

# Supplement: efficient weak second-order stochastic Runge-Kutta methods for non-commutative Stratonovich stochastic differential equations

Yoshio Komori<sup>a</sup>, Kevin Burrage<sup>b</sup>

<sup>a</sup>*Department of Systems Design and Informatics, Kyushu Institute of Technology, Iizuka 820-8502, Japan*

<sup>b</sup>*Oxford University Computing Laboratory, Wolfson Building, Parks Road, Oxford, UK*

---

## Abstract

This paper gives a modification of a class of stochastic Runge-Kutta methods proposed in a paper by Komori (2007). The slight modification can reduce the computational costs of the methods significantly.

*Keywords:* Order conditions, Multiplicative noise, Weak approximation  
*2008 MSC:* 60H10, 65L05, 65L06

---

## 1. Introduction

Runge-Kutta type methods for stochastic differential equations (SDEs) have been recently developed by many researchers [1, 2]. As an example of such methods, Komori [3] derived a stochastic Runge-Kutta (SRK) scheme with weak order 2 for non-commutative Stratonovich SDEs from a framework of SRK methods. Compared with other previous schemes, the scheme had the advantage that it can reduce the number of random variables that need to be simulated. Rößler [4], however, has pointed out that for this scheme the computational costs linearly depend on the dimension of the Wiener process for each diffusion coefficient, and has proposed new schemes without this drawback. But this requires 55 order conditions to be solved in order to construct weak second-order methods.

---

*Email addresses:* komori@ces.kyutech.ac.jp (Yoshio Komori), kevin.burrage@comlab.ox.ac.uk, kevin.burrage@gmail.com (Kevin Burrage)

*Preprint submitted to Journal of Computational and Applied Mathematics April 15, 2011*

*This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).*

In the present paper, we show that the drawback can be also removed in Komori's framework of SRK methods and only 38 order conditions need to be solved. The paper is organized as follows. In Section 2, we will briefly introduce this new class of SRK methods and the expression of their order conditions with rooted trees. In Section 3 we will concretely seek the order conditions under a modified setting on parameters and random variables. Lastly, we will give a brief discussion.

## 2. Preliminary

As preparation for the following sections, we give a brief introduction to a framework of our SRK methods and expressions for the order conditions in order to attain weak order two. Consider a  $d$ -dimensional Stratonovich stochastic differential equation

$$d\mathbf{y}(\tau) = \mathbf{g}_0(\mathbf{y}(\tau))d\tau + \sum_{j=1}^m \mathbf{g}_j(\mathbf{y}(\tau)) \circ dW_j(\tau), \quad 0 \leq \tau \leq T_{end}, \quad \mathbf{y}(0) = \mathbf{x}_0,$$

where  $W_j(\tau)$  is a scalar Wiener process. Let  $\tau_n$  be an equidistant grid point  $nh$  ( $n = 0, 1, \dots, M$ ) with step size  $h \stackrel{\text{def}}{=} T_{end}/M < 1$  ( $M$  is a natural number) and  $\mathbf{y}_n$  a discrete approximation to the solution  $\mathbf{y}(\tau_n)$ . In addition, suppose that the initial approximate random variable  $\mathbf{y}_0$  has the same probability law with all moments finite as that of  $\mathbf{x}_0$ , and define weak order in a usual way [2, 3]. As numerical methods for weak approximations, our SRK methods are given by

$$\begin{aligned} \mathbf{y}_{n+1} &= \mathbf{y}_n + \sum_{i=1}^s \sum_{j_a, j_b=0}^m c_i^{(j_a, j_b)} \mathbf{Y}_i^{(j_a, j_b)}, \\ \mathbf{Y}_{i_a}^{(j_a, j_b)} &= \tilde{\eta}_{i_a}^{(j_a, j_b)} \mathbf{g}_{j_b}(\mathbf{y}_n + \sum_{i_b=1}^s \sum_{j_c, j_d=0}^m \alpha_{i_a i_b}^{(j_a, j_b, j_c, j_d)} \mathbf{Y}_{i_b}^{(j_c, j_d)}) \end{aligned} \quad (1)$$

( $1 \leq i_a \leq s$ ,  $0 \leq j_a, j_b \leq m$ ), where  $c_i^{(j_a, j_b)}$  and  $\alpha_{i_a i_b}^{(j_a, j_b, j_c, j_d)}$  are constant parameters and where each  $\tilde{\eta}_{i_a}^{(j_a, j_b)}$  is a random variable independent of  $\mathbf{y}_n$  and its  $2k$ th moment is supposed to be equal to  $K_1 h^{2k}$  if  $j_b = 0$ , or  $K_2 h^k$  otherwise for constants  $K_1, K_2$  and  $k = 1, 2, \dots$

We can express the weak order conditions by multi-colored rooted trees (MRTs), whose definition is given in [3] and whose totality is denoted by  $T$ .

