Low-momentum nucleon-nucleon interaction and its application to few-nucleon systems

S. Fujii*
Department of Physics, University of Tokyo, Tokyo 113-0033, Japan

E. Epelbaum†
Jefferson Laboratory, Newport News, Virginia 23606, USA

H. Kamada,‡ R. Okamoto,§ and K. Suzuki‖
Department of Physics, Kyushu Institute of Technology, Kitakyushu 804-8550, Japan

W. Glöckle¶
Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany
(Received 28 April 2004; published 26 August 2004)

Low-momentum nucleon-nucleon interactions are derived within the framework of a unitary-transformation theory, starting with realistic nucleon-nucleon interactions. A cutoff momentum Λ is introduced to specify a border between the low- and high-momentum spaces. By Faddeev-Yakubovsky calculations the low-momentum interactions are investigated with respect to the dependence of ground-state energies of ^{3}H and ^{4}He on the parameter Λ. It is found that we need the momentum cutoff parameter $\Lambda \geqslant 5$ fm^{-1} in order to reproduce satisfactorily the exact values of the binding energies for ^{3}H and ^{4}He. The calculation with $\Lambda = 2$ fm^{-1} recommended by Bogner *et al.* leads to considerable overbinding at least for few-nucleon systems.

DOI: 10.1103/PhysRevC.70.024003 PACS number(s): 21.30.−x, 21.45+v, 27.10+h

I. INTRODUCTION

One of the fundamental problems in nuclear structure calculations is to describe nuclear properties, starting with realistic nucleon-nucleon (NN) interactions. However, since this kind of interaction has a repulsive core at a short distance, one has to derive an effective interaction in a model space from the realistic interaction, except for the case of precise few-nucleon structure calculations.

Recently, Bogner *et al.* have proposed a low-momentum nucleon-nucleon (LMNN) interaction which is constructed in momentum space for the two-nucleon system from a realistic nucleon-nucleon interaction, using conventional effective interaction techniques or renormalization group ones [1]. In the construction of the LMNN interaction a cutoff momentum Λ is introduced to specify a border between the low- and high-momentum spaces.

The LMNN interaction is constructed in order to account for the short-range correlations of the two nucleons interacting in the vacuum. So the question is to what extent the obtained LMNN interaction is a good approximation also for describing correlation of nucleons interacting in a nuclear many-body medium. The medium effect could appear through the single-particle potential and three-or-more-body correlations. From a practical point of view it is of high interest to explore the sensitivity of calculated results to the cutoff momentum Λ.

Bogner *et al.* have constructed their LMNN interaction in a way to conserve in the low-momentum region not only the on-shell properties of the original interaction (i.e., phase shifts and the deuteron binding energy) but also the half-on-shell T matrix [1]. They found that the LMNN interactions for $\Lambda = 2.1$ fm^{-1} corresponding to $E_{\rm lab} \simeq 350$ MeV become nearly universal, not (or only weakly) dependent on the choice of realistic interactions employed. They suggested using their LMNN interaction directly in nuclear structure calculations, such as shell-model [2] and Hartree-Fock calculations [3]. They claimed that for the momentum cutoff in the vicinity of $\Lambda = 2.1$ fm^{-1} the calculated low-lying spectra for ^{18}O, ^{134}Te, and ^{135}I are in good agreement with the experimental data and depend weakly on Λ. Their results have been found to agree with the data as well as or even slightly better than the results based upon the G matrix which is constructed by taking into account the short-range correlations, the Pauli blocking effect, and the state dependence for each nucleus.

Kuckei *et al.* [4] investigated the nuclear matter and the closed-shell nucleus ^{16}O by using the LMNN interaction obtained by Bogner *et al.* They concluded that the LMNN interaction can be a very useful tool for low-energy nuclear structure calculations, and one should be cautious if the observable of interest is sensitive to the single-particle spectrum at energies above the cutoff momentum.

However, one had better compare with the exact solutions once for all. In cases of three- and four-nucleon systems we can directly make such a comparison by solving the Faddeev-Yakubovsky equations [5].

We investigate the LMNN interaction by means of a unitary transformation using two independent (but equivalent)

*Electronic address: sfujii@nt.phys.s.u-tokyo.ac.jp
†Electronic address: epelbaum@jlab.org
‡Electronic address: kamada@mns.kyutech.ac.jp
§Electronic address: okamoto@mns.kyutech.ac.jp
‖Electronic address: suzuki@mns.kyutech.ac.jp
¶Electronic address: Walter.Gloeckle@tp2.ruhr-uni-bochum.de

approaches. One approach is based on the unitary transformation of the Ōkubo form [6] in which a LMNN interaction is obtained from the scattering amplitude in momentum space [7–9]. Another is the unitary-model-operator approach (UMOA) [10,11]. In contrast to the *G*-matrix theory the UMOA leads to an energy-independent and Hermitian effective interaction in a many-body system. Contrary to the LMNN of Ref. [1], we will not require conservation of the half-on-shell *T* matrix, which does not represent an observable quantity. Only low-momentum NN observables such as the on-shell *T* matrix, phase shifts and binding energies are guaranteed to remain unchanged under a unitary transformation. In principle, one could achieve the equivalence of the half-on-shell *T* matrix by performing an additional unitary transformation in the low-momentum space. We, however, refrain from doing that since we do not see any conceptual or practical advantage in requiring the equivalence of the half-on-shell *T* matrix.

