In the present paper, we investigate a sufficient condition for blowup of solutions to the system.

1. Introduction

Cellular slime molds exhibit the oriented movement in response to a certain chemical substance released by themselves in their environment, and form an aggregation. Such oriented movement is called chemotaxis. Keller and Segel [5] proposed a mathematical model to describe the initiation of chemotactic aggregation of cellular slime molds. With the cell density $u(x,t)$ and the concentration of the chemical substance $v(x,t)$ at place x and time t, a simplified Keller-Segel model to be considered in this paper is described by the two partial differential equations

$$\begin{cases} u_t = \nabla \cdot (a\nabla u - b\nabla v) & \text{in } \Omega \times (0,\infty), \\ \varepsilon v_t = d\Delta v - fv + gu & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.1}$$

subject to the homogeneous Neumann boundary conditions, where ε is a non-negative constant, d is a positive constant, and a, b, f and g are positive functions of x, t, u and v. Furthermore, Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$.

Keller and Segel discussed the initiation of cell aggregation as an instability of the spatially steady states. Schaaf [8, 9] developed the bifurcation structure for the system. In the case where $a = d = f = g = 1$ and $b = u$, the author and Suzuki [12] investigated the bifurcation structure.

As concerns dynamic aspects of solutions, the case where $a = d = f = g = 1$ is one of typical cases. When $a = d = f = g = 1$ and $b = u$, Nanjundiah [7] has given a conjecture that blow-up occurs in finite time in cell density $u(x,t)$ to form a δ-function singularity. Such a phenomenon is referred to as chemotactic collapse. Furthermore, Childress [1] and Childress and Percus [2] have studied the conjecture and suggested the following:

(i) Chemotactic collapse is not possible in one dimensional case.

(ii) In two dimensional case, chemotactic collapse can occur if a total cell number on Ω is larger than a critical number, but cannot occur for the total cell number on Ω less than the critical number.

(iii) In three or more dimensional case, chemotactic collapse can occur.

When solutions to the system (1.1) and (1.2) are radial, $a = d = f = g = 1$, $\varepsilon = 0$ and $b = u$, Nagai [6] has shown the following:

(1) Blowup cannot occur in one dimensional case.

(2) In two dimensional case, blowup can occur if the L^1-norm of the function u is larger than a critical number, but cannot occur for the L^1 norm less than the critical number.

(3) In three or more dimensional case, blowup can occur.

Here, the L^1 norm corresponds to the total cell number.

Furthermore, the author and Suzuki [13] have shown that δ-function singularities appear if the solution blows up at a finite time in two dimensional case. Therefore, we think that blowup of solutions corresponds to chemotactic aggregation of the living things.

In generally, the propositions (1), (2) and (3) may not hold in other cases. For instance, in the case where $a = d = f = g = 1$, $\varepsilon = 0$ and $b = u/v$, any radial solutions exist globally in time, when Ω is a disc in $\mathbf{R}^2$ (see [10]). Therefore, we must investigate the behavior of solutions in the several cases. In the present paper, we treat the following case:

(A1) a, f, g, $1/a$, $1/f$ and $1/g$ are positive, smooth and bounded functions of x, t, u and v, and are radial with respect to x. $|\nabla_x a|$ is bounded.

(A2) $\varepsilon = 0$ and d is a positive constant.

(A3) $b = \chi u/v$.

(A4) χ and $1/\chi$ are positive, smooth and bounded functions, and are radial with respect to x.

(A5) Ω is a open ball of radius L with center at the origin in $\mathbf{R}^N$.

We consider the behavior of solutions to (1.1) under the conditions

$$\begin{cases} \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

where n denotes the outer unit normal vector.

(A6) u_0 is positive, radial and smooth in $\overline{\Omega}$.

Under the assumptions (A1)–(A6), there exists a unique classical solution (u, v) to (1.1) and (1.2) in $\Omega \times (0, T)$ with some $T > 0$, the functions u and v are positive in $\overline{\Omega} \times (0, T)$ and these are radial with respect to x. Let T_{max} be the maximum existence time of the classical solution. If T_{max} is finite, the solution (u,v) satisfies

$$\limsup_{t \to T_{max}} (\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty) = \infty,$$

by which we mean that (u,v) blows up in finite time. Furthermore, the function u satisfies $\|u(\cdot,t)\|_1 = \|u_0\|_1$ for $t \in [0, T_{max})$. These are shown by the standard argument (see [10]). Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm for $1 \le p \le \infty$.