The weak order conditions are given as follows. Let  $\rho(t)$  be the number of vertices of  $t \in T$  and  $r(t)$  the number of vertices of  $t$  with the color 0, and suppose that any component of  $\mathbf{g}_j$  is sufficiently smooth and the regularity of the time discrete approximation is satisfied. If the following conditions are satisfied, the time discrete approximation  $\mathbf{y}_M$  converges to the  $\mathbf{y}(\tau_M)$  with weak (global) order  $q$  as  $h \rightarrow 0$ :

$$E \left[ \prod_{j=1}^L \tilde{\Phi}(t_j) \right] = E \left[ \prod_{j=1}^L \Phi(t_j) \right] \quad (2)$$

for any  $t_1, \dots, t_L \in T$  ( $1 \leq L \leq 2q$ ) satisfying  $\sum_{j=1}^L (\rho(t_j) + r(t_j)) \leq 2q$  and

$$E \left[ \tilde{\Phi}(t) \right] = 0 \quad (3)$$

for any  $t \in T$  satisfying  $\rho(t) + r(t) = 2q + 1$ , where the elementary weight  $\Phi(t)$  and the elementary numerical weight  $\tilde{\Phi}(t)$  are defined in [3].

### 3. Weak second order conditions for our SRK methods

In (1) we seek weak second order conditions that lead to a reduction in the number of evaluations on the diffusion coefficients. We can achieve this by slightly changing the parameter settings considered in [3]. Taking generality into account, we will leave implicitness in parameters as much as possible.

We use the same simplifying assumptions as those in [3], which are given by seven equalities for  $\tilde{\Phi}(t)$ . Four of them are as follows: for  $\tau^{(j)}$ ,  $[\tau^{(0)}]^{(j)}$ ,  $[\tau^{(l)}]^{(j)}$ ,  $[\tau^{(j)}]^{(l)} \in T$

$$\begin{aligned} \sum_{i_a=1}^s \sum_{j_a=0}^m c_{i_a}^{(j_a,j)} \tilde{\eta}_{i_a}^{(j_a,j)} &= \Delta W_j, \\ \sum_{i_a, i_b=1}^s \sum_{j_a, j_b=0}^m c_{i_a}^{(j_a,j)} \tilde{\eta}_{i_a}^{(j_a,j)} \alpha_{i_a i_b}^{(j_a, j, j_b, l)} \tilde{\eta}_{i_b}^{(j_b, l)} &= \frac{\Delta W_j (\Delta W_l + \Delta \tilde{W}_l)}{2}, \\ \sum_{i_a, i_b=1}^s \sum_{j_a, j_b=0}^m c_{i_a}^{(j_a,j)} \tilde{\eta}_{i_a}^{(j_a,j)} \alpha_{i_a i_b}^{(j_a, j, j_b, 0)} \tilde{\eta}_{i_b}^{(j_b, 0)} &= \frac{h \Delta W_j}{2}, \\ \sum_{i_a, i_b=1}^s \sum_{j_a, j_b=0}^m c_{i_a}^{(j_a, l)} \tilde{\eta}_{i_a}^{(j_a, l)} \alpha_{i_a i_b}^{(j_a, l, j_b, j)} \tilde{\eta}_{i_b}^{(j_b, j)} &= \frac{\Delta W_j (\Delta W_l - \Delta \tilde{W}_l)}{2} \end{aligned} \quad (4)$$

where  $0 < j < l$  and  $\Delta W_j$  ( $1 \leq j \leq m$ ) and  $\Delta \tilde{W}_l$  ( $2 \leq l \leq m$ ) are given as in [3]. Further, as in the same as [3] we set

$$\begin{aligned} \tilde{\eta}_i^{(0,0)} &= h, & \tilde{\eta}_i^{(j,j)} &= \Delta W_j \quad (j > 0), & c_i^{(j_a,0)} &= c_i^{(0,j_b)} = 0 \quad (j_a, j_b \neq 0), \\ \alpha_{i_a i_b}^{(j_a,0,j_c,0)} &= 0 \quad (j_a \neq 0 \text{ or } j_c \neq 0), & \alpha_{i_a i_b}^{(j_a,0,j_c,j)} &= 0 \quad (j_a \neq 0 \text{ or } j_c \neq j). \end{aligned}$$