In the present study we first apply the above-mentioned methods to the two-nucleon system in momentum space to construct the LMNN interaction. Although both approaches are based on the same idea of a unitary transformation, the calculation procedures for deriving the LMNN interaction are independent and quite different from each other. We calculate selected properties of the two-nucleon system using both schemes to confirm the numerical accuracy. This cross check is useful to ensure the reliability of the numerical calculations. Secondly, we investigate the Λ dependence in structure calculations of few-nucleon systems, where (numerically) exact calculations can be performed [12,13], and discuss the validity of the LMNN interaction by comparing the obtained results with the *exact* values.

This paper is organized as follows. In Sec. II, after the basic formulation of the unitary transformation is presented, the two methods are given with emphasis on the different calculation procedures for deriving the LMNN interaction. In Sec. III, LMNN interactions are constructed using both methods from realistic nucleon-nucleon interactions such as the CD-Bonn [14] and the Nijm-I [15] potentials. Then, the Faddeev and Yakubovsky equations are solved using the LMNN interactions for various values of the cutoff parameter Λ. Finally, we summarize our results in Sec. IV.

II. UNITARY TRANSFORMATION OF THE HAMILTONIAN FOR THE TWO-NUCLEON SYSTEM IN MOMENTUM SPACE

We consider a quantum mechanical system described by a Hamiltonian H. The Schrödinger equation reads

$$H\Psi = E\Psi. \tag{1}$$

Introducing a unitary transformation U with $UU^\dagger = 1$ we obtain a transformed Schrödinger equation

$$H'\Psi' = E\Psi' \tag{2}$$

with the transformed Hamiltonian and state, $H' = U^\dagger H U$ and $\Psi' = U^\dagger \Psi$, respectively. We introduce the concept of the effective Hamiltonian by means of the partition technique. The Hilbert space is divided into two subspaces, the model space (P space) and its complement (Q space), such that the Schrödinger equation becomes a 2×2 block matrix equation

$$\begin{bmatrix} PH'P & PH'Q \\ QH'P & QH'Q \end{bmatrix} \begin{bmatrix} P\Psi' \\ Q\Psi' \end{bmatrix} = E \begin{bmatrix} P\Psi' \\ Q\Psi' \end{bmatrix}. \tag{3}$$

Here P and Q are the projection operators of a state onto the model space and its complement, respectively, and they satisfy $P+Q=1$, $P^2=P$, $Q^2=Q$, and $PQ=QP=0$. The Q-space state is easily eliminated to produce the projected Schrödinger equation

$$\left[PH'P + PH'Q \frac{1}{E - QH'Q} QH'P \right] (P\Psi') = E(P\Psi'). \tag{4}$$

In general, the effective Hamiltonian which is given in parentheses depends on the energy E to be determined. However, if the decoupling equation

$$QH'P = 0 \tag{5}$$

is satisfied, then we have the equation for the energy-independent effective Hamiltonian in the P space

$$PH'P(P\Psi') = E(P\Psi'). \tag{6}$$

A unitary transformation can be parametrized as

$$U = \begin{pmatrix} P(1+\omega^\dagger\omega)^{-1/2}P & -P\omega^\dagger(1+\omega\omega^\dagger)^{-1/2}Q \\ Q\omega(1+\omega^\dagger\omega)^{-1/2}P & Q(1+\omega\omega^\dagger)^{-1/2}Q \end{pmatrix}, \tag{7}$$

and the wave operator ω satisfies the condition $\omega = Q\omega P$. Equation (7) is well known as the Ōkubo form [6]. Notice that the unitary transformation given in Eq. (7) is by no means unique: in fact one can construct infinitely many different unitary transformations which decouple the P and Q subspaces. For example, performing subsequently any additional transformation, which is unitary in the P subspace, one would get a different LMNN interaction [16]. The transformation in Eq. (7) depends only on the operator ω which mixes the P and Q subspaces and is in some sense "the minimal possible" unitary transformation. For more discussion the reader is referred to Ref. [6].

In the present study, the above unitary transformation is used in two different methods to derive LMNN interactions. In the following sections, we shall give details of the two methods.

A. Method 1

Consider a momentum-space Hamiltonian for the two-nucleon system of the form

$$H(\vec{p},\vec{p}') = H_0(\vec{p})\,\delta(\vec{p}-\vec{p}') + V(\vec{p},\vec{p}'), \tag{8}$$

where $H_0(\vec{p}) = \vec{p}^{\,2}/(2M)$ with the reduced mass M stands for the kinetic energy, and $V(\vec{p},\vec{p}')$ is the bare two-body interaction. Our aim is to decouple the low- and high-momentum components of this two-nucleon potential using the method of unitary transformation. To achieve that, we introduce the projection operators

$$P = \int d^3p |\vec{p}\rangle\langle\vec{p}|, \quad |\vec{p}| \leq \Lambda,$$

$$Q = \int d^3q |\vec{q}\rangle\langle\vec{q}|, \quad |\vec{q}| > \Lambda, \tag{9}$$

where Λ is a momentum cutoff whose value will be specified later, and P (Q) is a projection operator onto low-(high-)momentum states. Based on the unitary-transformation operator given in Eq. (7), the effective Hamiltonian in the P space takes the form

$$PH'P = P(1+\omega^\dagger\omega)^{-1/2}(H + \omega^\dagger H + H\omega + \omega^\dagger H\omega) \times (1+\omega^\dagger\omega)^{-1/2}P. \tag{10}$$

This interaction is by its very construction Hermitian [6].