For a positive number k we put

$$M_k(t) = \int_\Omega u(x,t)|x|^k dx.$$

Let us put

$$a^* = \sup\{a(x,t,u,v) : x \in \overline{\Omega}, t \ge 0, u \ge 0, v \ge 0\},$$

$$g^* = \sup\{g(x,t,u,v) : x \in \overline{\Omega}, t \ge 0, u \ge 0, v \ge 0\},$$

$$g_* = \inf\{g(x,t,u,v) : x \in \overline{\Omega}, t \ge 0, u \ge 0, v \ge 0\}$$

and

$$\chi_* = \inf\{\chi(x,t,u,v) : x \in \overline{\Omega}, t \ge 0, u \ge 0, v \ge 0\}.$$

Theorem 1. *Assume* (A1)–(A6). *Let* $N \ge 3$ *and* $2Na^*g^* < (N-2)\chi_* g_*$. *Then the solution* (u,v) *to* (1.1) *and* (1.2) *blows up at a finite time* T_{max} *and satisfies*

$$\limsup_{t \to T_{max}} (\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty) = \infty,$$

if $M_2(0)$ *is sufficiently small.*

2. Proof of Theorem 1

The propose in this section is to prove Theorem 1.

Let us put

$$f^* = \sup\{f(x,t,u,v) : x \in \overline{\Omega}, t \ge 0, u \ge 0, v \ge 0\}$$

and

$$f_* = \inf\{f(x,t,u,v) : x \in \overline{\Omega}, t \ge 0, u \ge 0, v \ge 0\}.$$

For a smooth function h, put

$$U(r,t;h) = \int_{|x|<r} h(x)u(x,t)dx \qquad \text{and} \qquad V(r,t;h) = \int_{|x|<r} h(x)v(x,t)dx \tag{2.1}$$

for $(r,t) \in [0,L] \times (0, T_{max})$.

Since the following lemma can be shown by using an argument similar to that of [11, Lemma 2.1], we omit the proof.

LEMMA 2.1. *Assume* (A1)–(A6). *Let* (u,v) *be a solution to* (1.1) *and* (1.2). *Then the function* v *satisfies*

$$v(x,t) \geq c_1 \qquad \text{for } x \in \overline{\Omega} \text{ and } t \in [0, T_{max}),$$

where c_1 *is a positive constant depending only on* f^*, g_*, g^*, $\|u_0\|_1$ *and* Ω.

The following lemma can be shown by using (A1) and an argument similar to that of [6].

LEMMA 2.2. *Let* $N \geq 3$ *and* (u,v) *be a solution to* (1.1) *and* (1.2). *Then the inequality holds:*

$$\frac{d}{dt}M_2(t) \leq 2Na^*\|u_0\|_1 + 2A^*M_1(t)$$
$$+ \frac{2}{d\omega_N}\int_\Omega \chi\frac{u(x,t)}{v(x,t)}\{V(|x|,t;f) - U(|x|,t;g)\}\frac{1}{|x|^{N-2}}\,dx$$

on $(0, T_{max})$, *where* ω_N *is the surface area of the unit sphere* S^{N-1} *in* $\mathbf{R}^N$ *and*

$$A^* = \sup\{|\nabla_x a(x,t,u,v)| : x \in \Omega, t \geq 0, u \geq 0, v \geq 0\}.$$

PROOF. Multiplying the first equation of (1.1) by $|x|^2$, integrating on Ω and using (A1), we get the following inequality

$$\begin{aligned}\frac{d}{dt}M(t) &= -\int_\Omega 2ax\cdot\nabla u(x,t)dx - \int_\Omega \frac{\chi u(x,t)}{v(x,t)}(2x)\cdot\nabla v(x,t)dx \\ &\leq 2Na^*\int_\Omega u(x,t)dx + 2A^*\int_\Omega u(x,t)|x|dx \\ &\quad - \int_\Omega \frac{\chi u(x,t)}{v(x,t)}(2x)\cdot\nabla v(x,t)dx.\end{aligned}$$

For $x \in \Omega\backslash\{0\}$, integrating the second equation of (1.1) on $\{y \in \Omega : |y| \leq |x|\}$, we get

$$\begin{aligned}d\omega_N|x|^{N-2}x\cdot\nabla v(x,t) &= \int_{|y|=|x|} d\frac{y}{|y|}\cdot\nabla_y v(y,t)d\sigma = \int_{|y|\leq|x|} d\Delta v(y,t)dy \\ &= \int_{|y|\leq|x|}(fv(y,t) - gu(y,t))dy \\ &= V(|x|,t;f) - U(|x|,t;g).\end{aligned}$$

Combining these leads to this lemma. $\square$

For a positive constant k let $G(\cdot,\cdot;k)$ be the Green's function of $-\Delta+k$ in Ω with the homogeneous Neumann boundary condition, and $E(\cdot;k)$ be the fundamental solution of $-\Delta+k$ in $\mathbf{R}^N$. We say that $K(x,y;k)=G(x,y;k)-E(x-y;k)$ is the compensating function.