On the other hand, for each ( $1 \leq j \leq m$ ), chose a value in  $\{1, 2, \dots, j-1, j+1, \dots, m\}$ , say  $k(j)$ , and assume

$$\begin{aligned} c_i^{(j_a,j)} &= 0 \quad (j_a \neq j, k(j)), & c_i^{(k(j),j)} &= 0 \quad (i \leq s-3), \\ \alpha_{i_a i_b}^{(j_a,j,j_c,j_d)} &= 0 \quad (j_a \neq j, k(j) \text{ and } (j_c \neq 0 \text{ or } j_d \neq 0)), \\ \alpha_{i_a i_b}^{(j,j,j_c,j_d)} &= 0 \quad (j_c \neq j_d), & \alpha_{i_a i_b}^{(k(j),j,j_c,j_d)} &= 0 \quad (j_c \neq 0, j), \\ \alpha_{i_a i_b}^{(k(j),j,0,j_d)} &= 0 \quad (j_d \neq 0), & \alpha_{i_a i_b}^{(k(j),j,j,j_d)} &= 0 \quad (j_d = 0, j) \end{aligned} \quad (5)$$

for  $j > 0$  and

$$\begin{aligned} \tilde{\eta}_{s-2}^{(j,l)} &= \begin{cases} \Delta W_j \Delta \tilde{W}_l / \sqrt{h} & (l > j), \\ -\Delta \tilde{W}_j \Delta W_l / \sqrt{h} & (j > l), \end{cases} & \tilde{\eta}_i^{(j,l)} &= \sqrt{h} \quad (i > s-2), \\ \alpha_{i_a i_b}^{(k(j),j,j,l)} &= 0 \quad (i_a, i_b \leq s-3 \text{ or } i_a \leq i_b) \end{aligned} \quad (6)$$

for  $j \neq l$  and  $j, l > 0$  (we always assume the restrictions for  $j, l$  in the sequel). Note that  $\tilde{\eta}_i^{(j,0)}$ ,  $\tilde{\eta}_i^{(0,j)}$  ( $1 \leq i \leq s$ ) and  $\tilde{\eta}_i^{(j,l)}$  ( $1 \leq i \leq s-3$ ) do not need to be set since they are not used below.

From the first equations in (4) and (5) we obtain

$$\sum_{i_a=1}^s c_{i_a}^{(j,j)} = 1, \quad c_{s-2}^{(k(j),j)} = c_{s-1}^{(k(j),j)} + c_s^{(k(j),j)} = 0. \quad (7)$$

Similarly, from this and the other three equations in (4) we have

$$\begin{aligned} \sum_{i_a, i_b=1}^s c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,0,0)} &= \frac{1}{2}, & \sum_{i_a, i_b=1}^s c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,l,l)} &= \frac{1}{2}, \\ \sum_{i_a=s-1}^s \sum_{i_b=1}^s c_{i_a}^{(k(j),j)} \alpha_{i_a i_b}^{(k(j),j,0,0)} &= 0, \\ c_{s-1}^{(k(j),j)} \alpha_{s-1, s-2}^{(k(j),j,j,l)} + c_s^{(k(j),j)} \alpha_{s, s-2}^{(k(j),j,j,l)} &= \frac{1}{2}, & c_s^{(k(j),j)} \alpha_{s, s-1}^{(k(j),j,j,l)} &= 0. \end{aligned} \quad (8)$$

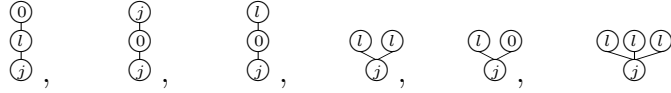
Here, note that the last equation in (7) and the last two equations in (8) yield

