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AN ELEMENTARY PROOF OF THE 2-DIMENSIONAL VERSION OF THE BROUWER FIXED POINT THEOREM

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Abstract

We give an elementary proof of the 2-dimensional version of the Brouwer fixed point theorem. The proof can be extended to the n-dimensional version naturally. In our proof, we only use the Bolzano-Weierstrass theorem and the fact that an odd number plus an even number is odd.

1. Introduction and preliminaries

Brouwer [1] and Hadamard [3] proved the following theorem which is referred to as the *Brouwer fixed point theorem.*

The Brouwer Fixed Point Theorem. *Let $n \in \mathbf{N}$ and let g be a continuous mapping from $[0,1]^n$ into itself. Then there exists $z \in [0,1]^n$ such that $g(z) = z$.*

This theorem is often called one of the most useful theorems because this theorem is used in numerous fields of mathematics. Also there are many proofs of this theorem; see Stuckless [6] and references therein. Franklin [2] wrote "The Brouwer fixed-point theorem seemed to me a priceless jewel guarded by a dragon called topology. Whenever I got near the jewel, the dragon would breathe fire on me. It was frustrating. The statement of Brouwer's theorem is so simple. Why was the proof so hard?"

Kulpa [4] gave a very elementary proof, and Takeuchi and Suzuki [7] gave a proof which is a version of Kulpa's. The authors think that most of the mathematicians can verify the proof in [7]. In this paper, we give a 2-dimensional version of the proof in [7]. The purpose of this paper is to give more people—for example, undergraduate students—the opportunity to verify the proof of the Brouwer fixed point theorem.

Indeed, in order to understand our labeling theorem (Theorem 1), readers only need the fact that an odd number plus an even number is odd. In order to understand the Brouwer fixed point theorem, readers only need the Bolzano-Weierstrass theorem.

The Bolzano-Weierstrass Theorem. *Let $n \in \mathbf{N}$. Let $\{x_m\}$ be a sequence in $[0,1]^n$. Then there is a subsequence $\{x_{m_i}\}$ of $\{x_m\}$ which converges to some $z \in [0,1]^n$.*

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Throughout this paper, we denote by $\mathbf{N}$ the set of all positive integers and by $\mathbf{N}_0$ the set of all nonnegative integers. For $k, l \in \mathbf{N}_0$ with $k \le l$, we define $N(k,l)$ by

$$N(k,l) = \{i : i \in \mathbf{N}_0,\ k \le i \le l\}.$$

For a set B consisting of finite elements, $\#B$ denotes the number of elements of B.

2. A labeling theorem

In this section, we fix $m \in \mathbf{N}$. Define two subsets L_1 and L_2 of $\mathbf{N}_0^2$ by

$$\text{(L)} \qquad L_1 = \{(i,0) : i \in N(0,m)\} \qquad \text{and} \qquad L_2 = \{(i,j) : i,j \in N(0,m)\}.$$

We introduce some concepts which are needed in the sequel.

- A subset B of L_2 is said to be a 1-*string* if there exists $i \in N(1,m)$ such that $B = \{(i-1,0),(i,0)\}$.
- A subset B of L_2 is said to be a 2-*string* if there exist $i,j \in N(1,m)$ such that B equals either

$$\{(i-1,j-1),(i,j-1),(i,j)\} \qquad \text{or} \qquad \{(i-1,j-1),(i-1,j),(i,j)\}.$$

For example,

$\{(0,0),(1,0)\}$ is a 1-string.

$\{(0,0),(1,0),(1,1)\}$ and $\{(0,0),(0,1),(1,1)\}$ are 2-strings.

Let ℓ be a function from L_2 into $N(0,2)$. In this paper, such a ℓ is called a *labeling*. For $(i,j) \in L_2$, we write $\ell(i,j)$ instead of $\ell((i,j))$.

- A subset B of L_2 is called 1-*fully labeled* if there exist $a,b \in L_2$ such that $B = \{a,b\}$, $\ell(a) = 0$ and $\ell(b) = 1$.
- A subset B of L_2 is called 2-*fully labeled* if there exist $a,b,c \in L_2$ such that $B = \{a,b,c\}$, $\ell(a) = 0$, $\ell(b) = 1$ and $\ell(c) = 2$.