The requirement of decoupling the two spaces leads to the following nonlinear integral equation for the operator ω:

$$V(\vec{q},\vec{p}) - \int d^3p'\,\omega(\vec{q},\vec{p}')V(\vec{p}',\vec{p}) + \int d^3q' V(\vec{q},\vec{q}')\omega(\vec{q}',\vec{p}) - \int d^3p' d^3q' \omega(\vec{q},\vec{p}')V(\vec{p}',\vec{q}')\omega(\vec{q}',\vec{p}) = (E_p - E_q)\omega(\vec{q},\vec{p}), \tag{11}$$

where we have denoted by $\vec{p}$ ($\vec{q}$) a momentum of the P-space (Q-space) state and by E_p (E_q) the kinetic energy $E_p = p^2/(2M)$ [$E_q = q^2/(2M)$].

Alternatively, one can determine the operator ω from the following linear equation, as exhibited in Refs. [7,8]:

$$\omega(\vec{q},\vec{p}) = \frac{T(\vec{q},\vec{p},E_p)}{E_p - E_q} - \int d^3p' \frac{\omega(\vec{q},\vec{p}')T(\vec{p}',\vec{p},E_p)}{E_p - E_{p'} + i\epsilon}. \tag{12}$$

Here the integration over p' goes from 0 to Λ. Consequently, the dynamical input in this method is the T matrix $T(\vec{p}_1,\vec{p}_2,E_{p_2})$. Note that this is not a usual equation of the Lippmann-Schwinger (LS) type since the position of the pole E_p in the integration over p' is not fixed but moves with p. In solving the integral equation Eq. (12), the second argument p in ω varies, whereas the first one q is a parameter.

In this study we have used the linearized equation Eq. (12) to project out high-momentum components from the realistic potentials. Since the projection operators P and Q do not carry any angular dependence, the integral equation Eq. (12) can be solved for each partial wave independently. In the partial wave decomposed form it reads

$$\omega^{sj}_{ll'}(q,p) = \frac{T^{sj}_{ll'}(q,p,E_q)}{E_p - E_q} - \sum_{\tilde{l}} \int_0^\Lambda p'^2 dp' \frac{\omega^{sj}_{l\tilde{l}}(q,p')T^{sj}_{\tilde{l}l'}(p',p,E_p)}{E_p - E_{p'} + i\epsilon}, \tag{13}$$

where $V^{sj}_{ll'}(q,p) \equiv \langle lsj,q|V|l'sj,p\rangle$ and $\omega^{sj}_{ll'}(q,p) \equiv \langle lsj,q|\omega|l'sj,p\rangle$. In the uncoupled case l is conserved and equals j. In the coupled cases it takes the values $l = j \pm 1$.

Equation (12) and, consequently, also Eq. (13) have a so-called moving singularity, which makes it more difficult to handle than the LS equation. Indeed, one has to discretize both p and p' points in Eq. (13). This does not necessarily allow one to solve Eq. (13). The difference $E_p - E_{p'}$ can be exactly zero since p and p' belong now to the same set of quadrature points. Thus, one cannot calculate the principal value integral in the same manner as for the LS equation. To solve this equation we have used a method proposed by Glöckle *et al.* [9].

Another problem arising in solving the linear equation is caused by the fact that the driving term $T^{sj}_{ll'}(q,p,E_p)/(E_p - E_q)$ becomes very large when $p \equiv |\vec{p}|$ and $q \equiv |\vec{q}|$ go to Λ. Consequently, the equation becomes ill defined. To handle this problem we have regularized this equation by multiplying the original potential $V(\vec{k}',\vec{k})$ with some smooth functions $f(k')$ and $f(k)$ which are zero in a narrow neighborhood of the points $k' = \Lambda$ and $k = \Lambda$ [17]. The precise form of this regularization does, in fact, not matter [8].

Having determined the operator ω, one can calculate the effective Hamiltonian in the P space according to Eq. (10). To evaluate the operator $P(1+\omega^\dagger\omega)^{-1/2}$ entering this equation we first diagonalize the operator $(1+\omega^\dagger\omega)$ and then take the square root. Finally, the effective potential can be found by subtracting the (not transformed) kinetic energy term from the effective Hamiltonian.