$$\alpha_{s, s-1}^{(k(j),j,j,l)} = 0. \quad (9)$$

**Remark 1.** We also have the following.

- i) For the trees in which a node and its child node are colored with the same color, their elementary numerical weights do not have  $\tilde{\eta}_i^{(j,l)}$  because  $\alpha_{i_a i_b}^{(j_a, j, j_c, j)} = 0$  ( $j_a \neq j$  or  $j_c \neq j$ ) and  $\alpha_{i_a i_b}^{(j_a, 0, j_c, 0)} = 0$  ( $j_a \neq 0$  or  $j_c \neq 0$ ).
- ii) For the trees in which the root is colored with 0, their elementary numerical weights do not have  $\tilde{\eta}_i^{(j,l)}$  because  $\alpha_{i_a i_b}^{(j_a, 0, j_c, j_d)} = 0$  ( $j_a \neq 0$  or  $j_c \neq j_d$ ).
- iii) For the trees in which the root is colored with  $j$  and has a child node colored with  $l$  which has a child node colored with  $k$  ( $\neq 0, l$ ), their elementary numerical weights do not have  $\tilde{\eta}_i^{(j,l)}$  or  $\tilde{\eta}_i^{(l,k)}$  because  $\alpha_{i_a i_b}^{(j_a, j, j_b, l)} \alpha_{i_b i_c}^{(j_b, l, j_c, k)} = 0$  ( $j_a \neq j$  or  $j_b \neq l$  or  $j_c \neq k$ ) from (5), (6) and (9).

Consequently, concerning weak order 2, all the trees whose elementary numerical weights have  $\tilde{\eta}_i^{(j,l)}$  are



as well as  $[\tau^{(0)}]^{(j)}$  and  $[\tau^{(l)}]^{(j)}$  dealt with in (8). Let us seek the order conditions concerning the above MRTs. For the MRTs except the second and fourth ones, (2) holds automatically. In order to satisfy (2) for the others and (3) for the fourth one, we obtain

$$\begin{aligned}
 \sum_{i_a, i_b, i_c=1}^s c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,0,0)} \alpha_{i_b i_c}^{(0,0,j,j)} &= 0, & \sum_{i_a, i_b, i_c=1}^s c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,l,l)} \alpha_{i_a i_c}^{(j,j,l,l)} &= \frac{1}{2}, \\
 \sum_{i_a=s-1} c_{i_a}^{(k(j),j)} \left( \alpha_{i_a, s-2}^{(k(j),j,j,l)} \right)^2 &= 0. & & (10)
 \end{aligned}$$

Incidentally, in the same way as that in [3] we can obtain the order conditions concerning the trees whose elementary numerical weights do not have  $\tilde{\eta}_i^{(j,l)}$ . Summarizing all mentioned up to here, we have all 38 order conditions for weak order 2: 32 of them are the same as (11)–(42) in [3] and the other 6 order conditions are given by

$$\begin{aligned}
 c_{s-2}^{(k(j),j)} &= c_{s-1}^{(k(j),j)} + c_s^{(k(j),j)} = \alpha_{s, s-1}^{(k(j),j,j,l)} = \sum_{i_a=s-1} \sum_{i_b=1}^s c_{i_a}^{(k(j),j)} \alpha_{i_a i_b}^{(k(j),j,0,0)} = 0, \\
 \sum_{i_a=s-1} c_{i_a}^{(k(j),j)} \left( \alpha_{i_a, s-2}^{(k(j),j,j,l)} \right)^2 &= 0, & \sum_{i_a=s-1} c_{i_a}^{(k(j),j)} \alpha_{i_a, s-2}^{(k(j),j,j,l)} &= \frac{1}{2}.
 \end{aligned}$$

#### 4. Discussion

The difference in the order conditions between the present paper and [3] is only the last six equality relationships. For example, let us set  $s = 4$  and use the same values for  $c_{i_a}^{(j_a, j_a)}$  ( $j_a = 0, j$ ) and  $\alpha_{i_a i_b}^{(j_a, j_a, j_b, j_b)}$  ( $j_b = 0, j, l$ ) as those in [3], which means that the first 32 order conditions are satisfied. Further, if we set  $\alpha_{i_a i_b}^{(j, l, 0, 0)} = 0$  and  $c_3^{(k(j), j)} = \gamma$  (a nonzero constant), then we obtain

$$c_2^{(k(j), j)} = \alpha_{43}^{(k(j), j, j, l)} = 0, \quad c_4^{(k(j), j)} = -\gamma, \quad \alpha_{32}^{(k(j), j, j, l)} = -\alpha_{42}^{(k(j), j, j, l)} = \frac{1}{4\gamma}$$

from the 6 order conditions. This new method leads to

$$\begin{aligned} \mathbf{Y}_3^{(k(j), j)} &= \sqrt{h} \mathbf{g}_j(\mathbf{y}_n + \frac{1}{4\gamma} \sum_{\substack{j_d=1 \\ j_d \neq j}}^m \tilde{\eta}_2^{(j, j_d)} \mathbf{g}_{j_d}(\mathbf{y}_n)), \\ \mathbf{Y}_4^{(k(j), j)} &= \sqrt{h} \mathbf{g}_j(\mathbf{y}_n - \frac{1}{4\gamma} \sum_{\substack{j_d=1 \\ j_d \neq j}}^m \tilde{\eta}_2^{(j, j_d)} \mathbf{g}_{j_d}(\mathbf{y}_n)). \end{aligned}$$

