CSSE-24 November 7, 2005

Weak Second Order Stochastic Runge-Kutta Methods for Non-commuting Stochastic Differential Equations

Yoshio Komori

Department of Systems Innovation and Informatics
Kyushu Institute of Technology

680-4 Kawazu
Iizuka, 820-8502, Japan
komori@ces.kyutech.ac.jp

Abstract

A new explicit stochastic Runge-Kutta scheme of weak order 2 is proposed for non-commuting stochastic differential equations (SDEs), which is derivative-free and which attains order 4 for ordinary differential equations. The scheme is directly applicable to Stratonovich SDEs and uses $2m-1$ random variables in the m-dimensional Wiener process case. It is compared with other derivative-free and weak second order schemes in numerical experiments.

1 Introduction

As the importance of stochastic differential equations (SDEs) increases, numerical methods for SDEs get studied more by many researchers. Especially, many numerical methods in the weak sense have been recently proposed for multi-dimensional SDEs with multiplicative noise in the multi-dimensional Wiener process case, whereas counterparts in the strong sense have been enormously developed in the last 10 years [3].

Among such weak methods, we are concerned with derivative-free methods. Let us introduce results concerning such methods, which attain weak order 2 at least. Kloeden and Platen [6, 10] have proposed a derivative-free scheme by replacing necessary derivatives by finite differences. Tocino and Vigo-Aguiar [15] have also proposed the scheme as an example in their Runge-Kutta family. Rößler [11, 12] has proposed other derivative-free schemes by assuming a commutativity condition [1, 13], which means

$$\boldsymbol{g}_j^{(1)}(\boldsymbol{y})\boldsymbol{g}_l(\boldsymbol{y}) = \boldsymbol{g}_l^{(1)}(\boldsymbol{y})\boldsymbol{g}_j(\boldsymbol{y}) \qquad (\forall \boldsymbol{y} \in \boldsymbol{R}^d,\ 1 \le j, l \le m,\ j \ne l)$$

in (2. 1). Here, $\boldsymbol{g}_j^{(1)}$ or $\boldsymbol{g}_l^{(1)}$ denotes the derivative of $\boldsymbol{g}_j$ or $\boldsymbol{g}_l$, respectively. On the other hand, Talay and Tubaro [14] have proposed the extrapolation method for SDEs. This method also makes it possible to obtain an approximate solution without using any derivative.

Komori [7] has also proposed a new stochastic Runge-Kutta family and developed Butcher's rooted tree analysis [4, 5] (which is for ordinary differential equations (ODEs)) to derive weak order conditions transparently for the new family. Then, utilizing the analysis, he has proposed a new explicit stochastic Runge-Kutta scheme of weak order 2, which is derivative-free and which attains order 4 for ODEs, under the commutativity condition [8].

In [7, 11, 15], it has been shown that each stochastic Runge-Kutta family includes the scheme proposed by Platen or its counterpart when the commutativity condition is not satisfied. It, however, still remains to find a solution of the order conditions in order to obtain another new scheme. Therefore, we aim at solving the order conditions and deriving a new explicit Runge-Kutta scheme of weak order 2 for non-commuting SDEs.

The organization of the present paper is as follows. In the next section we will give a brief introduction of our stochastic Runge-Kutta family as well as the expression of the order conditions of the family with rooted trees. In Section 3 we will find a solution of the order conditions after giving simplifying assumptions, and give some numerical experiments in the non-commutative case. In Section 4 we will give the summary and remarks. In the appendix, we will show the expectations of elementary numerical weights for weak order 2.

2 Stochastic Runge-Kutta family

In this section we introduce a stochastic Runge-Kutta family which gives approximate solutions for SDEs with a multi-dimensional Wiener process. To derive weak order conditions for the family, we utilize the multi-colored rooted tree analysis.

2.1 Weak order

First of all, we introduce the definition of weak (global) order. Let τ_n be an equidistant grid point nh $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and $\boldsymbol{y}_n$ a discrete approximation to the solution $\boldsymbol{y}(\tau_n)$ of the d-dimensional stochastic integral equation

$$\boldsymbol{y}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{g}_0(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_0^t \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s), \quad 0 \le t \le T_{end}, \tag{2. 1}$$

where $W_j(s)$ is a scalar Wiener process and $\circ$ means the Stratonovich formulation. The initial approximate random variable $\boldsymbol{y}_0$ is supposed to have the same probability law with all moments finite as that of $\boldsymbol{x}_0$. In addition, let $C_P^L(\boldsymbol{R}^d, \boldsymbol{R})$ be the totality of L times continuously differentiable $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^d$, all of whose partial derivatives of order less than or equal to L have polynomial growth. Then, the definition of weak order is given as follows [2].

Definition 2.1 *Suppose that discrete approximations $\boldsymbol{y}_n$ are given by a scheme. Then, we say that the scheme is of weak (global) order q if for each $G \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$, $C > 0$ (independent of h) and $\delta > 0$ exist such that*

$$|E[G(\boldsymbol{y}(\tau_M)] - E[G(\boldsymbol{y}_M)]| \le Ch^q, \quad h \in (0, \delta).$$

In order to obtain an approximate solution $\boldsymbol{y}_{n+1}$ of the solution $\boldsymbol{y}(t_{n+1})$ when $\boldsymbol{y}_n$ is given, we consider the stochastic Runge-Kutta family given by

$$\begin{aligned}
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s} \sum_{j_a, j_b=0}^{m} c_i^{(j_a,j_b)} \boldsymbol{Y}_i^{(j_a,j_b)}, \\
\boldsymbol{Y}_{i_a}^{(j_a,j_b)} &= \tilde{\eta}_{i_a}^{(j_a,j_b)} \Bigg\{ \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s} \sum_{j_c, j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \\
&\qquad + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s} \sum_{j_c, j_d=0}^{m} \tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)} \Bigg\}
\end{aligned} \tag{2. 2}$$

$(1 \le i_a \le s,\ 0 \le j_a, j_b \le m)$, where the constants $c_i^{(j_a,j_b)}$, $\alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)}$ and $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)}$ are defined by the Butcher tableau and where each $\tilde{\eta}_{i_a}^{(j_a,j_b)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\tilde{\eta}_{i_a}^{(j_a,j_b)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j_b = 0), \\ K_2 h^k & (j_b \ne 0) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. Note that this formulation includes stochastic Rosenbrock-Wanner methods [9].