B. Method 2

In Eq. (7), the unitary-transformation operator U has been given by the block form with respect to the projection operators P and Q. We notice here that the operator U can also be written in a more compact form as [18]

$$U = (1 + \omega - \omega^\dagger)(1 + \omega\omega^\dagger + \omega^\dagger\omega)^{-1/2}. \tag{14}$$

Using the above operator U, in general, the effective interaction $\tilde{V}$ for a many-nucleon system is defined through

$$\tilde{V} = U^{-1}(H_0 + V)U - H_0, \tag{15}$$

where H_0 is the kinetic energy of the constituent nucleons in the nuclear system, and V is the bare two-body interaction between the nucleons. Here we apply thus defined effective interaction to the two-nucleon problem. In that case H_0 becomes the relative kinetic energy of the two nucleons, and V the bare two-body interaction between the two nucleons. Then the LMNN interaction $V_{\text{low }k}$ of interest in the present work is given by

$$V_{\text{low }k} = P_{\text{low }k}\tilde{V}P_{\text{low }k}, \tag{16}$$

where $P_{\text{low }k}$ is the projection operator onto the low-momentum space for relative two-body states, and is the same as P in Eq. (9) of method 1. In order to obtain $V_{\text{low }k}$ in the form of the matrix elements using the plane-wave basis states, we shall present in the following a procedure for the numerical solution.

We first consider an eigenvalue equation for the relative motion of a two-nucleon system as

$$(H_0 + V)|\Psi_n\rangle = E_n|\Psi_n\rangle. \tag{17}$$

The above equation is written also in an integral form concerning relative momenta k and k' as

$$\int_0^\infty \langle k'|H_0 + V|k\rangle\langle k|\Psi_n\rangle k^2\, dk = E_n\langle k'|\Psi_n\rangle, \tag{18}$$

where

$$\int_0^\infty |k\rangle\langle k|k^2\, dk = 1. \tag{19}$$

We here make an approximation for Eq. (18) in a numerical integral form by introducing the adequate integral mesh points k_i and k_j and discretizing $k \to k_j$ and $k' \to k_i$ as

$$\sum_j \langle \overline{k_i}|H_0 + V|\overline{k_j}\rangle\langle \overline{k_j}|\Psi_n\rangle = E_n\langle \overline{k_i}|\Psi_n\rangle, \tag{20}$$

where $|\overline{k_i}\rangle$ and $|\overline{k_j}\rangle$ represent the plane-wave basis states in the matrices. Those grids are characterized by the mesh points k_i and k_j. Thus, $|\overline{k_i}\rangle$ is defined as

$$|\overline{k_i}\rangle = k_i\sqrt{W_i}|k_i\rangle \tag{21}$$

with the plane-wave states $|k_i\rangle$ and the weight factors for the numerical integral $\sqrt{W_i}$. They are normalized as

$$\langle \overline{k_i}|\overline{k_j}\rangle = \delta_{i,j}. \tag{22}$$

We note here that the eigenvectors $|\Psi_n\rangle$ in Eq. (17) can be expressed as $|\Psi_n\rangle = |\phi_n\rangle + \omega|\phi_n\rangle$ in terms of the operator ω and the P-space components $|\phi_n\rangle = P|\Psi_n\rangle$. Thus, the formal solution of ω is given by $\omega = \Sigma_n|\Psi_n\rangle\langle\tilde{\phi}_n|$ with the biorthogonal state $\langle\tilde{\phi}_n|$ of $|\phi_n\rangle$. In order to obtain the matrix elements of ω, we first solve Eq. (20) by diagonalizing the matrix elements, using the basis states $|\overline{k_i}\rangle$. Then, the matrix elements ω for the basis states $|\overline{k_p}\rangle$ in the P space and $|\overline{k_q}\rangle$ in the Q space are obtained as

$$\langle \overline{k_q}|\omega|\overline{k_p}\rangle = \sum_{n=1}^{d} \langle \overline{k_q}|Q|\Psi_n\rangle\langle\tilde{\phi}_n|P|\overline{k_p}\rangle, \tag{23}$$

where d is the number of the basis states (the integral points) in the P space. As shown in Eq. (9) the P space (low-momentum space) and Q space (high-momentum space) are defined with a cutoff momentum Λ as $0 < k \leqslant \Lambda$ and $\Lambda < k < \infty$, respectively. The bra states $\langle\tilde{\phi}_n|$ are obtained through matrix inversion as $[\langle\tilde{\phi}_n|\overline{k_p}\rangle] \equiv [\langle\overline{k_{p'}}|\phi_n\rangle]^{-1}$ and satisfy the relations $\Sigma_{k_p}\langle\tilde{\phi}_n|\overline{k_p}\rangle\langle\overline{k_p}|\phi_{n'}\rangle = \delta_{n,n'}$ and $\Sigma_n\langle\overline{k_{p'}}|\phi_n\rangle\langle\tilde{\phi}_n|\overline{k_p}\rangle = \delta_{k_{p'},k_p}$. It should be noted that the solution ω given in Eq. (23) is ambiguous in the sense that how to choose the set of eigenvectors $\{|\Psi_n\rangle, n = 1, 2, \ldots, d\}$ is not unique. We here select $\{|\Psi_n\rangle\}$ so that they have the largest P-space overlaps $O_n = \Sigma_{i=1}^{d} |\langle\Psi_n|P|\overline{k_i}\rangle|^2$ among all the eigenstates in Eq. (20). As we will show later, the numerical calculation shows that this selection of $\{|\Psi_n\rangle\}$ leads to the same solution obtained from method 1.

