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Lifetime Estimation of Defective Products from the Imaginal Mixture of Defective and Non-defective Products: The Trunsored Data Model

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Abstract. In solving the lifetime estimation problem of defective products using samples of size N from a mixture of defective and non-defective products, a new method of estimating the parameters of the underlying distribution function is proposed. We suppose that the ratio of the defective products to the non-defective products is unknown. The proposed method is to use an imaginal mixture model in which the non-defective products will never fail by the prescribed time T. If the non-defective products are dominant in the mixture, we can obtain the maximum likelihood estimates by dealing with the observed samples as truncated data with a conditional likelihood. If the defective products are dominant, we can no longer use the truncated data approach because the estimated sample size could end up being larger than N in which case the non-defective products would be empty. The imaginal mixture model, however, can estimate the parameters in either case. In addition, this model can test whether the non-defective products are empty or not because the likelihood functions for both the cases are of the same kind, whereas we cannot use the likelihood ratio test directly by using the likelihoods from the truncated model and the censored model due to the different kinds of likelihood. Thus, we call this versatile model which can be used for both the truncated and censored data models the trunsored data model. If it is not rejected that non-defective products are empty, we can regard the sampled data as censored, and can obtain smaller confidence intervals of the estimates of the parameters than those obtained by the trunsored model. After the introduction of this new mixture model, we apply it to the actual field data, and show how the proposed method works.

KEY WORDS: Truncated data; Censored data; Trunsored data; Imaginal mixture model; Durable population; Fragile population; Weibull distribution; Likelihood ratio test.

1. INTRODUCTION

General statistics textbooks delineate two kinds of incomplete data of importance in lifetime estimation: one is censored data and the other is truncated data. This paper introduces a third one, what we call the "trunsored" data.

In the ordinary lifetime estimation problem, we assume that all the samples will fail in time. However, if the mean lifetime of a population is too long (or the observation period is too short) to observe failures, we may not observe the failed samples by the prescribed time T at all. We define these kinds of samples as *non-defective* products, whereas we define samples where some fail and others may not fail by T as *defective* products. We define a population which consists of non-defective products as a *durable* population, while we define a population which consists of defective products as a *fragile* population. The problem we discuss here is how to estimate the lifetime in a mixture of these two populations. We usually assume only one fragile homogeneous population in the lifetime estimation problem, and we can then estimate the parameters which determine the characteristics of the population in parametric models by regarding the data as censored or as truncated when data are incomplete. A finite mixture model which consists of r cumulative distribution functions (cdfs), F_1, F_2, $\ldots$, F_r, can also be dealt with similarly to a homogeneous population model if each cdf corresponds to a fragile population. However, we do not always deal with such a homogeneous fragile population.

Hirose (2000) shows some claim data of failures observed from June 30th, 1998 to February 26th, 1999. The products are electronic boards used in communications. The data consist of numbers of claims over specific durations, thus the data are actually grouped. If we regard the data as continuous with failures occurring at the midpoint of each duration for simplicity, then the survival probability function of Kaplan-Meier illustrates that the failures will cease. This suggests that the homogeneity of the population is questionable and that there may be two distinct populations: a fragile population, and a durable population. By assuming that the failures are sampled from a (homogeneous) fragile population and that the underlying cdf is a (two-parameter) Weibull type, the estimated cdf by the maximum likelihood method (MLE) can be computed; we can suppose the existence of non-defective products because of the large discrepancy between the two functions. We now proceed to consider the lifetime estimation problem under such a condition.

Let us assume that there are two distinctive products: defective and non-defective products, and that they are mixed together. We draw samples of size N from this mixture, and the ratio of sample sizes between the two products is unknown; we also allow the possibility that the non-defective products are empty. We assume that n failures are observed by time T. In such a case, we first want to estimate the sample size M and the lifetime of the defective products by using the field data in order to make an appropriate production decision. Here, we define $p = M/N$.

