Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 59, 2012, pp. 1–5

AN EASILY VERIFIABLE PROOF OF THE BROUWER FIXED POINT THEOREM

Yukio TAKEUCHI and Tomonari SUZUKI

Abstract

We give an elementary proof of the Brouwer fixed point theorem. The proof is verifiable for most of the mathematicians.

1. The authors' remark

The authors have to inform that the proof in this paper is a version of Kulpa's proof in [3]. When they submitted this paper to some journal, the referee pointed out this fact. So they cannot tell that the result in this paper is new.

However, the authors think that there are some merits publishing this paper. First, Kulpa gave the proof of the Poincaré-Miranda theorem. So, when one searches for the proofs of the Brouwer theorem, it may be difficult to find Kulpa's paper. The authors consider that this paper helps to make Kulpa's paper [3] known wider. Second, Kulpa omitted the details of the proof. That is, the proof in [3] is not very easy to read. The third reason is connected with the second one. Kulpa mentioned that the concept of simplicial subdivision is unnecessary. However, we cannot judge clearly from Kulpa's paper whether the concept of simplex is unnecessary. The authors emphasize that the proof does not require the concepts of simplex nor simplicial subdivision. So we do not need a lot of logical steps associated with these concepts.

2. Introduction

The following theorem is referred to as the *Brouwer fixed point theorem*.

THEOREM 1 (Brouwer [1], Hadamard [2]). *Let* $n \in \mathbf{N}$ *and let* g *be a continuous mapping on* $[0,1]^n$. *Then there exists* $z \in [0,1]^n$ *such that* $g(z) = z$.

Theorem 1 is used in numerous fields of mathematics. So we can consider Theorem 1 is one of the most useful theorems in mathematics. There are many proofs of Theorem 1. For example, the proof based on the Sperner lemma [4] is very excellent. See also Stuckless [5] and references therein. In this paper, we give a proof

2000 *Mathematics Subject Classification.* Primary 55M20, Secondary 54C05.
Key words and phrases. the Brouwer fixed point theorem.

of Theorem 1. The authors think that most of the mathematicians can verify our proof. We do not need any geometric intuition.

3. Preliminaries

Throughout this paper, we denote by **N** the set of all positive integers and by **R** the set of all real numbers. We define $N(i,j)$ by

$$N(i,j) = \{k : k \in \mathbf{N} \cup \{0\}, i \le k \le j\}.$$

For $x \in \mathbf{R}^n$, $(x)_i$ denotes the i-th coordinate of x. For an arbitrary set B, we also denote by $\#B$ the cardinal number of B.

Let L be an arbitrary set, $n \in \mathbf{N}$ and $k \in N(0,n)$. Let ℓ be a mapping from L into $N(0,n)$. We call such a mapping a *labeling*. Then a subset B of L is called *k-fully labeled* if $\#B = k+1$ and $\ell(B) = N(0,k)$. The following lemma is very fundamental, however, it plays an important role.

LEMMA 2. *Let L be an arbitrary set, $n \in \mathbf{N}$ and $k \in N(1,n)$. Let ℓ be a labeling from L into $N(0,n)$. Let B be a subset of L with $\#B = k+1$ and $\ell(B) \subset N(0,k)$. Then the following hold:*

(i) *B includes at most two subsets which are $(k-1)$-fully labeled.*

(ii) *B includes exactly one subset which is $(k-1)$-fully labeled if and only if B is k-fully labeled.*

PROOF. Suppose there exists a subset C of B which is $(k-1)$-fully labeled. Since $\#C = k$, we can suppose $B = C \sqcup \{b\}$, where $\sqcup$ represents 'disjoint union'. In the case where $\ell(b) < k$, only two subsets C and $(C \backslash \ell^{-1}(\ell(b))) \cup \{b\}$ of B are $(k-1)$-fully labeled. In the other case, where $\ell(b) = k$, B is k-fully labeled and C is the only subset of B which is $(k-1)$-fully labeled. We have shown (i). From the above observation, (ii) is obvious. □

4. A labeling theorem

In this section, we fix $n, m \in \mathbf{N}$. We set n points $e_1, \dots, e_n \in \mathbf{R}^n$ by

$$e_1 = (1/m, 0, \dots, 0), \qquad e_2 = (0, 1/m, 0, \dots, 0), \qquad \dots, \qquad e_n = (0, \dots, 0, 1/m).$$

Define $n+1$ subsets $L_0, \dots, L_n$ of $[0,1]^n$ by $L_0 = \{0\}$ and

$$L_k = \left\{ \sum_{i=1}^{k} \alpha_i e_i : \alpha_i \in N(0,m) \right\} \tag{1}$$

for $k \in N(1,n)$. A labeling ℓ from L_n into $N(0,n)$ is called *Brouwer* if the following two conditions are satisfied:

(B1) If $(x)_k = 0$ for some $k \in N(1,n)$, then $\ell(x) \ne k$.