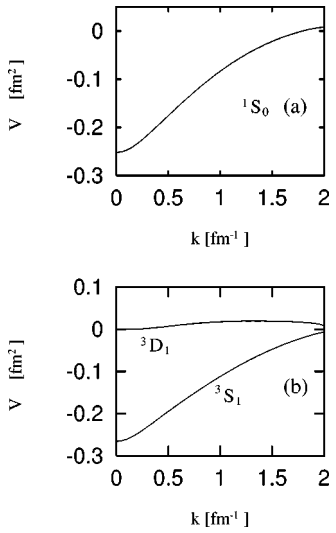

FIG. 1. Comparison of LMNN interactions from the CD-Bonn potential in the case of $\Lambda = 2.0$ fm^{-1}. The diagonal matrix elements $\langle k|V_{\text{low } k}|k\rangle$ for the $^1S_0$ (a) and $^3S_1$-$^3D_1$ (b) partial waves are shown. The lines depict method-1 (solid) and method-2 (long-dashed) results. Because they coincide very well, one cannot distinguish both lines by the eye.

In order to obtain the LMNN interaction (P-space effective interaction), we introduce the eigenvalue equation for $\omega^\dagger\omega$ in the P space as

$$\omega^\dagger\omega|\psi_\alpha\rangle = \mu_\alpha^2|\psi_\alpha\rangle. \tag{24}$$

Using the solutions to the above equation, the LMNN interaction of a Hermitian type is given by [18,19]

$$\langle\psi_\alpha|V_{\text{low}-k}|\psi_\beta\rangle = \frac{\sqrt{(1+\mu_\alpha^2)}\langle\psi_\alpha|R|\psi_\beta\rangle + \sqrt{(1+\mu_\beta^2)}\langle\psi_\alpha|R^\dagger|\psi_\beta\rangle}{\sqrt{(1+\mu_\alpha^2)} + \sqrt{(1+\mu_\beta^2)}}, \tag{25}$$

where

$$R = P(V + V\omega)P \tag{26}$$

is a low-momentum (effective) interaction of a non-Hermitian type. Finally, the matrix elements of the LMNN interaction using the plane-wave basis states $|k_i\rangle$ and $|k_j\rangle$ are obtained as

$$\langle k_i|V_{\text{low } k}|k_j\rangle = \frac{\sum_{\alpha,\beta}\langle\overline{k_i}|\psi_\alpha\rangle\langle\psi_\alpha|V_{\text{low } k}|\psi_\beta\rangle\langle\psi_\beta|\overline{k_j}\rangle}{k_i k_j\sqrt{W_i W_j}}. \tag{27}$$

By an interpolation technique the elements of the potential are prepared at arbitrary momenta in the P space.

III. RESULTS AND DISCUSSION

As mentioned in the previous section, we have two different methods based on the unitary transformation. Here we make the cross check using both methods.

In Fig. 1, the diagonal matrix elements of the LMNN interactions for the neutron-proton $^1S_0$ and $^3S_1$-$^3D_1$ channels

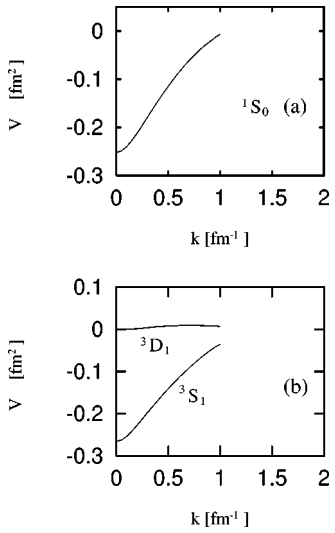

FIG. 2. Comparison of LMNN interactions for the off-diagonal elements $\langle k|V_{\mathrm{low}\ k}|2k\rangle$. Description is the same as in Fig. 1.

using the CD-Bonn potential [14] are shown in the case of $\Lambda=2.0$ fm^{-1}. In order to see the off-diagonal matrix elements, we also illustrate $\langle k|V_{\mathrm{low}\ k}|2k\rangle$ in Fig. 2. One can see that the results obtained by the two methods are almost the same for both the diagonal and nondiagonal matrix elements within 3–4 digits using typically 100 integral grid points. Having obtained the same results with high precision using two very different methods, we have confidence in our numerical results.

In Fig. 3, we show the same matrix elements of the original CD-Bonn potential and the LMNN interaction for the sake of comparison. The LMNN potential is very different from the original one. Nevertheless, the scattering phase shifts and the mixing parameter below $E_{\mathrm{lab}}=300$ MeV for

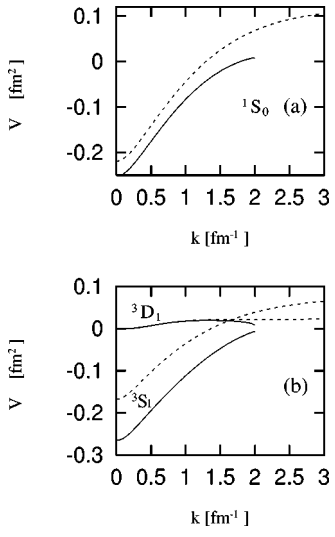

FIG. 3. Comparison of the LMNN interaction in the case of $\Lambda=2.0$ fm^{-1} and the original CD-Bonn potential for the $^1S_0$ (a) and $^3S_1$-$^3D_1$ (b) partial waves. The solid and dashed lines depict the diagonal matrix elements of the LMNN interaction and the original CD-Bonn potential, respectively.