2.2 Weak order conditions by multi-colored rooted trees

In this subsection we express weak order conditions by multi-colored rooted trees (MRTs). As preliminaries, we introduce several notations and definitions.

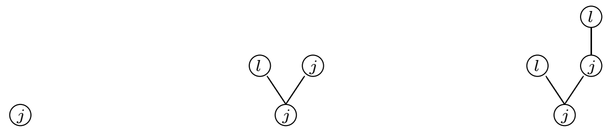
First, we introduce the multi-colored rooted tree (MRT) and a function on its set.

Definition 2.2 (Multi-colored rooted tree) *A multi-colored rooted tree with a root* ⓙ *(colored with a label j from 0 to m) is a tree recursively defined in the following manner:*

1) $\tau^{(j)}$ is the primitive tree having only a vertex ⓙ*.*

2) If $t_1, \dots, t_k$ are multi-colored trees, then $[t_1, \dots, t_k]^{(j)}$ is also a multi-colored rooted tree with the root ⓙ*.*

The totality of MRTs is denoted by T.



$\tau^{(j)}$ $\quad [\tau^{(l)}, \tau^{(j)}]^{(j)}$ $\quad [\tau^{(l)}, [\tau^{(l)}]^{(j)}]^{(j)}$

Figure 1: Examples of MRTs

Definition 2.3 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows:*

$$\Phi(\tau^{(j)}; s) = \int_{\tau_n}^{s} \circ \mathrm{d}W_j(s_1),$$

$$\Phi(t; s) = \int_{\tau_n}^{s} \prod_{i=1}^{k} \Phi(t_i; s_1) \circ \mathrm{d}W_j(s_1) \qquad \text{for } t = [t_1, \dots, t_k]^{(j)},$$

where $\circ \mathrm{d}W_0(s_1) \stackrel{\text{def}}{=} \mathrm{d}s_1$.

For ease of notation we will denote $\Phi(t; \tau_{n+1})$ by $\Phi(t)$.

Next, we introduce several matrices related to the formula parameters of (2. 2), the multi-colored rooted tree with labels (MRTL) and a function on its set. Let us adopt nominal symbols $\tilde{\eta}_{s+1}^{(j_a,j_b)}$, $\alpha_{s+1,i_b}^{(j_a,j_b,j_c,j_d)}$ and $\tilde{\gamma}_{s+1,i_b}^{(j_a,j_b,j_c,j_d)}$ and define $\alpha_{s+1,i_b}^{(0,0,j_c,j_d)} \stackrel{\text{def}}{=} c_{i_b}^{(j_c,j_d)}$ for $i_b \geq 1$ and

$$A^{(j,j')} \stackrel{\text{def}}{=} \begin{bmatrix} \alpha_{11}^{(0,j,0,j')} & \cdots & \alpha_{11}^{(m,j,0,j')} & \cdots & \alpha_{s+1,1}^{(0,j,0,j')} & \cdots & \alpha_{s+1,1}^{(m,j,0,j')} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{11}^{(0,j,m,j')} & \cdots & \alpha_{11}^{(m,j,m,j')} & \cdots & \alpha_{s+1,1}^{(0,j,m,j')} & \cdots & \alpha_{s+1,1}^{(m,j,m,j')} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{1s}^{(0,j,0,j')} & \cdots & \alpha_{1s}^{(m,j,0,j')} & \cdots & \alpha_{s+1,s}^{(0,j,0,j')} & \cdots & \alpha_{s+1,s}^{(m,j,0,j')} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{1s}^{(0,j,m,j')} & \cdots & \alpha_{1s}^{(m,j,m,j')} & \cdots & \alpha_{s+1,s}^{(0,j,m,j')} & \cdots & \alpha_{s+1,s}^{(m,j,m,j')} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

for $\alpha_{i_a i_b}^{(j_a,j,j_c,j')}$, where $\mathbf{0}$ stands for an $m+1$-dimensional column vector of 0's. Similarly, define the matrix $\tilde{\Gamma}^{(j,j')}$ for $\tilde{\gamma}_{i_a i_b}^{(j_a,j,j_c,j')}$ and set $\tilde{A}^{(j,j')} \stackrel{\text{def}}{=} A^{(j,j')} + \tilde{\Gamma}^{(j,j')}$. In addition, define the $(m+1)(s+1) \times (m+1)(s+1)$ diagonal matrix $D^{(j)}$ by

$$D^{(j)} \stackrel{\text{def}}{=} \mathrm{diag}(\tilde{\eta}_1^{(0,j)}, \dots, \tilde{\eta}_1^{(m,j)}, \dots, \tilde{\eta}_{s+1}^{(0,j)}, \dots, \tilde{\eta}_{s+1}^{(m,j)}).$$

In the sequel, let us use a label $X^{(j)} \in \{A^{(j)}, \tilde{A}^{(j)}\}$ as well as a matrix $X^{(j,j')} \in \{A^{(j,j')}, \tilde{A}^{(j,j')}\}$.

Definition 2.4 (Multi-colored rooted tree with labels) *A multi-colored rooted tree with labels, denoted by $t_{X^{(j)}}$, is one attached by labels according to the following rules:*

1) *The label of the root is $X^{(j)}$.*

2) *The label of the other vertices is decided by the number of branches and the color of the parent vertex:*

   - *the label is $\tilde{A}^{(j)}$ if the parent vertex has a single branch and it is colored with j,*
   - *the label is $A^{(j)}$ if the parent vertex has more than one branch and it is colored with j.*

The totality of MRTL's whose roots are labeled with $X^{(j)}$, is denoted by $\mathcal{T}_{X^{(j)}}$. For example, some MRTL's are listed in Fig. 2.