Lemma 2.3. *Solutions* (u,v) *to* (1.1) *and* (1.2) *satisfy*

$$\frac{1}{d}E(x;f^*/d)\int_{|y|\le|x|}gu(y,t)dy\le v(x,t)\le\frac{1}{d\omega_N|x|^{N-1}}\int_{|y|=|x|}E(x-y;f_*/d)d\sigma\|gu(\cdot,t)\|_1+\frac{1}{d}\int_\Omega K(x,y;f_*/d)gu(y,t)dy$$

for $(x,t)\in\Omega\backslash\{0\}\times(0,T_{max})$.

Proof. Since v is positive, we get

$$\begin{aligned}v(x,t)&=\frac{1}{d}\int_\Omega G(x,y;f^*/d)\{gu(y,t)+(f^*-f)v(x,t)\}dx\\&\ge\frac{1}{d}\int_\Omega G(x,y;f^*/d)gu(x,t)dx\qquad\text{for }(x,t)\in\Omega\times(0,T_{max}).\end{aligned}$$

It is shown that

$$\begin{aligned}&v(x,t)-\frac{1}{d}\int_\Omega K(x,y;f^*/d)gu(y,t)dy\\&\ge\frac{1}{d\omega_N}\int_{\mathbf{R}^N}E(|x-y|;f^*/d)\frac{y}{|y|^N}\cdot\nabla_y\tilde{U}(|y|,t;g)dy\\&=\lim_{\varepsilon\to0}\left(\frac{1}{d\omega_N}\int_{|y|=\varepsilon}E(|x-y|;f^*/d)\frac{\tilde{U}(|y|,t;g)}{|y|^{N-1}}\,d\sigma\right.\\&\qquad\left.-\frac{1}{d\omega_N}\int_{\varepsilon<|y|}\nabla_y\left(\frac{y}{|y|^N}E(|x-y|)\right)\tilde{U}(|y|,t;g)dy\right)\\&=-\lim_{\varepsilon\to0}\frac{1}{d\omega_N}\int_{\varepsilon<|y|}\nabla_y\cdot\left(\frac{y}{|y|^N}E(|x-y|;f^*/d)\right)\tilde{U}(|y|,t;g)dy\end{aligned}\tag{2.2}$$

for $(x,t)\in\Omega\backslash\{0\}\times[0,T_{max})$, where

$$\tilde{u}(x,t)=\begin{cases}u(x,t)&\text{if }x\in\Omega,\\0&\text{if }x\in\mathbf{R}^N\backslash\Omega,\end{cases}$$

$$\tilde{g}(x,t,u(x,t),v(x,t))=\begin{cases}g(x,t,u(x,t),v(x,t))&\text{if }x\in\Omega,\\0&\text{if }x\in\mathbf{R}^N\backslash\Omega\end{cases}$$

and

$$\tilde{U}(r,t;g) = \int_{|x|\le r} \tilde{g}\tilde{u}(x,t)dx.$$

By using the above inequality and

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|} \nabla_y \cdot \left(E(x-y;f^*/d)\frac{y}{|y|^N}\right) dy = -\omega_N E(x;f^*/d) \qquad \text{for } x \in \Omega\backslash\{0\},$$

we have that

$$\begin{aligned} v(x,t) \ge{}& \frac{1}{d}E(x;f^*/d)\|gu(\cdot,t)\|_1 + \frac{1}{d}\int_\Omega K(x,y;f^*/d)gu(y,t)dy \\ &+ \lim_{\varepsilon\to 0}\frac{1}{d\omega_N}\int_{\varepsilon<|y|}\nabla_y\cdot\left(\frac{y}{|y|^N}E(x-y;f^*/d)\right)(\|gu(\cdot,t)\|_1 - \tilde{U}(|y|,t;g)) \\ &\text{for } (x,t)\in\Omega\backslash\{0\}\times[0,T_{max}). \end{aligned}$$