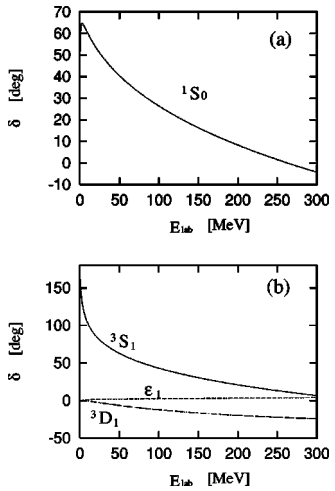

FIG. 4. Phase shifts for the $^1S_0$ (a) and the $^3S_1$-$^3D_1$ (b) channels below $E_{\mathrm{lab}}=300$ MeV. Because the lines from the LMNN interactions and the original CD-Bonn potential coincide, one cannot distinguish both lines by the eye. In (b) the upper, middle, and lower lines depict the $^3S_1$ phase shift, the mixing parameter ϵ_1, and the $^3D_1$ phase shift, respectively.

the $^1S_0$ and $^3S_1$-$^3D_1$ channels, as shown in Fig. 4, reproduce exactly the ones obtained from the original interaction. This has also been shown by Bogner *et al.* [1].

We now regard deuteron properties. In the light of the unitary transformation for the two-nucleon system, all the calculated deuteron binding energies for various values of Λ must reproduce the exact value using the original interaction. In Table I, calculated deuteron binding energies for various Λ using the CD-Bonn [14] and the Nijm-I [15] potentials are tabulated together with deuteron *D*-state probabilities. Note that the *D*-state probability is not observable [20]. The values in the last row are the exact values quoted from the original papers of the CD-Bonn and the Nijm-I potentials. Indeed, we see that the calculated binding energy for each Λ reproduces the exact value with high accuracy. However, as for the *D*-state probability, the difference between the results using the LMNN interaction and the exact value becomes larger as the value of Λ becomes smaller. If one uses the corresponding effective operator in calculating the *D*-state probability, i.e., the unitarily transformed projection operator onto the *D* state, then one would reproduce the original value for this quantity.

As shown in Table I, for the deuteron, the difference of the wave functions does not affect the binding energy since we perform the unitary transformation for the two-nucleon system. However, if we apply the LMNN interaction to the calculation of the ground-state energies of many-body systems, the situation will change. This is because the unitary transformation in the two-nucleon Hilbert space is not unitary any more in the Hilbert space of three and more nucleons. As a consequence, calculated binding energies will depend on Λ. In order to examine the Λ dependence, we perform the Faddeev and Yakubovsky calculations for few-nucleon systems. Recent precise calculations for few-nucleon systems are reviewed in Refs. [12,13].

TABLE I. Calculated binding energies E_b (MeV) and D-state probabilities P_D (%) of the deuteron for various values of Λ. The values in the last row are those quoted from the original papers of the CD-Bonn and the Nijm-I potentials.

	CD Bonn		Nijm I	
Λ (fm^{-1})	E_b (MeV)	P_D (%)	E_b (MeV)	P_D (%)
1.0	−2.224576	1.21	−2.224575	1.24
2.0	−2.224576	3.55	−2.224575	3.83
3.0	−2.224576	4.55	−2.224575	5.12
4.0	−2.224576	4.79	−2.224575	5.53
5.0	−2.224576	4.83	−2.224575	5.64
6.0	−2.224576	4.83	−2.224575	5.66
7.0	−2.224576	4.83	−2.224575	5.66
Quoted	−2.224575	4.83	−2.224575	5.66

Figure 5(a) exhibits the energy shift $\Delta E_b \equiv E_b(\Lambda) - E_b(\infty)$ from the ground-state energy $E_b(\infty)$ of ^{3}H as a function of Λ based on the CD-Bonn potential (solid) and the Nijm-I one (short-dashed) by a 34-channel Faddeev calculation. In the present study, only the neutron-proton interaction is used for all the channels for simplicity. The exact value $E_b(\infty)$ using the original potential CD Bonn (Nijm I) is −8.25 (−8.01) MeV. The long-dashed line depicts the Faddeev calculation using the original interaction, where the high-momentum components beyond Λ are simply truncated. The numerical stability is lost within the area $\Lambda \leqslant 1$ fm^{-1}. For the case of the CD-Bonn potential, we need $\Lambda > 8$ fm^{-1} to reach the exact value if the accuracy of 100 KeV is required for the case of simple truncation. This situation is greatly improved if we use the LMNN interaction. Even if we require the accuracy of 1 keV, we do not need the high-momentum components beyond $\Lambda \sim 8$ fm^{-1}. However, it should be noted that the results using the LMNN interaction for the values smaller than $\Lambda \sim 5$ fm^{-1} vary considerably, and there occurs the energy minimum around $\Lambda \sim 2$ fm^{-1}. The cutoff value, which produces the minimal value of ΔE_b, is close to the value proposed by Bogner *et al.* [1].