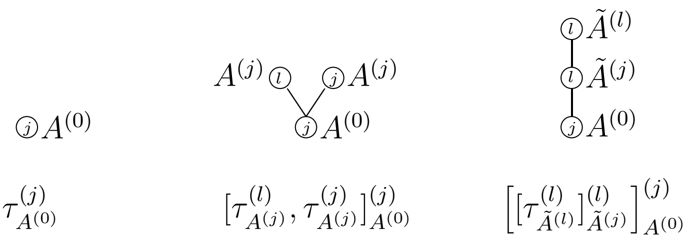


Figure 2: Examples of trees in $\mathcal{T}_{A^{(0)}}$

Definition 2.5 (Elementary numerical weight $\bar{\Phi}(t)$ on $\mathcal{T}_{X^{(j)}}$) *An elementary numerical weight of $t \in \mathcal{T}_{X^{(j)}}$ is given recursively as follows:*

$$\bar{\Phi}(\tau^{(j')}_{X^{(j)}}) = \mathbf{1} D^{(j')} X^{(j,j')},$$
$$\bar{\Phi}(t) = (\prod_{i=1}^{k}\!\cdot\, \bar{\Phi}(t_i)) D^{(j')} X^{(j,j')} \quad \text{for } t = [t_1, \ldots, t_k]^{(j')}_{X^{(j)}}$$

$(0 \le j, j' \le m)$, where $\tau^{(j')}_{X^{(j)}}$ and $[t_1, \ldots, t_k]^{(j')}_{X^{(j)}}$ express MRTL's whose roots are labeled by $X^{(j)}$. In addition, $\mathbf{1}$ stands for an $(m+1)(s+1)$-dimensional row vector of 1's, and $\prod_{i=1}^{k}\!\cdot\, \bar{\Phi}(t_i)$ means the elementwise product of row vectors $\bar{\Phi}(t_i)$.

Now, we can give weak order conditions. Let $\rho(t)$ be the number of vertices of $t \in T$ and $r(t)$ the number of vertices of t with the color 0, and suppose that any component of $\boldsymbol{g}_j$ belongs to $C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$ $(0 \le j \le m)$ and the regularity of the time discrete approximation are satisfied [6, 7]. If the following are satisfied, the time discrete approximation $\boldsymbol{y}_M$ converges to the $\boldsymbol{y}(\tau_M)$ with weak (global) order q as $h \to 0$:

$$E\left[\prod_{j=1}^{L} \bar{\Phi}_{(m+1)s+1}(t_j)\right] = E\left[\prod_{j=1}^{L} \Phi(\hat{t}_j)\right] \qquad (2.\ 3)$$

for any $t_1, \ldots, t_L \in \mathcal{T}_{A^{(0)}}$ $(1 \le L \le 2q)$ satisfying $\sum_{j=1}^{L}(\rho(\hat{t}_j) + r(\hat{t}_j)) \le 2q$ and

$$E\left[\bar{\Phi}_{(m+1)s+1}(t)\right] = 0 \qquad (2.\ 4)$$

for any $t \in \mathcal{T}_{A^{(0)}}$ satisfying $\rho(\hat{t}) + r(\hat{t}) = 2q + 1$.

2.3 Expectations of elementary weights

We show a way of seeking the expectation in the right-hand side of (2. 3) with the help of MRTs. In the multiple Stratonovich integrals, the usual chain rule holds as in the deterministic case. Hence, we can rewrite the product of elementary weights or the composition of subtrees in a elementary weight by the following rules:

- The product of elementary weights of two MRTs t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to the root of t_2 and an MRT generated by grafting t_2 to the root of t_1.
- The elementary weight of an MRT having subtrees t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to t_2's own root and an MRT generated by grafting t_2 to t_1's own root.

For example, we have

$$\Phi\left(\textcircled{0}\right)\Phi\begin{pmatrix}\textcircled{l}\\ |\\ \textcircled{j}\end{pmatrix}=\Phi\begin{pmatrix}\textcircled{l}\\ |\\ \textcircled{j}\\ |\\ \textcircled{0}\end{pmatrix}+\Phi\begin{pmatrix}\textcircled{0} & & \textcircled{l}\\ & \diagdown\ \diagup & \\ & \textcircled{j} & \end{pmatrix}=\Phi\begin{pmatrix}\textcircled{l}\\ |\\ \textcircled{j}\\ |\\ \textcircled{0}\end{pmatrix}+\Phi\begin{pmatrix}\textcircled{l}\\ |\\ \textcircled{0}\\ |\\ \textcircled{j}\end{pmatrix}+\Phi\begin{pmatrix}\textcircled{0}\\ |\\ \textcircled{l}\\ |\\ \textcircled{0}\end{pmatrix}.$$

In addition, by utilizing the relationship between multiple Stratonovich integrals and multiple Itô integrals ([6], p. 173), we can rewrite the expectations of the elementary weights of MRTs whose each vertex has no more than one branch as follows:

- The expectation of an elementary weight vanishes unless the even number of vertices are of colors different from 0 and each of these vertices has a parent or child vertex with the same color.
- Trace vertices in the direction from the root to upper vertices. Then, the expectation of an elementary weight of an MRT in which a vertex colored by $j \neq 0$ has a child vertex with the same color is equal to a half of that of another MRT given by replacing the two vertices with one vertex with the color 0. For example,

  $$E\left[\Phi\begin{pmatrix}\textcircled{j}\\ |\\ \textcircled{j}\\ |\\ \textcircled{0}\end{pmatrix}\right]=\frac{1}{2}E\left[\Phi\begin{pmatrix}\textcircled{0}\\ |\\ \textcircled{0}\end{pmatrix}\right]=\frac{1}{2}\Phi\begin{pmatrix}\textcircled{0}\\ |\\ \textcircled{0}\end{pmatrix}.$$

  Note that there is no longer need of the expectation in the right-hand side.