If the non-defective products are dominant, we need not care about the information of the sample size N. Therefore, a truncated data approach would suit the situation in principle because the primary objective is to know the sample size of M. In the case, the lifetime estimation for the defective products can be done by using the conditional likelihood if a parametric model of the underlying probability distribution is assumed. This kind of problem was dealt with by Bain and Weeks (1964), Blumenthal and Marcus (1975), Sanathanan (1977), Nath (1975), Hegde and Dahiya (1989), and Kalbfleish and Lawless (1992); some serious estimating problems were discussed by Deemer and Votaw (1955), and Mittal and Dahiya (1989) in exponential and Weibull distributions. However, the problem outlined above is different from those referred to in the literature. In our problem, the total sample size of defective and non-defective products is known to be N in advance and there may be non-defective products or may not be any non-defective products at all, while the problems previously treated in the literature do not take into consideration the existence of non-defective products, where the information of N is redundant. As far as I know, this defective and non-defective mixture problem has not been dealt with up to now (e.g., in Johnson, Kotz and Balakrishnan (1994), Meeker and Escobar (1998), Wallace, Blischke and Murthy (2000), and Klein and Moeschberger (1997)). This kind of mixture model and the truncated model allow the boundary of defective products move, but the sample size of the defective products is limited to N at most in the mixture model, while the truncated model does not require that constraint. In other words, this mixture model may be viewed as the truncated model with a total sample size constraint. If the boundary is strictly fixed, we are now solving the censored data problem.

If the defective products are estimated to be dominant, an investigation into whether any non-defective products exist or not is then useful. When the existence of the non-defective products is rejected, we may obtain the estimates of the parameters by regarding the data as censored because the confidence intervals of the estimates with the censored model are smaller those with the mixture model (see section 4); obviously the truncated model provides far worse confidence intervals of the estimates (Kalbfleisch and Lawless (1988)).

In the next section, we introduce the *imaginal* mixture model of the defective and non-defective products, and show the maximum likelihood estimates (MLE) of the parameters obtained from this imaginal mixture model are almost the same to those obtained from the truncated model if the non-defective products are dominant. On the other hand, the MLE obtained from the imaginal mixture model are the combination of those obtained from the censored and the truncated models if the defective products are dominant. Section 3 deals with the confidence intervals of the estimates obtained by the imaginal mixture model, and section 4 deals with the hypothesis test whether the non-defective products are empty or not by using the likelihood ratio test, and use the Bartlett correction method. In section 5, we apply the proposed model to the field data shown above. Some concluding remarks are shown in section 6.

2. IMAGINAL MIXTURE MODEL

Assume that the cdfs for fragile and durable populations are $F(t;\theta)$ and $G(t;\phi)$, respectively. We also assume that the corresponding probability density functions (pdfs) are smooth. A cdf and a pdf which correspond to the mixture population of fragile and durable populations are

$$\begin{aligned} H(t;\psi) =& pF(t;\theta) + (1-p)G(t;\phi), \\ h(t;\psi) =& pf(t;\theta) + (1-p)g(t;\phi), \\ & (t \geq 0), \end{aligned} \tag{1}$$

respectively, under the restriction, $0 < p \leq 1$, in general. If the observation is finished by the prescribed time T, and failure time, $t_1, t_2, \ldots, t_n (\leq T)$, is observed, then we can express the likelihood function for the (continuous) mixture model in the form,

$$L_m(\psi) = \{1 - H(T;\psi)\}^{N-n} \cdot \prod_{i=1}^{n} h(t_i;\psi), \tag{2}$$

and the restriction for p becomes,

$$n/N \leq p \leq 1. \tag{3}$$

We define $L_m(\theta, p)$ as the likelihood on the *imaginal* mixture model. If we know a priori that $p = 1$, we have the likelihood for the censored data model,

$$L_c(\theta) = \{1 - F(T;\theta)\}^{N-n} \cdot \prod_{i=1}^{n} f(t_i;\theta). \tag{5}$$

On the other hand, the conditional likelihood for truncated data can be expressed (Kalbfleish and Sprott (1970)) by

$$L_t(\theta) = \prod_{i=1}^{n} f(t_i;\theta)/F(T;\theta). \tag{6}$$

Then, we have an intriguing property regarding the solutions of the likelihood equations corresponding to (2), (5), and (6).

THEOREM *The maximum likelihood estimate $\hat{\theta}$ corresponding to the imaginal mixture model (2) is the same to that corresponding to the truncated model (6) if $\hat{p} < 1$, and it is the same to that to the censored model (5) if $\hat{p} = 1$.*

EXAMPLE 1 By using the simulated data: .41, 1.77, .68, .48, .38, .34, .72, 2.96, .63, .54, .31, 1.24, .04, 3.14, 1.92, 1.10, .08, 1.74, generated from an exponential model, $F(t;\eta) = 1 - e^{-t/\eta}$, with $\eta = 1, N = 20, p = .9, T = 2$, we obtain the MLE, $\hat{\eta}_{im} = 1.274, \hat{p} = 1$ ($\log L_{im} = -19.871$); $\hat{\eta}_{im} = 1.274$ is the same MLE to that obtained from the censored model. The maximum value of the likelihood function is obtained at the boundary $p = 1$. However, if we do not impose the constraint $n/N \leq p \leq 1$, then we have the local maximum log-likelihood point, $(\hat{\eta}, \hat{p}) = (1.427, 1.061)$ with $\log L = -19.850$; $\hat{\eta}_t = 1.427$ can be obtained by solving the likelihood equation corresponding to (6). The solution of the likelihood equation in the imaginal mixture model can be obtained either by the solution in the truncated model or by the solution in the censored model, and therefore we may call the imaginal mixture model the trunsored model.

