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Failure Probability Estimation from Inferior and Quality Mixed Populations

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Abstract

A new approach, for the estimation of the failure probability from a mixed model of inferior and quality populations for the case that the ratio of the inferior population to the quality populations is unknown, is introduced. The possible existence of a quality population in the mixed populations is investigated using the Akaike information criterion. The probability of failure after the screening test is then obtained.

Key Words : *Binomial distribution, Mixture model, Akaike information criterion, Bootstrap*

1. Introduction

If the underlying distribution of starting failure of a computer product is assumed to follow a single homogeneous population and we only observe the failure counts for each test sample; we also assume that the product can be started again after the failure of starting. Then the failure probability is easily estimated by the binomial failure probability computation, e.g. $\hat{p} = K/N$ where K is the total count of failures and N is the number of total trials.,

Table 1 shows a test result of failure counts for 142 test samples. We can see that some samples fail very easily, while others very hardly fail. Typically, a and c lines and e-i lines in the table show very different features from each other; line e expresses that there are five samples that fail once in ten trials. Therefore, we cannot simply suppose an existence of only one homogeneous population for that situation.

The following situations are considered in general:

(1) A single homogeneous population with (constant) failure probability p is assumed, as mentioned above.

(2) Failure probability p is assumed to follow a certain (continuous) probability distribution.

(3) The products are mixed from more than 2 kinds of populations where one population may have a failure probability $p = 0$.

(4) The situations (2) and (3) are combined.

It would be difficult to consider the situations (2) and (4) by looking at Table 1 because the failure probability p is not explicitly expressed by some (continuous) random variables. We are, thus, going to consider the situation (1) and (3) in this paper. When we assume that samples drawn from a population will never be broken, we define such a population as a *quality* population, whereas a population where the failures may or may not be observed is defined as an *inferior* population. Many conventional mixture problems, i.e., failure populations alone, have been considered up to now, but a mixed population of inferior and quality populations has not been discussed except for continuous cases (e.g., Hirose (2002), Meeker (1987)); this is a new point of the paper.

Suppose, first, that we have a mixed population of inferior and quality populations for the case that the ratio, r, of the inferior population to the quality populations is unknown and the failure count, k_j, for each test sample is observed. We first estimate the ratio r, and the failure probability p by a single test for the inferior population. We next want to estimate the failure probability of a product during repeated starting conditions after a screening test in which no failures are observed in the prescribed trial times.

2. Mixture Model

Assume that group G_1 is a population of quality products and G_2 inferior. Assume also that r_i denotes the probability that a sample belongs to group G_i and p_i denotes the failure probability for group G_i; we suppose that $p_1 = 0$ first (i.e., quality group), but we will also deal with the cases that $p_1 > 0$ (i.e., inferior group) later.

Since we cannot determine whether a sample is drawn from G_1 or G_2, we set the conditional expectation, P, of the failure probability for a sample as

$$\begin{aligned} P =& Pr(failure\ G_1|G_1) + Pr(failure\ G_2|G_2) \\ =& \binom{n}{k} p_1^k (1-p_1)^{n-k} \cdot r_1 + \binom{n}{k} p_2^k (1-p_2)^{n-k} \cdot r_2, \end{aligned} \tag{1}$$

if k starting failures are observed in n starting trials. Since

$$p_1^k (1-p_1)^{n-k} = 1, \quad (k=0,\ p_1 \to 0), \tag{2}$$

$$p_1^k (1-p_1)^{n-k} \to 0, \quad (k>0,\ p_1 \to 0), \tag{3}$$

hold,

$$P \to r_1 + (1-p_2)^n \cdot r_2, \quad (k=0,\ p_1 \to 0), \tag{4}$$

$$P \to \binom{n}{k} p_2^k (1-p_2)^{n-k} \cdot r_2, \quad (k>0,\ p_1 \to 0). \tag{5}$$

Then, the likelihood becomes

$$\begin{aligned} L =& Pr(no\ failures) \times Pr(at\ least\ one\ failure) \\ =& \left(\prod_j \{ r_1 + (1-p_2)^{n_j} \cdot r_2 \} \right)_{no\ failures} \\ & \times \left(\prod_j \binom{n_j}{k_j} p_2^{k_j} (1-p_2)^{n_j - k_j} \cdot r_2 \right)_{at\ least\ one\ failure} \end{aligned} \tag{6}$$