3 Solution of order conditions

In the previous section we have shown the order conditions with MRTL's, and demonstrated the calculation of the expectations of elementary weights and elementary numerical weights appearing in the order conditions. In this section we will find a solution of the conditions for weak order 2 in the non-commutative case.

3.1 Simplifying assumption

As seen in (2. 3) and (2. 4), the conditions for weak order are generally given in the form of expectations. By replacing expectations with monomials for trees which have only a few vertices, however, we can reduce the number of the order conditions. In relation to

$\tau^{(0)}_{A^{(0)}}$, $\tau^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$, $[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$ and $[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(0)}}$ $(j < l)$, let us assume that the following equations hold (simplifying assumptions):

$$
\begin{aligned}
&\sum c^{(j_1',0)}_{i_1} \tilde{\eta}^{(j_1',0)}_{i_1} = h, \\
&\sum c^{(j_1',j)}_{i_1} \tilde{\eta}^{(j_1',j)}_{i_1} = \triangle W_j, && && (3.\ 1)\\
&\sum c^{(j_1',j)}_{i_1} \tilde{\eta}^{(j_1',j)}_{i_1} \tilde{\alpha}^{(j_1',j,j_2',j)}_{i_1 i_2} \tilde{\eta}^{(j_2',j)}_{i_2} = \frac{(\triangle W_j)^2}{2}, \\
&\sum c^{(j_1',0)}_{i_1} \tilde{\eta}^{(j_1',0)}_{i_1} \tilde{\alpha}^{(j_1',0,j_2',j)}_{i_1 i_2} \tilde{\eta}^{(j_2',j)}_{i_2} = \frac{h\triangle W_j}{2}, \\
&\sum c^{(j_1',j)}_{i_1} \tilde{\eta}^{(j_1',j)}_{i_1} \tilde{\alpha}^{(j_1',j,j_2',0)}_{i_1 i_2} \tilde{\eta}^{(j_2',0)}_{i_2} = \frac{h\triangle W_j}{2}, \\
&\sum c^{(j_1',j)}_{i_1} \tilde{\eta}^{(j_1',j)}_{i_1} \tilde{\alpha}^{(j_1',j,j_2',l)}_{i_1 i_2} \tilde{\eta}^{(j_2',l)}_{i_2} = \frac{\triangle W_j(\triangle W_l + \triangle \tilde{W}_l)}{2} && (j < l), && (3.\ 2)\\
&\sum c^{(j_1',l)}_{i_1} \tilde{\eta}^{(j_1',l)}_{i_1} \tilde{\alpha}^{(j_1',l,j_2',j)}_{i_1 i_2} \tilde{\eta}^{(j_2',j)}_{i_2} = \frac{\triangle W_j(\triangle W_l - \triangle \tilde{W}_l)}{2} && (j < l), && (3.\ 3)
\end{aligned}
$$

where $\triangle W_j$'s $(j = 1, \ldots, m)$ and $\triangle \tilde{W}_l$'s $(l = 2, \ldots, m)$ are mutually independent random variables satisfying

$$
E\left[(\triangle W_j)^k\right] = \begin{cases} 0 & (k = 1, 3, 5), \\ (k-1)h^{k/2} & (k = 2, 4), \\ O(h^3) & (k \geq 6), \end{cases} \quad E\left[(\triangle \tilde{W}_l)^k\right] = \begin{cases} 0 & (k = 1, 3), \\ h & (k = 2), \\ O(h^2) & (k \geq 4). \end{cases} \qquad (3.\ 4)
$$

Note that the expressions in the right-hand side of (3. 2) and (3. 3) come from the approximation

$$
\Phi\begin{pmatrix} l \\ j \end{pmatrix} \approx \begin{cases} \dfrac{\triangle W_j(\triangle W_l + \triangle \tilde{W}_l)}{2} & (j < l), \\ \dfrac{\triangle W_l(\triangle W_j - \triangle \tilde{W}_j)}{2} & (j > l). \end{cases}
$$

Then, the next order conditions are satisfied:

$$
\begin{aligned}
&E\left[\bar{\Phi}_{(m+1)s+1}\left(0_{A^{(0)}}\right)\right] = h, \qquad E\left[\left\{\bar{\Phi}_{(m+1)s+1}\left(0_{A^{(0)}}\right)\right\}^2\right] = h^2, \\
&E\left[\bar{\Phi}_{(m+1)s+1}\left(0_{A^{(0)}}\right)\left\{\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right\}^2\right] = h^2, \\
&E\left[\left\{\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right\}^2\right] = h, \qquad E\left[\left\{\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right\}^4\right] = 3h^2, \\
&E\left[\left\{\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right\}^2\left\{\bar{\Phi}_{(m+1)s+1}\left(l_{A^{(0)}}\right)\right\}^2\right] = h^2, \\
&E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\right] = \frac{h}{2}, \quad E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(0_{A^{(0)}}\right)\right] = \frac{h^2}{2}, \\
&E\left[\left\{\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\right\}^2\right] = \frac{3h^2}{4}, \\
&E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} l_{\tilde{A}^{(l)}} \\ l_{A^{(0)}} \end{pmatrix}\right] = \frac{h^2}{4}, \\
&E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\left\{\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right\}^2\right] = \frac{3h^2}{2},
\end{aligned}
$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\left\{\bar{\Phi}_{(m+1)s+1}\left(l_{A^{(0)}}\right)\right\}^2\right]=\frac{h^2}{2},$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(0)}} \\ 0_{A^{(0)}} \end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right]=\frac{h^2}{2},$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} 0_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\right]=\frac{h^2}{2},$$

$$E\left[\left\{\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} l_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\right\}^2\right]=\frac{h^2}{2}\qquad(j\neq l),\tag{3. 5}$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} l_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} j_{\tilde{A}^{(l)}} \\ l_{A^{(0)}} \end{pmatrix}\right]=0\qquad(j\neq l),\tag{3. 6}$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} l_{\tilde{A}^{(j)}} \\ j_{A^{(0)}} \end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(j_{A^{(0)}}\right)\bar{\Phi}_{(m+1)s+1}\left(l_{A^{(0)}}\right)\right]=\frac{h^2}{2}\qquad(j\neq l).\tag{3. 7}$$

Because (3. 5), (3. 6) and (3. 7) cause difficulties in the construction of weak second order schemes for non-commutative SDEs, it is remarkable that the virtue of the simplifying assumptions (3. 2) and (3. 3) ensures that the equations hold.