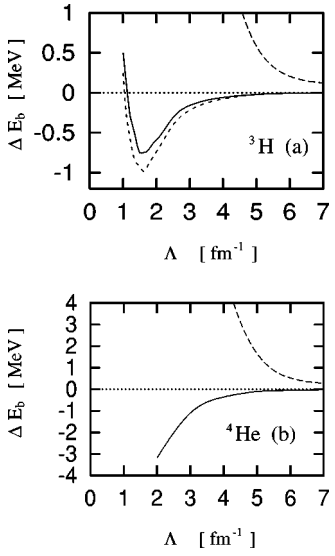

FIG. 5. Energy shifts ΔE_b of ^{3}H (a) and ^{4}He (b) as a function of Λ using the LMNN interaction (solid) from the CD-Bonn potential. The long-dashed lines in (a) and (b) are plotted for the case of simple momentum cutoff calculations. The short-dashed line in (a) depicts the result for the LMNN interaction from the Nijm-I potential.

We note here that the wave function using the LMNN interactions is very close to the true wave function if the Jacobi momentum set (p,q) of the three-nucleon system is smaller than the adopted value of Λ when we take $\Lambda \geqslant 4$ fm^{-1}.

By solving the Yakubovsky equations we can calculate the binding energy of ^{4}He. A similar tendency of the Λ dependence for ^{3}H can be also seen in the results for ^{4}He. In Fig. 5(b), the Λ dependence of the energy shift ΔE_b of the ground state of ^{4}He is illustrated. We demonstrate only the case of the CD-Bonn potential. Here we adopt the S-wave [(5+5)-channel] approximation and do not include the Coulomb force for simplicity [12]. We also use only the neutron-proton interaction for all the channels. The exact value $E_b(\infty)$ using the original CD-Bonn potential in the above-mentioned approximations is −27.74 MeV. The shape of the curve is similar to that for ^{3}H up to $\Lambda \sim 2$ fm^{-1}. The numerical instability already starts at $\Lambda \leqslant 2$ fm^{-1}. Here it seems to conserve the strong correlation relation [12] between $\Delta E_b(^4\mathrm{He})$ and $\Delta E_b(^3\mathrm{H})$ where the ratio is about 5 in the region $\Lambda \geqslant 3$ fm^{-1}.

In the case of ^{3}H, the energy shift ΔE_b is about 750 keV (1 MeV in the case of the Nijm-I potential) at the minimum point. On the other hand, the difference amounts to about 3 MeV for ^{4}He at $\Lambda = 2.0$ fm^{-1}. We remark that the minimal value of the cutoff Λ which leads only to small deviations of the ^{3}H and ^{4}He binding energies from their exact values is $\Lambda \sim 5$ fm^{-1}. As far as the excitation spectra of low-lying states from the ground state are concerned, the Λ dependence at $\Lambda \sim 2$ fm^{-1}may be weak and the LMNN interactions could be useful as has been shown in shell-model calculations [2].

It is noted that for LMNN interactions considered as a purely computational tool, which allows to get rid of the hard core and thus enables many-body calculations based on realistic NN potentials without using the G-matrix formalism, we recommend the value of the cutoff Λ of the order of (or

higher than) ~ 5 fm^{-1}. One then ensures that the binding energies do not deviate significantly from their exact values in the case of few-nucleon systems. Further reducing the values of the cutoff Λ leads to a significant suppression of the repulsive part of the interaction which becomes visible. In other words, if one would be able to extend the unitary transformation formalism to more-than-two-nucleon Hilbert space, additional three- and more-nucleon forces would be generated through eliminating the high-momentum components. These forces would probably be repulsive[1] (at least for the systems discussed in this work) and would restore the values of the few- and many-nucleon binding energies and other observables found in calculations with the original, not transformed, NN forces.

On the other hand, it is well known that all realistic NN interactions underbind light nuclei such as ^{3}H and ^{4}He. As a consequence, attractive three-nucleon forces are needed in order to reproduce the experimental numbers. It is, therefore, in principle possible that the repulsive effective many-nucleon forces generated through elimination of the high-momentum components compensate, to some extent, these missing attractive forces, minimizing the total effect of the many-nucleon interactions. If this oversimplified picture reflects the real situation, one would observe a better agreement with the data for calculations based on the LMNN interactions as compared to the ones based on realistic NN forces. Although the results provided by existing many-body calculations as well as by the present few-body calculations with LMNN interactions appear to be quite promising in this respect,[2] and thus might speak in favor of the above-mentioned tendency of counteracting these two kinds of many-body forces, to the best of our knowledge, no general proof of the validity of the above-mentioned picture has yet been offered.

In addition, one should keep in mind that only restricted information about many-nucleon forces, which might have very complicated spin and spatial structures, is provided by the discrete spectrum. A much better testing ground is served by a variety of scattering observables. It would therefore be interesting to test the LMNN forces in the three-nucleon continuum. Further, we notice that the choice of the LMNN is not unique. Preserving the half-on-shell T matrix does not seem to be advantageous in any respect, at least from the conceptual point of view. If this condition is not required, infinitely many equivalent LMNN's can be constructed by means of a unitary transformation in the low-momentum subspace. Even if many-body effects appear to be small for one particular choice of LMNN forces, this might not be the case for a different choice.

Certainly, the LMNN interactions with small values of the cutoff, $\Lambda \sim 2$ fm^{-1}, are not disadvantageous compared to existing realistic NN forces regarding the nuclear structure calculations, if these realistic forces are viewed as purely phenomenological parametrizations with no physical content, underlying the only requirement of reproducing properly the low-energy NN data.