(B2) If $(x)_k = 1$ for some $k \in N(1,n)$, then $\ell(x) \ge k$.

It is obvious that $\ell(x) \le k$ for $k \in N(0,n)$ and $x \in L_k$. A subset B of L_k is said to be a *k-string* if there exist $x_0, \dots, x_k \in L_k$ and a bijection σ on $N(1,k)$ such that $B = \{x_0, \dots, x_k\}$ and $x_j = x_0 + \sum_{i=1}^{j} e_{\sigma(i)}$ for $j \in N(1,k)$. The following are obvious:

$$x_k = x_0 + \sum_{i=1}^{k} e_i, \tag{2}$$

$$\sum_{i=1}^{n} (x_j)_i = \sum_{i=1}^{n} (x_0)_i + j/m \qquad \text{for } j \in N(1,k). \tag{3}$$

We sometimes write $B = \langle x_0, \dots, x_k \rangle$. We note that B is a 0-string if and only if $B = \{0\}$. We also note that from (3) the order of $x_0, \dots, x_k$ is unique.

Theorem 3. *Let ℓ be a Brouwer labeling from L_n into $N(0,n)$. Then there exists an n-string which is n-fully labeled.*

We note that Theorem 3 is connected with the Sperner lemma [4]. Before proving Theorem 3, we need some preliminaries.

Lemma 4. *There exists exactly one* 0*-string which is* 0*-fully labeled.*

Proof. Since $\ell(0) = 0$, $\langle 0 \rangle$ is a 0-string which is 0-fully labeled. □

Lemma 5. *Let C be a $(k-1)$-string for some $k \in N(1,n)$. Then there exists exactly one k-string which includes C.*

Proof. Suppose $C = \langle x_0, \dots, x_{k-1} \rangle$. It follows from (2) and $(x_i)_k = 0$ for $i \in N(0, k-1)$ that only

$$B = \langle x_0, \dots, x_{k-1}, x_{k-1} + e_k \rangle$$

is a k-string which includes C. □

Lemma 6. *Let B be a k-string for some $k \in N(1,n)$ and let C be a subset of B which is $(k-1)$-fully labeled. Then the following hold:*

- (i) *If $C \subset L_{k-1}$, then C is a $(k-1)$-string and there exists exactly one k-string which includes C.*
- (ii) *If $C \not\subset L_{k-1}$, then there exist exactly two k-strings which include C.*

Proof. Suppose $B = \langle x_0, \dots, x_k \rangle$, $x_j = x_0 + \sum_{i=1}^{j} e_{\sigma(i)}$ and $B = C \sqcup \{x_h\}$. In the case of (i), it is obvious that $h = k$ and $\sigma(k) = k$. So C is a $(k-1)$-string. By Lemma 5, only B is a k-string which includes C. In the case of (ii), we consider three cases:

- $h = 0$
- $0 < h < k$
- $h = k$

In the first case, arguing by contradiction, we assume $(x_k)_{\sigma(1)} = 1$. Then $(x_1)_{\sigma(1)} = \cdots = (x_k)_{\sigma(1)} = 1$, which imply $\ell(x_1) \ge \sigma(1) > 0, \dots, \ell(x_k) \ge \sigma(1) > 0$ by (B2). Thus C is not $(k-1)$-fully labeled. This is a contradiction. Thus $(x_k)_{\sigma(1)} < 1$.

Therefore

$$B' = \langle x_1, \dots, x_k, x_k + e_{\sigma(1)} \rangle$$

is another k-string and includes C. In the second case,

$$B' = \langle x_0, \dots, x_{h-1}, x_{h-1} + e_{\sigma(h+1)}, x_{h+1}, \dots, x_k \rangle$$

is another k-string and includes C. In the third case, arguing by contradiction, we assume $(x_0)_{\sigma(k)} = 0$. Then $(x_0)_{\sigma(k)} = \cdots = (x_{k-1})_{\sigma(k)} = 0$, which imply $\ell(x_0) \neq \sigma(k), \dots, \ell(x_{k-1}) \neq \sigma(k)$ by (B1). Since $\sigma(k) = k$ implies $C \subset L_{k-1}$, we have $\sigma(k) < k$. So C is not $(k-1)$-fully labeled. This is a contradiction. Thus $(x_0)_{\sigma(k)} > 0$. Therefore

$$B' = \langle x_0 - e_{\sigma(k)}, x_0, \dots, x_{k-1} \rangle$$

is another k-string and includes C. By (2) and (3), it is impossible that three k-strings include C. □

By Lemmas 5 and 6, we obtain the following.