3.2 Explicit stochastic Runge-Kutta methods

We consider the explicit stochastic Runge-Kutta methods and show how to solve the order conditions.

First of all, we set

$$\tilde{\eta}_i^{(0,0)}=h,\qquad \tilde{\eta}_i^{(j_a,j_b)}=\begin{cases}\triangle\tilde{W}_{j_b} & (j_b>j_a>0),\\ \triangle W_{j_b} & (j_a\geq j_b>0).\end{cases}\tag{3. 8}$$

Next, let us set $c_i^{(j_a,0)}=0$ $(j_a\neq 0)$, $\alpha_{i_ai_b}^{(j_a,j,j_c,j)}=0$ $(j_a\neq j$ or $j_c\neq j)$, $\alpha_{i_ai_b}^{(j_a,0,j_c,j)}=0$ $(j_a\neq 0$ or $j_c\neq j)$, $\alpha_{i_ai_b}^{(j_a,j,j_c,0)}=0$ $(j_a\neq j$ or $j_c\neq 0)$, $\alpha_{i_ai_b}^{(j_a,0,j_c,0)}=0$ $(j_a\neq 0$ or $j_c\neq 0)$ and $\alpha_{i_ai_b}^{(j_a,j,j_c,l)}=0$ if $j_a=j_c$ or $j_a,j_c\neq j,l$ when $l>j>0$, or if $j_a\neq j,l$ or $j_c\neq l$ when $j>l>0$. These settings, (3. 1) and (3. 4) imply that the following statement holds as far as concerning weak order 2:

> The expectation of the $((m+1)s+1)$-st element of an elementary numerical weight or the product of those is equal to 0 if the odd number of vertices are of the same color $j(\neq 0)$.

As we have seen in Subsection 2.3, the expectation of an elementary weight or the product of those vanishes if the odd number of vertices are of the same color $j(\neq 0)$. The above statement ensures that (2. 3) holds for such MRTL's and (2. 4) holds.

Then, let us introduce

$$c_i^{(j)}\overset{\text{def}}{=}c_i^{(j,j)},\qquad \alpha_{i_ai_b}^{(j,j')}\overset{\text{def}}{=}\alpha_{i_ai_b}^{(j,j,j',j')},\qquad A_{i_a}^{(j,j')}\overset{\text{def}}{=}\sum_{i_b=1}^{i_a-1}\alpha_{i_ai_b}^{(j,j')}\qquad(j,j'\geq 0),$$

$$A_{i_a}^{(l,j,j,l)}\overset{\text{def}}{=}\sum_{i_b=1}^{i_a-1}\alpha_{i_ai_b}^{(l,j,j,l)},\qquad A_{i_a}^{(j,l,j,j)}\overset{\text{def}}{=}\sum_{i_b=1}^{i_a-1}\alpha_{i_ai_b}^{(j,l,j,j)}\qquad(l>j>0)$$

for ease of notation.

From (3. 8) we obtain

$$\sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} = \sum_{i_1} c_{i_1}^{(j)} \triangle W_j + \sum_{\substack{i_1 \\ j_1'>j}} c_{i_1}^{(j_1',j)} \triangle W_j + \sum_{\substack{i_1 \\ j_1'<j}} c_{i_1}^{(j_1',j)} \triangle \tilde{W}_j.$$

Hence, if

$$\sum c_{i_1}^{(j)} = 1, \qquad \sum c_{i_1}^{(l,j)} = 0 \quad (j<l), \qquad \sum c_{i_1}^{(j,l)} = 0 \quad (j<l),$$

then, (3. 1) holds.

When $j < l$, we also obtain

$$\sum c_{i_1}^{(j_1',j)} \tilde{\eta}_{i_1}^{(j_1',j)} \alpha_{i_1 i_2}^{(j_1',j,j_2',l)} \tilde{\eta}_{i_2}^{(j_2',l)} = \sum_{i_1,i_2} c_{i_1}^{(j)} \triangle W_j \alpha_{i_1 i_2}^{(j,l)} \triangle W_l + \sum_{i_1,i_2} c_{i_1}^{(l,j)} \triangle W_j \alpha_{i_1 i_2}^{(l,j,j,l)} \triangle \tilde{W}_l.$$

Hence, (3. 2) is equivalent to

$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} = \frac{1}{2}, \qquad \sum c_{i_1}^{(l,j)} A_{i_1}^{(l,j,j,l)} = \frac{1}{2}$$

for $j < l$. Here, note that $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} = 0$ $(\forall i_a, i_b, j_a, j_b, j_c, j_d)$ because we consider explicit stochastic Runge-Kutta methods. Similarly, (3. 3) is equivalent to

$$\sum c_{i_1}^{(l)} A_{i_1}^{(l,j)} = \frac{1}{2}, \qquad \sum c_{i_1}^{(j,l)} A_{i_1}^{(j,l,j,j)} = -\frac{1}{2}$$

for $j < l$.