Lemma 3.4 yields that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|}\nabla_y\cdot\left(\frac{y}{|y|^N}E(x-y;f^*/d)\right)\int_{|y|\le|z|}\tilde{g}\tilde{u}(z,t)dzdy \\ &\ge \int_{|x|<|y|}\nabla_y\cdot\left(\frac{y}{|y|^N}E(x-y;f^*/d)\right)\int_{|x|\le|z|}\tilde{g}\tilde{u}(z,t)dzdy \\ &\quad+ \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<|x|}\nabla_y\cdot\left(\frac{y}{|y|^N}E(x-y;f^*/d)\right)\int_{|x|\le|z|}\tilde{g}\tilde{u}(z,t)dzdy \\ &= -\omega_N E(x;f^*/d)\int_{|x|\le|z|}\tilde{g}\tilde{u}(z,t)dz \qquad \text{for } (x,t)\in\Omega\backslash\{0\}\times[0,T_{max}). \end{aligned}$$

Combining these with Lemma 3.2 implies the lower estimate of v.

Since v satisfies

$$\begin{aligned} v(x,t) &= \frac{1}{d}\int_\Omega G(x,y;f_*/d)\{gu(y,t)+(f_*-f)v(x,t)\}dy \\ &\le \frac{1}{d}\int_\Omega G(x,y;f_*/d)gu(x,t)dy, \end{aligned}$$

we can show that

$$\begin{aligned} &v(x,t) - \frac{1}{d}\int_\Omega K(x,y;f_*/d)gu(y,t)dy \\ &\quad\le -\lim_{\varepsilon\to 0}\frac{1}{d\omega_N}\int_{\varepsilon<|y|}\nabla_y\cdot\left(\frac{y}{|y|^N}E(x-y;f_*/d)\right)\tilde{U}(|y|,t;g)dy \end{aligned}$$

for $(x, t) \in \Omega \backslash \{0\} \times [0, T_{max})$. Lemma 3.4 leads to

$$-\lim_{\varepsilon \to 0} \int_{\varepsilon < |y|} \nabla_y \cdot \left(\frac{y}{|y|^N} E(x - y; f^*/d) \right) \int_{|z| \le |y|} \tilde{g}\tilde{u}(z, t) dz dy$$

$$\le -\lim_{\varepsilon \to 0} \int_{|x| < |y|} \nabla_y \cdot \frac{y}{|y|^N} E(x - y; f^*/d) \int_{|z| \le |y|} \tilde{g}\tilde{u}(z, t) dz dy$$

$$\le \frac{1}{|x|^{N-1}} \int_{|y| = |x|} E(x - y; f_*/d) d\sigma \|gu(\cdot, t)\|_1$$

$$\text{for } (x, t) \in \Omega \backslash \{0\} \times [0, T_{max}).$$

These means the upper estimate of v. Thus the proof is complete. □

LEMMA 2.4. *Let $N \ge 3$ and (u, v) be a solution to* (1.1) *and* (1.2). *There exists a constant $\delta \in (0, L)$ such that*

$$x \cdot \nabla(|x|^{N-1} v(x, t)) \ge 0 \qquad \text{for } (x, t) \in B(\delta) \times (0, T_{max}),$$

where $B(\delta) = \{x \in \mathbf{R}^N : |x| < \delta\}$.

PROOF. As v is radial, v satisfies

$$\frac{x}{|x|} \cdot \nabla_x(|x|^{N-1} v(x, t)) = |x|^{N-2} x \cdot \nabla_x v(x, t) + (N - 1)|x|^{N-2} v(x, t)$$

$$= \frac{1}{d\omega_N} (V(|x|, t; f) - U(|x|, t; g)) + (N - 1)|x|^{N-2} v(x, t)$$

for $(x, t) \in \Omega \times (0, T_{max})$. Combining this with Lemmas 2.3 and 3.3 implies that

$$\frac{x}{|x|} \cdot \nabla_x(|x|^{N-1} v(x, t)) \ge \frac{1}{d\omega_N} V(|x|, t; f) + \frac{1}{d} \left(\frac{1}{\omega_N(N - 2)} - C|x| \right) U(|x|, t; g)$$

$$\ge 0 \qquad \text{for } (x, t) \in B(\delta) \times [0, T_{max})$$

with some $\delta \in (0, L)$. Thus the proof is complete. □

PROOF OF THEOREM 1. It follows from Lemmas 2.3, 3.2, 3.3 and 3.5 that for $(x, t) \in B(L/2) \times [0, T_{max})$

$$v(x, t) \le \frac{g^*}{d} \left\{ \frac{1}{\omega_N(N - 2)|x|^{N-2}} + \frac{C}{|x|^{N-3}} \right\} \|u_0\|_1.$$