LEMMA 7. *Let C be a subset of L_k which is $(k-1)$-fully labeled for some $k \in N(1, n)$. Then the following hold:*

(i) *C is included by at most two k-strings.*

(ii) *C is included by exactly one k-string if and only if C is a $(k-1)$-string.*

LEMMA 8. *Let $k \in N(1, n)$. Assume there exist exactly odd $(k-1)$-strings which are $(k-1)$-fully labeled. Then there exist exactly odd k-strings which are k-fully labeled.*

PROOF. Define four sets S_1, S_2, T_1 and T_2 as follows: $B \in S_1$ if and only if B is a k-string which includes exactly one $(k-1)$-fully labeled subset. $B \in S_2$ if and only if B is a k-string which includes exactly two $(k-1)$-fully labeled subsets. $C \in T_1$ if and only if C is a $(k-1)$-fully labeled subset included by exactly one k-string. $C \in T_2$ if and only if C is a $(k-1)$-fully labeled subset included by exactly two k-strings. By Lemma 2 (ii), we note that $B \in S_1$ if and only if B is a k-string which is k-fully labeled. By Lemma 7 (ii), we also note that $C \in T_1$ if and only if C is a $(k-1)$-string which is $(k-1)$-fully labeled. With double-counting, we count the number of $(k-1)$-fully labeled subsets in k-strings. Then by Lemmas 2 (i) and 7 (i), we have

$$\#S_1 + 2\#S_2 = \#T_1 + 2\#T_2.$$

Since $\#T_1$ is odd, we obtain $\#S_1$ is odd. □

PROOF OF THEOREM 3. By Lemmas 4 and 8, There exist exactly odd n-strings which are n-fully labeled. Since 0 is not odd, there exists at least one n-string which is n-fully labeled. □

5. A proof

We give a proof of Theorem 1.

Proof of Theorem 1. Define n functions $g_1, \ldots, g_n$ from $[0,1]^n$ into $[0,1]$ by $g_k(x) = (g(x))_k$ for $k \in N(1,n)$. It is obvious that g_k is continuous.

We fix $m \in \mathbf{N}$ and define L_n by (1). Define a labeling ℓ from L_n into $N(0,n)$ by

$$\ell(x) = \max\{k \in N(1,n) : (x)_k > 0, g_k(x) \le (x)_k\},$$

where $\max \emptyset = 0$. From the definition of ℓ, (B1) obviously holds. If $(x)_k = 1$, then $g_k(x) \le (x)_k$, which implies $\ell(x) \ge k$. So (B2) holds and hence ℓ is Brouwer. We note that if $\ell(x) = 0$, then $(x)_j \le g_j(x)$ for $j \in N(1,n)$. By Theorem 3, there exists an n-string $B^{(m)}$ which is n-fully labeled. Let $y_0^{(m)}, \ldots, y_n^{(m)} \in [0,1]^n$ satisfy $B^{(m)} = \{y_0^{(m)}, \ldots, y_n^{(m)}\}$ and $\ell(y_k^{(m)}) = k$ for $k \in N(0,n)$.

Since $[0,1]^n$ is compact, by the Bolzano-Weierstrass theorem, $\{y_0^{(m)}\}$ has a convergent subsequence. Without loss of generality, we may assume $\{y_0^{(m)}\}$ itself converges to some $z \in [0,1]^n$. Since the diameter of $B^{(m)}$ is $\sqrt{n}/m$, $\{y_k^{(m)}\}$ also converges to z for $k \in N(1,n)$. Since $(y_0^{(m)})_j \le g_j(y_0^{(m)})$ for $j \in N(1,n)$ and $g_k(y_k^{(m)}) \le (y_k^{(m)})_k$ for $k \in N(1,n)$, we have $g_k(z) = (z)_k$ for all $k \in N(1,n)$. Thus $g(z) = z$. □

References

[1] L. E. J. Brouwer, Uber Abbildung der Mannigfaltigkeiten, Math. Ann., **71** (1912), 97–115.

[2] J. Hadamard, Sur quelques applications de l'indice de Kronecker, in J. Tannery (Ed.): Introduction à la théorie des fonctions d'une variable, Vol. 2, Hermann (1910), pp. 875–915.

[3] W. Kulpa, The Poincare-Miranda theorem, Amer. Math. Monthly, **104** (1997), 545–550. MR1453657

[4] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Sem. Univ. Hamburg **6** (1928), 265–272.

[5] T. Stuckless, Brouwer's fixed point theorem: Methods of proof and generalizations, Thesis, Simon Fraser University (2003). available online at http://oldweb.cecm.sfu.ca/Preprints03/preprints03.html

Takahashi Institute for Nonlinear Analysis
1-11-11, *Nakazato, Minami*
Yokohama 232-0063, *Japan*
E-mail address: aho31415@yahoo.co.jp

Department of Basic Sciences
Kyushu Institute of Technology
Tobata, Kitakyushu 804-8550, *Japan*
E-mail address: suzuki-t@mns.kyutech.ac.jp