As we have seen, each of (3. 1), (3. 2) and (3. 3) yields at least two algebraic equations as a sufficient or equivalent condition. In analogy, each of the following two order conditions also yields two algebraic equations. The order condition

$$E\left[\bar{\Phi}_{(m+1)s+1}\left([\tau_{A^{(j)}}^{(l)}, [\tau_{A^{(l)}}^{(j)}]_{A^{(j)}}^{(l)}]_{A^{(0)}}^{(j)}\right)\right] = 0 \qquad (j \neq l)$$

yields

$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,j)} = 0 \quad (j \neq l), \qquad \sum c_{i_1}^{(l,j)} A_{i_1}^{(l,j,j,l)} \alpha_{i_1 i_2}^{(l,j,j,l)} A_{i_2}^{(j,l,j,j)} = 0 \quad (j<l),$$

and the order condition

$$E\left[\bar{\Phi}_{(m+1)s+1}\left([\tau_{A^{(j)}}^{(l)}, \tau_{A^{(j)}}^{(l)}]_{A^{(0)}}^{(j)}\right) \bar{\Phi}_{(m+1)s+1}\left(\tau_{A^{(0)}}^{(j)}\right)\right] = \frac{h^2}{2} \qquad (j \neq l)$$

yields

$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,l)}\right)^2 = \frac{1}{2} \quad (j \neq l), \qquad \sum c_{i_1}^{(l,j)} \left(A_{i_1}^{(l,j,j,l)}\right)^2 = 0 \quad (j<l).$$

On the other hand, the other order conditions yield just one algebraic equation, respectively.

Ultimately, in order to find a solution that satisfies the simplifying conditions and the order conditions, all we have to do is to solve the following equations (Appendix). In the sequel, we suppose $j, l \neq 0$ and omit to write $j \neq l$ as far as it does not cause a confusion.

$$\sum c_{i_1}^{(0)} = 1, \tag{3. 9}$$
$$\sum c_{i_1}^{(j)} = 1, \tag{3. 10}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} = \frac{1}{2}, \tag{3. 11}$$
$$\sum c_{i_1}^{(0)} A_{i_1}^{(0,j)} = \frac{1}{2}, \tag{3. 12}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,0)} = \frac{1}{2}, \tag{3. 13}$$
$$\sum c_{i_1}^{(0)} A_{i_1}^{(0,0)} = \frac{1}{2}, \tag{3. 14}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,0)} = \frac{1}{4}, \tag{3. 15}$$
$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,j)} A_{i_2}^{(j,j)} = \frac{1}{4}, \tag{3. 16}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,0)} A_{i_2}^{(0,j)} = 0, \tag{3. 17}$$
$$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,j)}\right)^2 = \frac{1}{2}, \tag{3. 18}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,0)} A_{i_1}^{(j,j)} = \frac{1}{4}, \tag{3. 19}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \alpha_{i_2 i_3}^{(j,j)} A_{i_3}^{(j,j)} = \frac{1}{24}, \tag{3. 20}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \left(A_{i_2}^{(j,j)}\right)^2 = \frac{1}{12}, \tag{3. 21}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,j)} = \frac{1}{8}, \tag{3. 22}$$
$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,j)}\right)^3 = \frac{1}{4}, \tag{3. 23}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,j)} = \frac{1}{6}, \tag{3. 24}$$
$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,j)}\right)^2 = \frac{1}{3}, \tag{3. 25}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} = \frac{1}{2}, \tag{3. 26}$$
$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,l)}\right)^2 = \frac{1}{2}, \tag{3. 27}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} A_{i_1}^{(j,l)} = \frac{1}{4}, \tag{3. 28}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,l)} = \frac{1}{4}, \tag{3. 29}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,j)} = 0, \tag{3. 30}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,l)} = \frac{1}{4}, \tag{3. 31}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,l)} = \frac{1}{4}, \tag{3. 32}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,j)} = 0, \tag{3. 33}$$

$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,l)} = \frac{1}{8}, \tag{3. 34}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,l)} A_{i_2}^{(l,j)} = 0, \tag{3. 35}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \left(A_{i_2}^{(j,l)}\right)^2 = \frac{1}{4}, \tag{3. 36}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} \left(A_{i_1}^{(j,l)}\right)^2 = \frac{1}{4}, \tag{3. 37}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} \alpha_{i_2 i_3}^{(l,j)} A_{i_3}^{(j,l)} = 0, \tag{3. 38}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} \alpha_{i_2 i_3}^{(l,l)} A_{i_3}^{(l,j)} = 0, \tag{3. 39}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \alpha_{i_2 i_3}^{(j,l)} A_{i_3}^{(l,l)} = \frac{1}{8}, \tag{3. 40}$$
$$\sum c_{i_1}^{(l,j)} = 0 \qquad (j < l), \tag{3. 41}$$
$$\sum c_{i_1}^{(j,l)} = 0 \qquad (j < l), \tag{3. 42}$$
$$\sum c_{i_1}^{(l,j)} A_{i_1}^{(l,j,j,l)} = \frac{1}{2} \qquad (j < l), \tag{3. 43}$$
$$\sum c_{i_1}^{(j,l)} A_{i_1}^{(j,l,j,j)} = -\frac{1}{2} \qquad (j < l), \tag{3. 44}$$
$$\sum c_{i_1}^{(l,j)} A_{i_1}^{(l,j,j,l)} \alpha_{i_1 i_2}^{(l,j,j,l)} A_{i_2}^{(j,l,j,j)} = 0 \qquad (j < l), \tag{3. 45}$$
$$\sum c_{i_1}^{(l,j)} \left(A_{i_1}^{(l,j,j,l)}\right)^2 = 0 \qquad (j < l). \tag{3. 46}$$

Note that $\alpha_{i_a i_b}^{(j,j')} = 0$ ($i_a \leq i_b,\ \forall j, j'$) and $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} = 0$ ($\forall i_a, i_b, j_a, j_b, j_c, j_d$) because we consider explicit stochastic Runge-Kutta methods.

The system of the conditions (3. 10), (3. 11), (3. 20), (3. 21), (3. 22), (3. 23), (3. 24) and (3. 25) has the same algebraic structure as that of the order conditions for ordinary Runge-Kutta methods to attain order 4 for ODEs ([4], pp. 90-91). Hence, because the stage number s has to be at least 4, let us suppose $s = 4$ in the sequel.