Taking $\eta \in (0, L/2)$, this yields that

$$
\begin{aligned}
&\int_\Omega \chi \frac{u(x,t)}{v(x,t)} U(|x|,t;g) \frac{1}{|x|^{N-2}}\,dx \\
&\quad \geq \frac{d\chi_* g_*}{g^*\|u_0\|_1} \left\{ \frac{1}{(N-2)\omega_N} + C\eta \right\}^{-1} \int_{|x|<\eta} u(x,t)U(|x|,t;1)dx \\
&\quad = \frac{d\chi_* g_*(N-2)\omega_N}{2g^*\|u_0\|_1(1+C(N-2)\omega_N\eta)} U(\eta,t;1)^2 \\
&\quad \geq \frac{d\chi_* g_*(N-2)\omega_N}{2g^*\|u_0\|_1(1+C(N-2)\omega_N\eta)} \left(\|u_0\|_1 - \frac{1}{\eta^2} M_2(t) \right)_+^2 \\
&\quad \geq \frac{d\chi_* g_*(N-2)\omega_N}{2g^*\|u_0\|_1(1+C(N-2)\omega_N\eta)} \left(\|u_0\|_1^2 - \frac{2}{\eta^2} \|u_0\|_1 M_2(t) \right), \qquad (2.3)
\end{aligned}
$$

where $(\cdot)_+ = \max(\cdot, 0)$. Take $\eta = \min\{\delta, L/4\}$, where δ is the constant in Lemma 2.4. Lemma 2.4 entails

$$
V(|x|,t;f) \leq \omega_N f^* |x|^N v(x,t) \qquad \text{for } (x,t) \in B(\eta) \times [0, T_{max}).
$$

It follows from the second equation of (1.1) and the boundary condition (1.2) that

$$
V(|x|,t;f) \leq V(L,t;f) = U(L,t;g) \leq g^*\|u_0\|_1 \qquad \text{for } t \in (0, T_{max}).
$$

These leads to

$$
\begin{aligned}
&\int_\Omega \chi u(x,t) V(|x|,t;f) \frac{1}{|x|^{N-2} v(x,t)}\,dx \\
&\quad \leq \omega_N \chi^* f^* \int_{|x|\leq\eta} |x|^2 u(x,t)dx + \frac{\chi^* g^* \|u_0\|_1}{\eta^N c_1} \int_{\eta\leq|x|\leq L} |x|^2 u(x,t)dx \\
&\quad \leq C_\eta M_2(t) \qquad \text{for } t \in [0, T_{max})
\end{aligned}
$$

with some positive constant C_η depending on η, where c_1 is the constant in Lemma 2.1. Hölder inequality yields

$$
\begin{aligned}
M_1(t) &= \int_\Omega |x| u(x,t)dx \\
&\leq \left(\int_\Omega u(x,t)dx \right)^{1/2} \left(\int_\Omega |x|^2 u(x,t)dx \right)^{1/2} \\
&= \|u_0\|_1^{1/2} M_2(t)^{1/2}.
\end{aligned}
$$

Substituting the inequality in Lemma 2.2 into this and (2.3) and using the above inequality, we have

$$\frac{d}{dt}M_2(t) \le \left(2Na^* - \frac{\chi_* g_*(N-2)}{g^*(1+C(N-2)\omega_N\eta)}\right)\|u_0\|_1 + 2A^*\|u_0\|_1^{1/2}M_2(t)^{1/2} + \left(\frac{2C_\eta}{d\omega_N} + \frac{2\chi_* g_*(N-2)}{\eta^2 g^*(1+C(N-2)\omega_N\eta)}\right)M_2(t)$$

for $t \in (0, T_{max})$. Since $2Na^*g^* < \chi_* g_*(N-2)$, we have

$$2Na^* - \frac{\chi_* g_*(N-2)}{g^*(1+C(N-2)\omega_N\eta)} < 0$$

for a sufficiently small $\eta > 0$. Take a sufficiently small $\eta > 0$. If $M_2(0)$ is sufficiently small, the right-hand side of the above inequality is negative at $t = 0$. Then $M_2(t)$ decreases. If we assume $T_{max} = \infty$, there exists $T_0 > 0$ such that $M_2(T_0) < 0$. It contradicts the positivity of solutions in $\Omega \times (0, T_{max})$. Then we conclude $T_{max} < \infty$. Thus the proof is complete. □

3. Properties of Green's functions, fundamental solutions and compensation functions

The purpose in this section is to give some properties of the Green's functions G, the fundamental solutions E and the compensating functions K mentioned in the previous section.