Now, the necessary intermediate stage values for  $\mathbf{y}_{n+1}$  in (1) are only  $\mathbf{Y}_3^{(k(j), j)}$  and  $\mathbf{Y}_4^{(k(j), j)}$  in addition to  $\mathbf{Y}_i^{(0, 0)}$  and  $\mathbf{Y}_i^{(j, j)}$  ( $1 \leq i \leq 4$ ), whereas the  $\mathbf{Y}_i^{(j, l)}$ 's are necessary in [3].

Finally, in order to show the computational advantages of our method, we deal with the last example in [5]. That is, we transform (38) in [5] into the Stratonovich form and apply numerical schemes to it. Here, note that (39) in [5] is incorrect and its correct expression is given by

$$\begin{aligned} E[(X(t))^4] &= (74342479604283 + 1749302625065840e^t \\ &\quad - 24798885546415218e^{2t} - 263952793100784216e^{3t} \\ &\quad + 1531088033542529311e^{4t}) / (124416 \times 10^{13}). \end{aligned}$$

Using the Mersenne twister [6], we simulate  $256 \times 10^6$  independent trajectories for a given  $h$ . The results are indicated in Fig. 1. Solid, dash or dotted lines mean our efficient scheme for  $\gamma = 1$ , the NON scheme [3] or the RS1 scheme [4], respectively. In addition,  $S_a$  stands for the sum of the number of evaluations on the drift or diffusion coefficients and the number of generated pseudo random numbers. We can see that the efficient scheme and the NON scheme are almost the same in the relative errors, but the efficient scheme is superior in the computational costs.

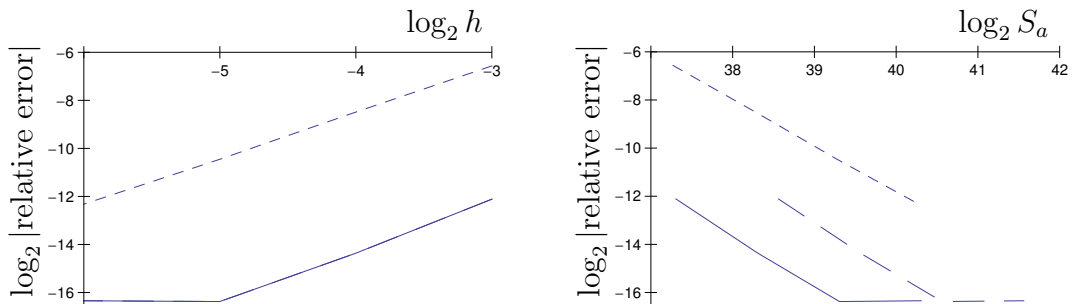


Figure 1: Relative errors about the 4th moment at  $t = 1$ .

## Acknowledgments

The authors thank the referee for his valuable comments to improve earlier manuscripts of this paper. This work was partially supported by JSPS Grant-in-Aid for Scientific Research No. 19540139.

## References

- [1] P. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, New York, 1999. Corrected 3rd printing.
- [2] K. Burrage, P. Burrage, Order conditions of stochastic Runge-Kutta methods by B-series, SIAM J. Numer. Anal. **38** (2000) 1626–1646.
- [3] Y. Komori, Weak second-order stochastic Runge-Kutta methods for non-commutative stochastic differential equations, J. Comput. Appl. Math. **206** (2007) 158–173.
- [4] A. Röbler, Second order Runge-Kutta methods for stratonovich stochastic differential equations, BIT **47** (2007) 657–680.
- [5] K. Debrabant, A. Röbler, Families of efficient second order Runge-Kutta methods for the weak approximation of Itô stochastic differential equations with scalar noise, Appl. Numer. Math. **59** (2009) 582–594.
- [6] M. Matsumoto, T. Nishimura, Mersenne twister: a 623-dimensionally equidistributed uniform pseudo-random number generator, ACM Transactions on Modeling and Computer Simulation (TOMACS) **8** (1998) 3–30.