For stochastic Runge-Kutta schemes, Rößler ([11], p. 99) has proposed taking account of not only weak order but also order for ODEs. Now, for $s = 4$, we can let (2. 2) attain order 4 for ODEs. For this, we add the following six conditions:

$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} \alpha_{i_2 i_3}^{(0,0)} A_{i_3}^{(0,0)} = \frac{1}{24}, \tag{3. 47}$$
$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} \left(A_{i_2}^{(0,0)}\right)^2 = \frac{1}{12}, \tag{3. 48}$$
$$\sum c_{i_1}^{(0)} A_{i_1}^{(0,0)} \alpha_{i_1 i_2}^{(0,0)} A_{i_2}^{(0,0)} = \frac{1}{8}, \tag{3. 49}$$
$$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,0)}\right)^3 = \frac{1}{4}, \tag{3. 50}$$
$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} A_{i_2}^{(0,0)} = \frac{1}{6}, \tag{3. 51}$$
$$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,0)}\right)^2 = \frac{1}{3}, \tag{3. 52}$$

which come from $[[[\tau_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[\tau_{A^{(0)}}^{(0)}, [\tau_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[[\tau_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$ and $[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$.

To find a solution, we first simplify the equations from (3. 26) to (3. 40). By noting that we can suppose $\alpha_{32}^{(j,l)} = \alpha_{32}^{(l,j)}$, we have $\alpha_{43}^{(j,l)} A_2^{(j,l)} = 0$ from (3. 38) and (3. 40). If

$A_2^{(j,l)} = 0$, by noting that we can suppose $A_i^{(j,l)} = A_i^{(l,j)}$ for any i, we have $\alpha_{43}^{(j,l)} = 0$ from (3. 29) and (3. 30). Similarly, if $\alpha_{43}^{(j,l)} = 0$, we have $A_2^{(l,j)} = 0$ from (3. 31) and (3. 35). Hence, $\alpha_{43}^{(j,l)} = A_2^{(j,l)} = 0$. Then, $A_3^{(j,l)} = A_4^{(j,l)} = 1$ holds from (3. 29), (3. 32) and (3. 36). Consequently, we have

$$\alpha_{43}^{(j,l)} = A_2^{(j,l)} = 0, \qquad A_3^{(j,l)} = A_4^{(j,l)} = 1.$$

By substituting these into the equations from (3. 26) to (3. 40) and rewriting them, we obtain

$$c_3^{(j)} + c_4^{(j)} = \frac{1}{2}, \tag{3. 53}$$

$$c_3^{(j)} A_3^{(j,j)} + c_4^{(j)} A_4^{(j,j)} = \frac{1}{4}, \tag{3. 54}$$

$$c_4^{(j)} \alpha_{43}^{(j,j)} = \frac{1}{4}, \tag{3. 55}$$

$$\alpha_{42}^{(j,l)} A_2^{(l,l)} = \frac{1}{2}, \tag{3. 56}$$

$$\alpha_{32}^{(j,l)} = \alpha_{42}^{(j,l)}. \tag{3. 57}$$

As we have mentioned, the system of the conditions (3. 10), (3. 11), (3. 20), (3. 21), (3. 22), (3. 23), (3. 24) and (3. 25) has the same algebraic structure as that of the order conditions for ordinary Runge-Kutta methods of order 4. Hence, we can utilize the results known in the deterministic case to solve the system of the order conditions. The following five special cases where a solution surely exists are known for ordinary Runge-Kutta methods of order 4 with 4 stages ([4], pp. 164–165):

Case I	$A_2^{(j,j)} \notin \{0, \frac{1}{2}, \frac{1}{2} \pm \frac{\sqrt{3}}{6}, 1\}, \quad A_3^{(j,j)} = 1 - A_2^{(j,j)},$
Case II	$c_2^{(j)} = 0, \quad A_2^{(j,j)} \neq 0, \quad A_3^{(j,j)} = \frac{1}{2},$
Case III	$c_3^{(j)} \neq 0, \quad A_2^{(j,j)} = \frac{1}{2}, \quad A_3^{(j,j)} = 0,$
Case IV	$c_4^{(j)} \neq 0, \quad A_2^{(j,j)} = 1, \quad A_3^{(j,j)} = \frac{1}{2},$
Case V	$c_3^{(j)} \neq 0, \quad A_2^{(j,j)} = A_3^{(j,j)} = \frac{1}{2}.$

In Cases I and V, for example, the solutions are given by the following Butcher tableaux

$$\begin{array}{c|c} \boldsymbol{A}^{(j,j)} & \left[\alpha_{i_a i_b}^{(j,j)}\right] \\ \hline & \left(\boldsymbol{c}^{(j)}\right)^{\mathrm{T}} \end{array},$$

respectively:

Case I

$$\begin{array}{c|cccc}
0 & & & & \\
\delta_0 & \delta_0 & & & \\
A_3^{(j,j)} & \dfrac{A_3^{(j,j)}\delta_1}{2\delta_0} & \dfrac{A_3^{(j,j)}}{2\delta_0} & & \\
1 & \dfrac{12\left(A_3^{(j,j)}\right)^3 - 24\left(A_3^{(j,j)}\right)^2 + 17A_3^{(j,j)} - 4}{2\delta_0\delta_2} & \dfrac{A_3^{(j,j)}\delta_1}{2\delta_0\delta_2} & \dfrac{\delta_0}{\delta_2} & \\
\hline
 & \dfrac{\delta_2}{12A_3^{(j,j)}\delta_0} & \dfrac{1}{12A_3^{(j,j)}\delta_0} & \dfrac{1}{12A_3^{(j,j)}\delta_0} & \dfrac{\delta_2}{12A_3^{(j,j)}\delta_0}
\end{array},$$

where $\delta_0 \overset{\text{def}}{=} 1 - A_3^{(j,j)}$, $\delta_1 \overset{\text{def}}{=} 1 - 2A_3^{(j,j)}$ and $\delta_2 \overset{\text{def}}{=} 6A_3^{(j,j)} - 1 - 6\left(A_3^{(j,j)}\right)^2$, and

Case V

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & \frac{1}{2} - \frac{1}{6A_3^{(j,j)}} & \frac{1}{6A_3^{(j,j)}} & & \\
1 & 0 & 1 - 3A_3^{(j,j)} & 3A_3^{(j,j)} & \\
\hline
 & \frac{1}{6} & \frac{2}{3} - A_3^{(j,j)} & A_3^{(j,j)} & \frac{1}{6}
\end{array}.$$

The solutions in Cases II, III, IV and V, however, do not satisfy (3. 53) and (3. 55), simultaneously. Hence, the following are the steps we should carry out to find a solution of all the conditions:

Step 1) Select the solution in Case I as that of the system (3. 10), (3. 11), (3. 20), (3. 21), (3. 22), (3. 23), (3. 24) and (3. 25), substitute it into (3. 53), (3. 54), (3. 55), (3. 56) and (3. 57), and solve them.