A labeling ℓ from L_2 into $N(0,2)$ is called *Brouwer* if the following conditions hold:

(B1) $\ell(i,m) = 2$ for $i \in N(0,m)$.
(B2) $\ell(i,0) \ne 2$ for $i \in N(0,m)$.
(B3) $\ell(m,j) \ne 0$ for $j \in N(0,m)$.
(B4) $\ell(0,j) \ne 1$ for $j \in N(0,m)$.

The following is our labeling theorem, which is related to the Sperner lemma [5].

THEOREM 1 (Labeling theorem). *Let ℓ be a Brouwer labeling from L_2 into $N(0,2)$. Then there exists a 2-string which is 2-fully labeled.*

Before proving Theorem 1, we need some lemmas.

LEMMA 2. *Let ℓ be a Brouwer labeling from L_2 into $N(0,2)$. Then there exist exactly odd 1-strings which are 1-fully labeled.*

PROOF. By (B2), we have $\ell(i,0) \in \{0,1\}$ for $i \in N(0,m)$. So, for $i \in N(1,m)$, the following are equivalent:

- $\ell(i-1,0) + \ell(i,0)$ is odd.
- $\ell(i-1,0) + \ell(i,0) = 1$.
- The 1-string $\{(i-1,0),(i,0)\}$ is 1-fully labeled.

By (B2) and (B4), we have $\ell(0,0) = 0$. By (B2) and (B3), we have $\ell(m,0) = 1$. Hence the number

$$\sum_{i=1}^{m}(\ell(i-1,0) + \ell(i,0)) = \ell(0,0) + \ell(m,0) + 2\sum_{i=1}^{m-1}\ell(i,0)$$

is odd. So $\#\{i \in N(1,m) : \ell(i-1,0) + \ell(i,0) \text{ is odd}\}$ is odd. Therefore there exist exactly odd 1-strings which is 1-fully labeled. $\square$

LEMMA 3. *Let ℓ be a Brouwer labeling from L_2 into $N(0,2)$. Let B be a 2-string. Then the following hold:*

(i) *B includes at most two subsets which are 1-fully labeled.*

(ii) *B includes exactly one subset which is 1-fully labeled if and only if B is 2-fully labeled.*

PROOF. Suppose $B = \{a,b,c\}$ with $\ell(a) \le \ell(b) \le \ell(c)$.

- If $\ell(a) = \ell(b) = 0$ and $\ell(c) = 1$, then $\{a,c\}$ and $\{b,c\}$ are 1-fully labeled.
- If $\ell(a) = 0$ and $\ell(b) = \ell(c) = 1$, then $\{a,b\}$ and $\{a,c\}$ are 1-fully labeled.
- If $\ell(a) = 0$, $\ell(b) = 1$ and $\ell(c) = 2$, then $\{a,b\}$ is 1-fully labeled and B is 2-fully labeled.
- Otherwise, there does not exist any subset of B which is 1-fully labeled.

From this observation, (i) and (ii) hold. $\square$

LEMMA 4. *Let ℓ be a Brouwer labeling from L_2 into $N(0,2)$. Let C be a 1-fully labeled subset. Then the following hold*:

(i) *C is included by at most two 2-strings.*

(ii) *C is included by exactly one 2-string if and only if C is a 1-string.*

PROOF. Suppose C is included by some 2-string. We consider the following four cases:

- $C = \{(i-1,0),(i,0)\}$ for some $i \in N(1,m)$, that is, C is a 1-string.
- $C = \{(i-1,j),(i,j)\}$ for some $i,j \in N(1,m)$.
- $C = \{(i,j-1),(i,j)\}$ for some $i \in N(0,m)$ and $j \in N(1,m)$.
- $C = \{(i-1,j-1),(i,j)\}$ for some $i,j \in N(1,m)$.