Lemma 3.1. *Let k be a positive constant. Green's functions and fundamental solutions satisfy*

$$G(x, y; k) = k^{(N-2)/2}\bar{G}(\sqrt{k}x, \sqrt{k}y) > 0 \qquad \text{for } (x, y) \in \Omega \times \Omega$$

and

$$E(x; k) = k^{(N-2)/2}E(\sqrt{k}x; 1) > 0 \qquad \text{for } x \in \mathbf{R}^N,$$

where $\bar{G}$ is the Green's function of $-\Delta + 1$ in $\sqrt{k}\Omega = \{\sqrt{k}x : x \in \Omega\}$ with the homogeneous Neumann boundary condition.

Proof. Let h be a smooth function on $\bar{\Omega}$ and let

$$w(x) = \int_\Omega G(x, y; k)h(y)dy.$$

The function w is a solution to the problem

$$\begin{cases} -\Delta w + kw = h & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us put $X = \sqrt{k}x$, $\bar{h}(X) = h(x)$ and $\bar{w}(X) = w(x)$. Since the functions $\bar{h}$ and $\bar{w}$ satisfy

$$\Delta_X \overline{w}(X) = \sum_{j=1}^{N} \frac{\partial^2}{\partial X_j^2} \overline{w}(X) = \frac{1}{k} \Delta w(X/\sqrt{k})$$

$$= -\frac{1}{k} h(X/\sqrt{k}) + w(X/\sqrt{k}) = -\frac{1}{k} \bar{h}(X) + \overline{w}(X) \qquad \text{for } X \in \sqrt{k}\Omega$$

and

$$\frac{\partial}{\partial n} \overline{w}(X) = 0 \qquad \text{for } X \in \partial\sqrt{k}\Omega,$$

we have

$$\int_\Omega G(x, y; k) h(y) dy = w(x) = \overline{w}(X)$$

$$= \int_{\sqrt{k}\Omega} \overline{G}(X, Y) \frac{1}{k} \bar{h}(Y) dY$$

$$= \int_\Omega \overline{G}(\sqrt{k}x, \sqrt{k}y) k^{(N-2)/2} h(y) dy \qquad \text{for } x \in \Omega.$$

The positivity of the Green's function is a fundamental property (see [4]). These yield the first claim. A similar argument gives the second claim. Thus the proof is complete. □

Lemmas 3.2 and 3.3 follow from fundamental properties of fundamental solutions and compensating functions (see [4]).

Lemma 3.2. *For a positive constant k the compensating function $K(\cdot, \cdot; k)$ is positive in $\Omega \times \Omega$ and satisfies*

$$\sup_{|x|<L_0, |y|<L} K(x, y; k) < \infty \qquad \textit{for } 0 < L_0 < L.$$

Lemma 3.3. *Let $N \geq 3$ and k be a positive constant. Then the fundamental solution satisfies*

$$\left| E(x; k) - \frac{1}{(N-2)\omega_N |x|^{N-2}} \right| \leq \frac{C}{|x|^{N-3}},$$

where C is a positive constant depending only on k.

Lemma 3.4 is shown by using an argument similar to that of [10, Lemma 3.3].

Lemma 3.4. *Let $N \geq 3$ and k be a positive constant. For $x \in \mathbf{R}^N \backslash \{0\}$ the fundamental solution $E(\cdot; k)$ satisfies $x \cdot \nabla_x E(x; k) < 0$ and*

$$\int_{|y|=r} \nabla_y \cdot \left(\frac{y}{|y|^N} E(x-y;k) \right) d\sigma = \begin{cases} \dfrac{k}{r^{N-1}} \displaystyle\int_{|y|<r} E(x-y;k)dy & (0<r<|x|), \\ -\dfrac{k}{r^{N-1}} \displaystyle\int_{|y|>r} E(x-y;k)dy & (|x|<r). \end{cases}$$

Lemma 3.5 is the special case of [3, Proposition 3.19] concerning double layer potentials.

Lemma 3.5. *The equation*

$$\int_{|y|=1} \frac{1}{|x-y|^{N-2}}\, d\sigma = \omega_N$$

holds for $x \in S^{N-1}$.

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