The number of inferior groups might be larger than 1, say $g-1$, then (6) can be extended as

$$L = \left(\prod_j \left\{ r_1 + \sum_{i=2}^{g} (1-p_i)^{n_j} \cdot r_i \right\} \right)_{no\ failures} \times \left(\prod_j \left\{ \sum_{i=2}^{g} \binom{n_j}{k_j} p_i^{k_j} (1-p_i)^{n_j-k_j} \cdot r_i \right\} \right)_{at\ least\ one\ failure} . \tag{7}$$

3. Failure Probability After Screening

If the failure probability for the population is already estimated, then we can predict the failure probability during M times starting after m times screening test. The probability, $Q_{m,M}$, of M times no failures after m times no failures is

$$\begin{aligned} Q_{m,M} &= \frac{Pr(no\ failures\ in\ (m+M)\ trials)}{Pr(no\ failures\ in\ m\ trials)} \\ &= \frac{r_1 + (1-p_2)^{m+M} \cdot r_2}{r_1 + (1-p_2)^m \cdot r_2}. \end{aligned} \tag{8}$$

Therefore, the probability, $P_{m,M}$, that at least one failure occurs during M times starting after screening is

$$P_{m,M} = 1 - Q_{m,M} = 1 - \frac{r_1 + (1-p_2)^{m+M} \cdot r_2}{r_1 + (1-p_2)^m \cdot r_2}. \tag{9}$$

An extension of (9) to $g-1$ inferior groups is similarly obtained.

4. Application

By maximizing the likelihoods in (6) and (7), we can obtain the maximum likelihood estimates for r and p. The computational results for the data in Table 1 are shown in Table 2, where numbers of inferior groups are 1, 2, 3, 4. In the table, a simple homogeneous population model and conventional mixture models ($p_i > 0$, $i = 1, \cdots, g$) are also shown. As a typical case, the contour plot of the log-likelihood function for the case that one inferior and one quality populations are mixed is shown in Figure 1, in which a unique solution can be seen. From the table, we can see that (1) the assumption of the existence of a quality group is appropriate, and (2) the case that the number of groups is 4 is the most suitable, according to the Akaike information criterion, i.e., AIC, (Akaike (1973)). It can be seen from the table that the assumption of a single homogeneous population is very unrealistic; the AIC value to that model is 222.914, while the most appropriate model gives its value of 114.172.

The confidence intervals of the estimates r_i and p_i are computed using the bootstrap method (Efron, (1982)). For instance, the histogram for the estimate r_1 (quality population ratio) is shown in Figure 2. As the figure indicates, the estimate r_1 is not so informative. However, Figure 3, showing the relation between r_1 and r_2, suggests that $r_1 + r_1$ may be informative; the histogram for $r_1 + r_1$ shown in Figure 4 presents this tendency. We may say that that G_1 and G_2 consist of *almost entirely quality* groups. This is consistent with the results under the assumption that there are one inferior and one quality groups. The 95% confidence intervals of r_1 and $r_1 + r_1$ are shown in Table 3, along with the case of $g = 2$.

Next, the failure probabilities $Q_{m,M}$ after screening under the condition that $p_1 = 0$ and $g = 4$ are computed, and $P_{m,M}$ are shown in Table 4; for example, the no failure probability in 5000 trails ($M = 5000$) after m screenings is close to

$$Q_{m,\infty} = \frac{\sum_{i>1}(1-p_i)^m \cdot r_i}{r_1 + \sum_{i>1}(1-p_i)^m \cdot r_i}. \tag{10}$$

This tendency is also valid for 1000 trails.

The confidence interval of the estimates $P_{m,M}$ and $Q_{m,M}$ are also obtained using the bootstrap method. The histogram for the estimates $P_{30,1000}$ is shown in Figure 5. The figure indicates that the estimate $P_{30,1000}$ is not so informative. This tendency remains unchanged even when $m = 100$. The 95% confidence interval of $P_{30,1000}$ and $P_{100,1000}$ are shown in Table 5 along with the case of $g = 2$. In $g = 2$, $P_{m,M}$ is much more informative than that in $g = 4$.