In the first case, only

$$\{(i-1,0),(i,0),(i,1)\}$$

is a 2-string which includes C. In the second case, since C is 1-fully labeled, $\ell(i-1,j), \ell(i,j) \in \{0,1\}$ holds. So by (B1), we have $j < m$. Hence

$$\{(i-1,j-1),(i-1,j),(i,j)\} \qquad \text{and} \qquad \{(i-1,j),(i,j),(i,j+1)\}$$

are 2-strings which include C. In the third case, by (B3) and (B4), we have $0 < i < m$. So

$$\{(i-1,j-1),(i,j-1),(i,j)\} \qquad \text{and} \qquad \{(i,j-1),(i,j),(i+1,j)\}$$

are 2-strings which include C. In the fourth case,

$$\{(i-1,j-1),(i,j-1),(i,j)\} \qquad \text{and} \qquad \{(i-1,j-1),(i-1,j),(i+1,j)\}$$

are 2-strings which include C. It is impossible that three 2-strings include C. From this observation, (i) and (ii) hold. □

PROOF OF THEOREM 1. Define four sets S_1, S_2, T_1 and T_2 as follows:

- $B \in S_1$ if and only if
  B is a 2-string which includes exactly one 1-fully labeled subset.
- $B \in S_2$ if and only if
  B is a 2-string which includes exactly two 1-fully labeled subsets.
- $C \in T_1$ if and only if
  C is a 1-fully labeled subset included by exactly one 2-string.
- $C \in T_2$ if and only if
  C is a 1-fully labeled subset included by exactly two 2-strings.

By Lemma 3 (ii), we note that $B \in S_1$ if and only if B is a 2-string which is 2-fully labeled. By Lemma 4 (ii), we also note that $C \in T_1$ if and only if C is a 1-string which is 1-fully labeled. With double-counting, we count the number of 1-fully labeled subsets in 2-strings. Then by Lemmas 3 (i) and 4 (i), we have

$$\#S_1 + 2\#S_2 = \#T_1 + 2\#T_2.$$

By Lemma 2, $\#T_1$ is odd. Therefore we obtain $\#S_1$ is odd. This implies $\#S_1 \neq 0$, that is, there is a 2-string which is 2-fully labeled. □

3. The 2-dimensional version of the Brouwer fixed point theorem

The 2-dimensional version. *Let g be a continuous mapping from $[0,1]^2$ into itself. Then there exists $z \in [0,1]^2$ such that $g(z) = z$.*

PROOF. Define functions g_1 and g_2 from $[0,1]^2$ into $[0,1]$ by

$$g(s,t) = (g_1(s,t), g_2(s,t)) \qquad \text{for } (s,t) \in [0,1]^2.$$

That is, if $g(s,t) = (s', t')$, then $g_1(s,t) = s'$ and $g_2(s,t) = t'$. Since g is continuous, g_1 and g_2 are also continuous. It is obvious that

$$\text{(G)} \qquad g_2(s,1) \leq 1, \qquad g_2(s,0) \geq 0, \qquad g_1(1,t) \leq 1, \qquad g_1(0,t) \geq 0$$

for $s, t \in [0,1]$. We fix $m \in \mathbf{N}$ and define L_2 by (L). Define a subset M of L_2 by

$$M = \{(i,j) : j > 0,\, g_2(i/m, j/m) \leq j/m\}$$

and a labeling ℓ from L_2 into $N(0,2)$ by

$$\ell(i,j) = \begin{cases} 2 & \text{if } (i,j) \in M \\ 1 & \text{if } (i,j) \notin M,\, i > 0,\, g_1(i/m, j/m) \leq i/m \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of ℓ and (G), it is obvious that ℓ satisfies (B1)–(B4). That is, ℓ is a Brouwer labeling from L_2 into $N(0,2)$. We note that

$$\begin{aligned} \text{(GL)} \qquad & g_2(i/m, j/m) \leq j/m \qquad \text{if } \ell(i,j) = 2, \\ & g_1(i/m, j/m) \leq i/m \qquad \text{if } \ell(i,j) = 1, \\ & i/m \leq g_1(i/m, j/m) \quad \text{and} \quad j/m \leq g_2(i/m, j/m) \qquad \text{if } \ell(i,j) = 0. \end{aligned}$$