Because of this, we cannot simply use the asymptotic normal theory to obtain the confidence intervals of the estimates, in particular, when $\hat{p}$ is located in the vicinity of boundaries of p. Moreover, testing the hypothesis whether the durable population is empty or not seems to be difficult. In the following two sections, we consider these matters.

3. CONFIDENCE INTERVALS OF THE ESTIMATES

If the regularity conditions hold in the truncated and censored model, approximate confidence intervals of the estimates in both the models can be obtained by using the observed Fisher information matrix. In the imaginal mixture model, a similar method for obtaining the confidence interval of the estimate seems not so easy. If $\hat{p} < 1$ is obtained, the solution is the same as that corresponding to that from the truncated model; thus the dispersion of the MLE for (2) is similar to that for (6). If $\hat{p} = 1$ is obtained, the MLE is the same as that corresponding that from the censored model; thus the dispersion of the MLE for (2) is similar to that for (5). Consequently, we know that the confidence interval in the imaginal mixture model is a combination of the confidence intervals in the truncated and censored models, but how we combine them analytically seems to be difficult. We first do some simulation here.

EXAMPLE 2 The simulation results which will be presented below show what we have mentioned above. If the non-defective products are dominant in the mixture, we can obtain the maximum likelihood estimates by dealing with the observed samples as truncated data with a conditional likelihood. If the defective products are dominant, we can no longer use the truncated data approach because the estimated sample size may be larger than N (then non-defective products may be empty).

The simulation condition is, $F(t;\eta) = 1 - e^{-t/\eta}$, with $\eta = 1$, $N = 100$, $p = 1, .95, \cdots, .6$, $T = 2$, and the number of trials is 1000 for each case. The variances of the estimates in the imaginal mixture model are, in general, smaller than those in the truncated model and larger than those in the censored model. The smaller the value of p, the the larger the bias in the censored model, because of the discrepancy from the true model, although the variance is still smaller than that in the truncated model. When p is small enough, the estimates in the imaginal mixture model is very similar to those in the truncated model, e.g., $p = .6$ in this simulation result. The 90% and 95% confidence intervals in these models are the same.

We do simulation in order to more easily understand what happens in our estimation procedure; artificial points of $(\hat{p}, \hat{\eta})$ when $p = 1, .9, .6$, where $\hat{\eta}$ denotes the estimate in the p constrained imaginal mixture model and $\hat{p}$ denotes the estimate in the p unconstrained imaginal mixture model. The points, under the condition $\hat{p} < 1$, express the truncated model solutions in which $\hat{\eta}$ and $\hat{p}$ are strongly correlated, therefore $\hat{\eta}$, under the condition $\hat{p} < 1$, may be biased, which explain the bias of $\hat{\eta}$, especially when $p = 1$. A bunching of points which looks like a chain may reminds us that the random number generator did not appropriately work. This suspicion will be removed in noticing the points when $p = 1$ in the figure; they consist of the points such that the number of observed failures is just 88. A gap between the two chains is due to the discreteness of the number of observed failures. When $p = .9$, the estimates $\hat{\eta}$ are still the combination of the truncated and censored model solutions. However, we no longer care about the censored model solution when $p = .6$.

As a consequence of the simulation results, we can approximately classify the patterns of the confidence intervals of the estimates into three categories: the censored model confidence intervals (pattern A), the truncated model confidence intervals (pattern B), and the mixture of these two confidence intervals (pattern C). Pattern A should be done after we confirm that the data can be dealt with as the censored data by testing the hypothesis $p = 1$; how we test it is shown in the next section. For both patterns A and B, we may construct the confidence intervals of the estimates based on the observed Fisher information matrix or the likelihood ratio statistics. For pattern C, however, we have to do the Monte Carlo simulation, the bootstrap (Efron (1979, 1982)), in general. We will not get involved too deeply in the problem here.