5. Discussions

5.1 Discrete mixture model and continuous mixture model

Hirose (2002) proposes a mixture model of fragile and durable populations for (continuous) lifetime analysis. The mathematical formulation is expressed as

$$L(\theta, r) = \{1 - rF(T;\theta)\}^{n-k} \cdot \prod_{i=1}^{k} rf(t_i;\theta), \quad (0 \leq r \leq 1), \tag{11}$$

where F and f are the cumulative probability distribution function and the density function respectively, θ is the unknown parameter, T is the censoring time, t_i is the observed failure time. Here, r is the ratio of the fragile population to the mixed population. This reminds us of a similar treatment to a discrete model here. For example, if we assume that the starting failure probability is $F(T)$ and successful probability is $1 - F(T)$, (11) becomes

$$L \propto (1 - rp)^{n-k} \cdot (rp)^k. \tag{12}$$

We may assume that $F(t) = 1 - \exp(-\alpha t)$. By combining a constant term $\prod_j \binom{n_j}{k_j}$ to (12), L can be maximized when $rp = 27/4120$ and $\log L$ gives a value of -110.457 which is exactly the same as that given in the simple binomial model; see Table 2, the number of parameters is 1. However, we cannot determine the values of r and p uniquely by maximizing (12). Thus, we cannot find the ratio r in the discrete model unlike the continuous model.

5.2 Model selection

To select the most optimal model is a difficult problem. We have to use some justification methods. The AIC is one of the methods. To select a model which gives smaller confidence intervals for the estimates than any other models may be another choice. From the AIC viewpoint, the model $g = 4$ is the most appropriate. From the smaller confidence interval viewpoint for $P_{m,M}$, $g = 2$ may be an appropriate model. However, it would be dangerous to adopt the latter case. Even if $P_{m,M}$ is not so informative, adopting the case of $g = 4$ would be appropriate.

6. Summary and Concluding Remarks

Estimation method of the failure probability from a mixture model of inferior and quality populations for the case that the ratio of the inferior population to the quality populations is unknown is introduced. A failed data case indicates the possible existence of a quality population in the mixed populations using the Akaike information criterion. The probability of failure after the screening test is also obtained.

References

Akaike, H. (1973), "Information theory and an extension of the maximum likelihood principle", in *2nd International Symposium on Information Theory*, Akademiai Kiado, Budapest, pp. 267-281.

Efron, B. (1982), *The Jackknife, the Bootstrap and Other Resampling Plans*, Philadelphia: Society of Industrial and Applied Mathematics.

Hirose, H. (2002), "Trunsored data analysis with applications to field data", in *Hawaii International Conference on Statistics and Related Fields*, Honolulu, Hawaii.

Meeker, W.Q. (1987), "Limited failure population life tests: application to integrated circuit reliability", *Technometrics*, vol. 29, pp. 51-65.

Table 1. Test result of failure counts.

	trials	failure times	number of samples
a	1000	0	1
b	1000	4	1
c	100	0	8
d	10	0	123
e	10	1	5
f	10	2	1
g	10	3	1
h	10	4	1
i	10	9	1
total	4120	27	142

Table 2. Estimated parameters, $\log L$, and AIC.

number of parameters	p_1 r_1	p_2 r_2	p_3 r_3	p_4 r_4	p_5 r_5	$\log L$	AIC
2	0 0.6917	0.01830 0.3083				−96.5605	197.121
4	0 0.45071	0.004506 0.5214	0.4489 0.02788			−55.7097	119.419
6	0 0.5621	0.003774 0.3891	0.1896 0.04178	0.8998 0.007044		−51.0865	114.173
8	0 0.4918	0.003774 0.3891	0.1896 0.04178	0.8998 0.007044	0.191×10^{-12} 0.07025	−51.0865	118.173
1	0.006553 1					−110.457	222.914
3	0.4401 0.02880	0.002207 0.9712				−57.3370	120.674
5	0 0.45071	0.004506 0.5214	0.4489 0.02788			−55.7097	121.419

Table 3. 95% confidence intervals of r_1, $r_1 + r_2$.

	lower confidence limt	estimate	upper confidence limit	g
r_1	4.4×10^{-13}	0.5621	0.9470	4
$r_1 + r_2$	0.8975	0.9512	0.9893	4
r_1	0.4633	0.6917	0.8855	2

Table 4. The probability of failures $P_{m,M}$ after screening.