By Theorem 1, there exists a 2-string $B^{(m)}$ which is 2-fully labeled. Suppose

$$\begin{aligned} & B^{(m)} = \{(i_0^{(m)}, j_0^{(m)}), (i_1^{(m)}, j_1^{(m)}), (i_2^{(m)}, j_2^{(m)})\}, \\ & \ell(i_0^{(m)}, j_0^{(m)}) = 0, \qquad \ell(i_1^{(m)}, j_1^{(m)}) = 1, \qquad \ell(i_2^{(m)}, j_2^{(m)}) = 2. \end{aligned}$$

We put

$$x_m = (i_0^{(m)}/m, j_0^{(m)}/m), \qquad y_m = (i_1^{(m)}/m, j_1^{(m)}/m), \qquad z_m = (i_2^{(m)}/m, j_2^{(m)}/m).$$

Thus, we have sequences $\{x_m\}$, $\{y_m\}$ and $\{z_m\}$ in $[0,1]^2$.

By the Bolzano-Weierstrass theorem, $\{x_m\}$ has a convergent subsequence. Without loss of generality, we may assume $\{x_m\}$ itself converges to some $z = (s_0, t_0) \in [0,1]^2$. It is easy to see that

$$\|x_m - y_m\| \leq \sqrt{2}/m \qquad \text{and} \qquad \|x_m - z_m\| \leq \sqrt{2}/m$$

for each $m \in \mathbf{N}$, where $\|x_m - y_m\|$ is the distance between x_m and y_m. By these inequalities, we have $\{y_m\}$ and $\{z_m\}$ also converge to (s_0, t_0). By (GL), we have

$$\begin{aligned} & i_0^{(m)}/m \leq g_1(i_0^{(m)}/m, j_0^{(m)}/m), \qquad j_0^{(m)}/m \leq g_2(i_0^{(m)}/m, j_0^{(m)}/m), \\ & g_1(i_1^{(m)}/m, j_1^{(m)}/m) \leq i_1^{(m)}/m, \qquad g_2(i_2^{(m)}/m, j_2^{(m)}/m) \leq j_2^{(m)}/m \end{aligned}$$

for each $m \in \mathbf{N}$. Letting $n \to \infty$, we obtain

$$s_0 \leq g_1(s_0, t_0), \qquad t_0 \leq g_2(s_0, t_0), \qquad g_1(s_0, t_0) \leq s_0 \qquad \text{and} \qquad g_2(s_0, t_0) \leq t_0.$$

Thus $g_1(s_0, t_0) = s_0$ and $g_2(s_0, t_0) = t_0$, that is, $g(z) = g(s_0, t_0) = (s_0, t_0) = z$. $\square$

References

[1] L. E. J. Brouwer, Uber Abbildung der Mannigfaltigkeiten, Math. Ann., **71** (1912), 97–115.

[2] J. N. Franklin, Methods of mathematical economics: Linear and nonlinear programming, fixed-point theorems, Reprint of the 1980 original, Classics in Applied Mathematics 37, Society for Industrial and Applied Mathematics (SIAM), (2002). MR1875314

[3] J. Hadamard, Sur quelques applications de l'indice de Kronecker, in J. Tannery (Ed.): Introduction à la théorie des fonctions d'une variable, Vol. 2, Hermann (1910), pp. 875–915.

[4] W. Kulpa, The Poincare-Miranda theorem, Amer. Math. Monthly, **104** (1997), 545–550. MR1453657

[5] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Sem. Univ. Hamburg **6** (1928), 265–272.

[6] T. Stuckless, Brouwer's fixed point theorem: Methods of proof and generalizations, Thesis, Simon Fraser University (2003). available online at http://oldweb.cecm.sfu.ca/Preprints03/preprints03.html

[7] Y. Takeuchi and T. Suzuki, An easily verifiable proof of the Brouwer fixed point theorem, Bull. Kyushu Inst. Technol., **59** (2012), 1–5.

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