4. HYPOTHESIS TEST WHETHER THE DEFECTIVE PRODUCTS ARE EMPTY

As mentioned before, we can estimate the parameters of the imaginal mixture model by obtaining the MLE in the truncated model as long as the non-defective products are dominant. The information of N is void. However, it is very important to determine whether the non-defective products are empty or not when defective products are dominant because the confidence interval of the MLE in the censored model is still smaller than those in the imaginal mixture model. In addition, the estimates of the parameters can be biased if the non-defective products are not empty. Unlike the situation where the likelihoods in the truncated model and the censored model are different from each other, and where we cannot apply the likelihood ratio test to these models directly, we can use the likelihood ratio test on the imaginal mixture model and censored model because of the same kinds of likelihoods.

We now consider the properties of the following two functions (7) and (8):

$$\Lambda_{im} = 2\{\sup_{n/N \le p \le 1, \theta} \log L_{im}(\theta, p) - \sup_{\theta} \log L_{im}(\theta, p_0)\}, \tag{7}$$

$$\Lambda_t = 2\{\sup_{p>0,\theta} \log L_{im}(\theta, p) - \sup_{\theta} \log L_{im}(\theta, p_0)\}. \tag{8}$$

A chi-square approximation to (7) is questionable if p_0 is close to 1 because the maximizer of the log-likelihood function may be located on the boundary $p = 1$, where the log-likelihood function is no longer quadratically approximated. However, (8) can be approximated to chi-square distribution function for large samples because the maximizer of the log-likelihood function is located in the interior of the parameter space as long as we do not impose a constraint $n/N \le p \le 1$ (we have assumed that the regularity conditions hold to the fragile population). For moderate sample sizes, a chi-square approximation to (8) may become inaccurate because a quadratic approximation to the log-likelihood function that leads to the chi-square approximation sometimes fails; thus, a Bartlett-adjusted version Λ_{tB} may be useful; Bartlett (1937) noted that the χ^2_d distribution is a far better fit to the distribution of $d\Lambda/E(\Lambda)$ than to that of the likelihood ratio statistic Λ, where d is the degree of freedom.

The means, variances and quantiles of the sampling distributions of (7), (8) and its Bartlett-adjusted version based on 1000 simulations from an exponential model with $\eta = 1$, $p = 1$, and $N = 1000, 100$ are computed; Λ_{im} is heavily skewed and shows inaccurate approximation to the χ^2_1 distribution even for $N = 1000$ because almost half of the estimates are falling on the boundary $p = 1$, but Λ_t and its Bartlett-adjusted version Λ_{tB} show rather accurate approximation to it for both the cases $N = 1000$ and $N = 100$. Therefore, we can use (8) for a hypothesis test in which null hypothesis is $p = 1$ (i.e., the censored model). However, significance levels should be dealt with as a one-sided test.

Similarly, the following function,

$$l_t = 2\{\sup_{p>0,\theta} \log L_{im}(\theta, p) - \log L_{im}(\theta_0, p_0)\}, \tag{9}$$

can be approximated to chi-square distribution function with $d + 1$ degree of freedom, if the number of unknown parameters of θ equals to d. The means, variances and quantiles of the sampling distributions of (9) and its Bartlett-adjusted version based on 1000 simulations from an exponential model with $\eta = 1$, $p = 1$, and $N = 1000, 100$ are given.

EXAMPLE 3 In the case of example 1, $\hat{p} > 1$ is obtained, and therefore the null hypothesis $p = 1$ is not rejected, if the constraint $n/N \le p \le 1$ is not imposed. The approximate 95% confidence interval of $\hat{\eta}$ is computed to be $.65 \le \eta \le 1.90$ by using the observed Fisher information in the censored model because we may regard the problem as pattern A.

On the other hand, if we use the simulated data which are shown in Appendix 2, generated from an exponential model, $F(t; \eta) = 1 - e^{-t/\eta}$, with $\eta = 1, N = 400, p = .5, T = 2$, then we obtain $\hat{\eta}_c = 3.45$, $\log L_c = -382.86$, $\hat{\eta}_{im} = 1.43$, $\hat{p}_{im} = .567$, $\log L_{im} = -378.21$. The value $\Lambda_t = 9.29$ is clearly larger than $\chi^2_{1,.90} = 2.71$, and this means that the null hypothesis $p = 1$ is rejected with significance level .05. The number of defective products is estimated to be 227 ($\hat{p}_{im} \times N = .567 \times 400 = 226.8$). In this case, the 95% confidence region for (p, η) and interval for η can be obtained by using the likelihood ratio statistics such that:

$$\{(\eta, p) : \sup_{\eta>0, p>0} \log L_{im}(\eta, p) \ge \log L_{im}(\hat{\eta}_{im}, \hat{p}_{im}) - \frac{1}{2}\chi^2_{2,.95}\}, \tag{10}$$

$$\{\eta : \sup_{\eta>0} \log L_{im}(\eta, \hat{p}_{im}) \ge \log L_{im}(\hat{\eta}_{im}, \hat{p}_{im}) - \frac{1}{2}\chi^2_{1,.95}\}. \tag{11}$$

The confidence region by (10) is computed, and the confidence interval by (11) is $1.16 \le \eta \le 1.76$. This is the case of pattern B.