M	$m = 30$	$m = 50$	$m = 100$
1	0.001457	0.001375	0.001214
5	0.007208	0.006823	0.006025
10	0.01425	0.01352	0.01194
50	0.06588	0.06275	0.05542
100	0.1203	0.1147	0.1013
500	0.3244	0.3093	0.2732
1000	0.3733	0.3560	0.3144
5000	0.3820	0.3643	0.3217
10000	0.3820	0.3643	0.3217
50000	0.3820	0.3643	0.3217

$g = 4$

Table 5. 95% confidence intervals of $P_{m,M}$.

	lower confidence limt	estimate	upper confidence limit	g
$P_{30,1000}$	0.0001	0.3733	0.9435	4
$P_{100,1000}$	2.7×10^{-11}	0.3144	0.9394	4
$P_{30,1000}$	0.03879	0.2039	0.4215	2
$P_{100,1000}$	.001991	0.06566	0.2215	2

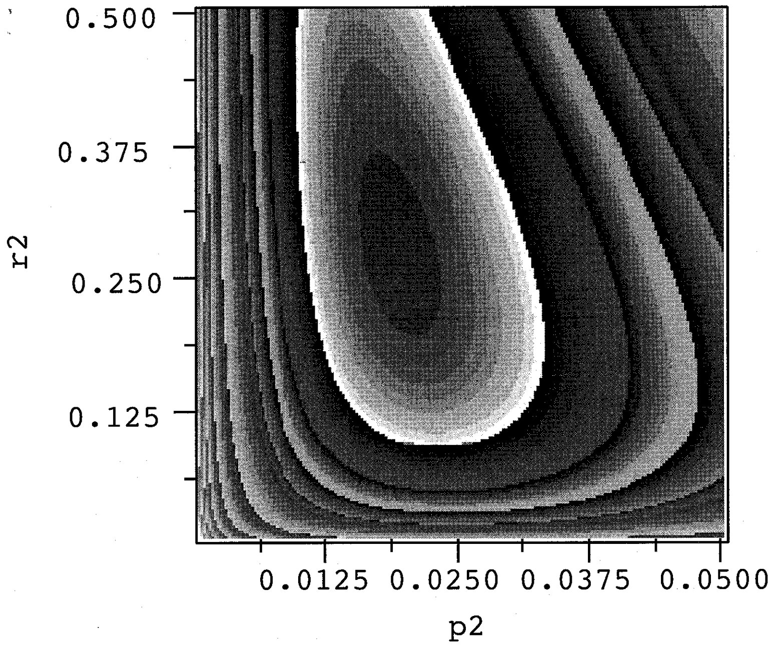

Fig. 1 The contour plot of the log-likelihood for the case that one inferior and one quality populations are mixed.

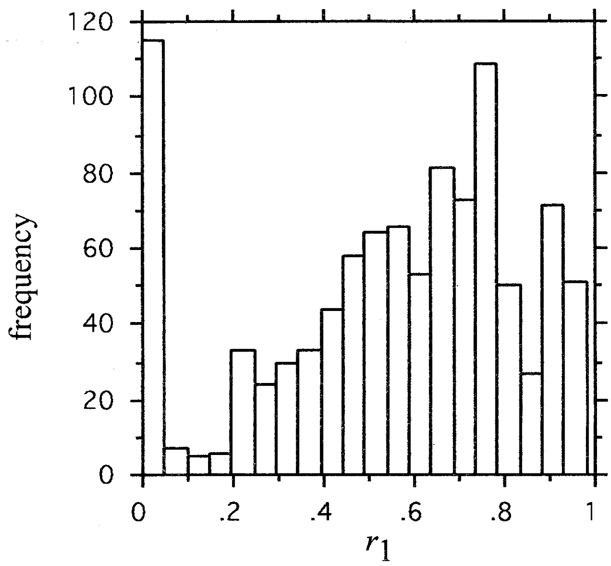

Fig. 2 The histogram of estimate r_1.