[1]Speaking more precisely, they would have an effect of decreasing the binding energies.

[2]Notice, however, the counterexample's mentioned before.

IV. SUMMARY

The LMNN interactions have been derived through a unitary-transformation theory from realistic nucleon-nucleon interactions such as the CD-Bonn and the Nijm-I potentials. We have constructed the LMNN interactions by two different methods which are based on the common unitary transformation. In order to have a cross check of both computer codes, we have shown that the LMNN interactions obtained by the two methods yield the same results. The LMNN interaction reproduces the low-energy observables in the two-nucleon system with high precision, which has been confirmed by the calculation of the deuteron binding energy and the phase shifts.

The LMNN interaction has been successfully applied to the Faddeev-Yakubovsky calculations for three- and four-nucleon systems. The calculated binding energies of the few-nucleon systems begin to deviate from the values calculated using the original NN potentials for Λ smaller than ~ 5 fm^{-1}, whereas the results obtained by simply cutting off the high momentum components without performing a unitary transformation deviate considerably even at much higher values of Λ. In an appropriately truncated (i.e., with $\Lambda \geqslant 5$ fm^{-1}) low-momentum space the LMNN interaction reproduces the exact values of the binding energies, at least, for the few-nucleon systems.

However, we should keep in mind that the calculations of ground-state energies using the LMNN interaction for $\Lambda \sim 2$ fm^{-1} yield considerably more attractive results than the exact values. We note that as shown in Fig. 1 of Kuckei *et al.*'s work [4] one needs more than 4.0 fm^{-1} as the cutoff value Λ in order to reproduce at least qualitatively the saturation property of nuclear matter. Thus, the application of the LMNN interaction to structure calculations should be done with care, though the LMNN interaction for $\Lambda \sim 2$ fm^{-1} may be suitable for the calculation of the excitation spectra of low-lying states as has been shown in the shell-model calculations [2].

ACKNOWLEDGMENTS

One of the authors (S.F.) acknowledges the Special Postdoctoral Researchers Program of RIKEN. This work was supported by a Grant-in-Aid for Scientific Research (C) (Grant No. 15540280) from the Japan Society for the Promotion of Science, a Grant-in-Aid for Specially Promoted Research (Grant No. 13002001) from the Ministry of Education, Culture, Sports, Science and Technology in Japan, and the U.S. Department of Energy Contract No. DE-AC05-84ER40150 under which the Southeastern Universities Research Association (SURA) operates the Thomas Jefferson Accelerator Facility. The numerical calculations were performed on a Hitachi SR8000 (Leibnitz-Rechenzentrum für die Münchener Hochschule) in Germany.

[1] S. K. Bogner, T. T. S. Kuo, and A. Schwenk, Phys. Rep. **386**, 1 (2003).
[2] Scott Bogner, T. T. S. Kuo, L. Coraggio, A. Covello, and N. Itaco, Phys. Rev. C **65**, 051301(R) (2002).
[3] L. Coraggio, N. Itaco, A. Covello, A. Gargano, and T. T. S. Kuo, Phys. Rev. C **68**, 034320 (2003).
[4] J. Kuckei, F. Montani, H. Müther, and A. Sedrakian, Nucl. Phys. **A723**, 32 (2003).
[5] For example, W. Glöckle, *The Quantum Mechanical Few-Body Problem* (Springer-Verlag, Berlin, 1983).
[6] S. Ōkubo, Prog. Theor. Phys. **12**, 603 (1954).
[7] E. Epelbaoum, W. Glöckle, and Ulf-G. Meißner, Phys. Lett. B **439**, 1 (1998).
[8] E. Epelbaoum, W. Glöckle, A. Krüger, and Ulf-G. Meißner, Nucl. Phys. **A645**, 413 (1999).
[9] W. Glöckle, G. Hasberg, and A. R. Neghabian, Z. Phys. A **305**, 217 (1982).
[10] K. Suzuki and R. Okamoto, Prog. Theor. Phys. **92**, 1045 (1994).
[11] S. Fujii, R. Okamoto, and K. Suzuki, Phys. Rev. C **69**, 034328 (2004).
[12] H. Kamada *et al.*, Phys. Rev. C **64**, 044001 (2001); H. Kamada and W. Glöckle, Nucl. Phys. **A548**, 205 (1992).
[13] A. Nogga, A. Kievsky, H. Kamada, W. Glöckle, L. E. Marcucci, S. Rosati, and M. Viviani, Phys. Rev. C **67**, 034004 (2003).
[14] R. Machleidt, F. Sammarruca, and Y. Song, Phys. Rev. C **53**, R1483 (1996).
[15] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C **49**, 2950 (1994).
[16] Jason D. Holt, T. T. S. Kuo, and G. E. Brown, Phys. Rev. C **69**, 034329 (2004).
[17] E. Epelbaum, Ph.D. thesis, Ruhr-Universität Bochum, 2000.
[18] K. Suzuki, Prog. Theor. Phys. **68**, 246 (1982).
[19] T. T. S. Kuo, P. J. Ellis, Jifa Hao, Zibang Li, K. Suzuki, R. Okamoto, and K. Kumagai, Nucl. Phys. **A560**, 621 (1993).
[20] J. L. Friar, Phys. Rev. C **20**, 325 (1979).