5. APPLICATION TO THE ELECTRONIC BOARD LIFETIME

The main objective in this section is to obtain the estimate of the number of defective products and its confidence interval for the electronic boards which were introduced previously. Although the data seems complex to some extent, we can now estimate the number of defective products even for this case based on the theory described in the previous sections.

Since 15 lots were shipped one after another on different dates as shown in Hirose (2000), each lot j has its own observing end-time T_j. By extending the log-likelihood function, which is used for grouped data for one lot, the log-likelihood for 15 lots can be expressed in the form:

$$\log L_m = \sum_{j=1}^{15} \Big((N_j - n_j) \log\{1 - H(T_j)\} + \sum_{i=1}^{7} n_{j,i} \log[\{H(t_{j,i+1}) - H(t_{j,i})\}] \Big), \qquad (12)$$

where, $t_{j,i}$ $(i = 1, \cdots, 8)$ denotes ith inspection time of lot j; N_j and $n_j = \sum_{i=1}^{7} n_{j,i}$ ($n_{j,i}$: the number of failures in duration $t_{j,i} < t \leq t_{j,i+1}$ for lot j) are the number of shipped products and the number of failed products of lot j. We assume here a two-parameter Weibull model for the fragile population,

$$F(t; \eta, \beta) = 1 - \exp\{-(t/\eta)^{\beta}\}, \quad (t \geq 0, \eta > 0, \beta > 0). \qquad (13)$$

By maximizing (12), the MLE of the parameters, $(\hat{\eta}, \hat{\beta}, \hat{p}) = (286, 3.10, .0491)$, are obtained with the maximum value of log-likelihood, $\log L_{im} = -769.73$. The number of defective products, M, is estimated to be 147. Thus, the defective population seems not to be dominant. Actually, the maximum value of $\log L_c$ for the censored model, -785.04 ($(\hat{\eta}, \hat{\beta}) = (2094, 1.84)$), rejects the null hypothesis $p = 1$ with significance level 5%. The 95% confidence interval for p obtained by using the likelihood ratio statistics based on (8) which is explained previously is computed to be $.0413 \leq p \leq .0577$, and its corresponding confidence interval for the number of defective products is $124 \leq M \leq 173$. Although this lower limit violates the feasible region of $M \geq 133$, this is not so informative and therefore we simply reset these confidence intervals to $.0444 \leq p \leq .0577$ and $133 \leq M \leq 173$, respectively. The 95% confidence region based on (9) is computed, where the upper and lower confidence limits are $.0444 \leq p \leq .0645$, and its corresponding confidence interval for the number of defective products is $124 \leq M \leq 193$. This is the case of pattern B.

In July, 2000, the number of failure data from March, 1999 to July, 2000 is newly obtained. It is 42, and the total number of failures becomes to be 175.

6. SUMMARY AND CONCLUDING REMARKS

In lifetime estimation, the finite mixture models which consist of fragile populations have been dealt with extensively. However, a mixture model which consists of both a fragile and a durable populations had not yet been noticed. We have introduced this new kind of mixture problem originating with the field data on electronic boards and have shown how to estimate the lifetime of them. When the number of sample size is known in advance and the truncation time is specified, we deal with the data as censored; when the sample size is unknown, we deal with the data as truncated. The problem we have dealt with here is the case where the total sample size is known but the number of defective products is unknown; in other words, we deal with the data as truncated with a limited number of samples. The estimates of the parameters in this case can be obtained by solving the censored data model in some cases and by solving the truncated data model in other cases, and therefore we call this versatile model which can be used for both the truncated and censored data models the trunsored data model.

We have shown how we deal with the case where the estimated number of defective products is close to the total number of samples by using the imaginal mixture likelihood function. We could not use the likelihoods of the censored and truncated data in the usual likelihood ratio test, but the imaginal mixture likelihood covers both the cases and the likelihood ratio statistics becomes useful.

The trunsored models may be useful not only when the durable population is included, but also when failures of some population are hidden.

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