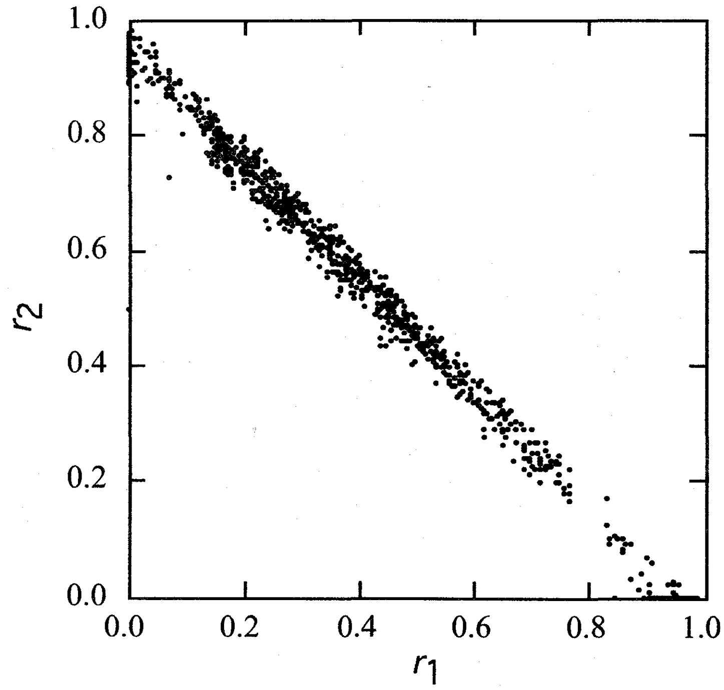

Fig. 3 The reration between r_1 and r_2

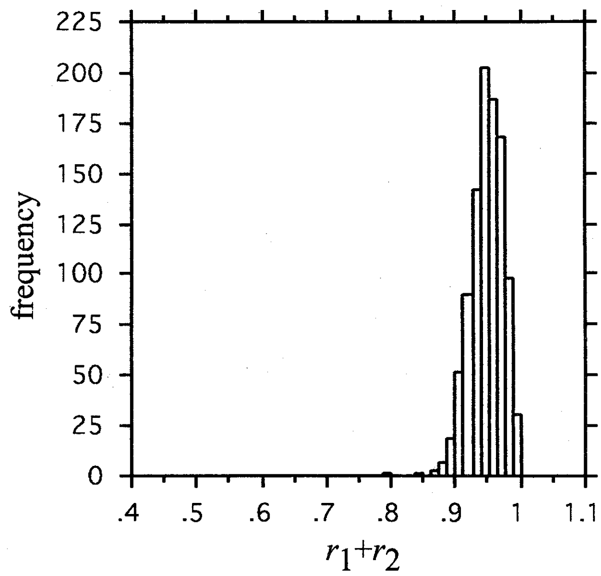

Fig. 4 The histogram of estimate r_1+r_2

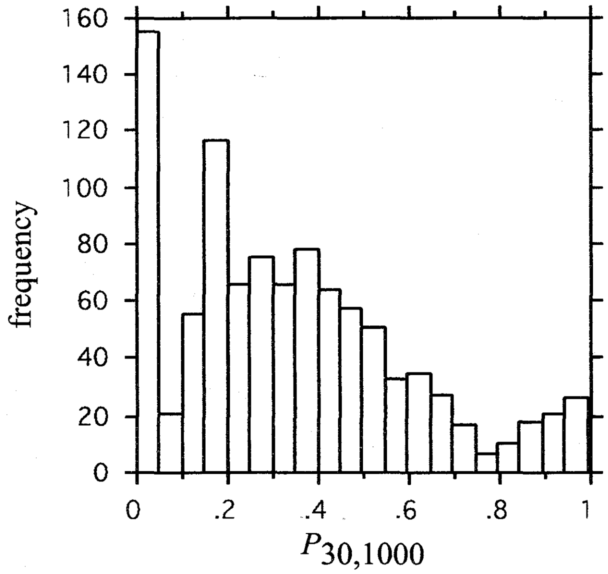

Fig. 5 The histogram of estimate $P_{30,1000}$.