Step 2) Select one among the five cases above and adopt its solution as that of the system (3. 9), (3. 14), (3. 47), (3. 48), (3. 49), (3. 50), (3. 51) and (3. 52) since those are exactly the conditions for ordinary Runge-Kutta methods of order 4 with 4 stages.

Step 3) Substitute the solution in Step 1) into (3. 13), (3. 15) and (3. 19), and seek $A_2^{(j,0)}$, $A_3^{(j,0)}$ or $A_4^{(j,0)}$. Here, note that the equations are not linearly independent with respect to the parameters.

Step 4) Substitute the solution in Step 2) into (3. 12) and (3. 18), and seek $A_2^{(0,j)}$, $A_3^{(0,j)}$ or $A_4^{(0,j)}$. Here, it is remarkable that the equations are equivalent when the parameters are 0 or 1 except $A_2^{(0,j)} = A_3^{(0,j)} = A_4^{(0,j)} = 0$.

Step 5) Substitute the results in Steps 1) and 4) into (3. 17), and seek $\alpha_{32}^{(j,0)}$, $\alpha_{42}^{(j,0)}$ or $\alpha_{43}^{(j,0)}$. Here, it is remarkable that the equation has the trivial solution $\alpha_{32}^{(j,0)} = \alpha_{42}^{(j,0)} = \alpha_{43}^{(j,0)} = 0$.

Step 6) Substitute the results in Steps 2) and 4) into (3. 16), and seek $\alpha_{32}^{(0,j)}$, $\alpha_{42}^{(0,j)}$ or $\alpha_{43}^{(0,j)}$.

Step 7) Solve (3. 41), (3. 43) and (3. 46) for three parameters among $c_i^{(l,j)}$ $(1 \le i \le 4)$ and $A_i^{(l,j,j,l)}$ $(2 \le i \le 4)$.

Step 8) Solve (3. 42), (3. 44) and (3. 45) for three parameters among $c_i^{(j,l)}$ $(1 \le i \le 4)$, $A_i^{(j,l,j,j)}$ $(2 \le i \le 4)$, $\alpha_{32}^{(l,j,j,l)}$, $\alpha_{42}^{(l,j,j,l)}$ and $\alpha_{42}^{(l,j,j,l)}$.

By following the steps, let us find a solution of all the conditions. In Step 1), from the solution in Case I and (3. 55) we have $A_3^{(j,j)} = \frac{1}{3}$. Then, (3. 53) and (3. 54) hold since $c_3^{(j)} = \frac{3}{8}$ and $c_4^{(j)} = \frac{1}{8}$. From (3. 56) and (3. 57), $\alpha_{32}^{(j,l)} = \alpha_{42}^{(j,l)} = \frac{3}{4}$. We chose the solution of Case V in Step 2). We obtain

$$A_2^{(j,0)} = -2A_3^{(j,0)} + 2, \quad A_4^{(j,0)} = 3A_3^{(j,0)} - 2$$

in Step 3). Let us set $A_2^{(0,j)} = A_4^{(0,j)} = 1$ and $A_3^{(0,j)} = 0$ in Step 4). This makes (3. 12) and (3. 18) equivalent and means $c_2^{(0)} + c_4^{(0)} = \frac{1}{2}$. Hence, $c_3^{(0)} = \frac{1}{3}$ in the present case. In Step 5) let us set $\alpha_{32}^{(j,0)} = \alpha_{42}^{(j,0)} = \alpha_{43}^{(j,0)} = 0$. In Step 6) we set $\alpha_{42}^{(0,j)} = \alpha_{43}^{(0,j)} = 0$ and obtain $\alpha_{32}^{(0,j)} = \frac{9}{8}$. In Step 7), when we set $c_1^{(l,j)} = c_4^{(l,j)} = 0$ for $j < l$, we obtain

$$A_2^{(l,j,j,l)} = -A_3^{(l,j,j,l)}, \quad c_2^{(l,j)} = -\frac{1}{4A_3^{(l,j,j,l)}}, \quad c_3^{(l,j)} = \frac{1}{4A_3^{(l,j,j,l)}} \qquad (j < l,\ A_3^{(l,j,j,l)} \neq 0)$$

from (3. 41), (3. 43) and (3. 46). In Step 8) let us set $c_1^{(j,l)} = c_4^{(j,l)} = A_2^{(j,l,j,j)} = 0$ for $j < l$. Then, (3. 45) holds automatically, and we obtain

$$c_2^{(j,l)} = \frac{1}{2A_3^{(j,l,j,j)}}, \qquad c_3^{(j,l)} = -\frac{1}{2A_3^{(j,l,j,j)}} \qquad (j < l,\ A_3^{(j,l,j,j)} \neq 0)$$

from (3. 42) and (3. 44).

We finally obtain

$$\begin{array}{c|c}
\left[\alpha_{i_a i_b}^{(0,0)}\right] & \left[\alpha_{i_a i_b}^{(j,0)}\right] \\
\left[\alpha_{i_a i_b}^{(0,j)}\right] & \left[\alpha_{i_a i_b}^{(j,j)}\right] \ \left[\alpha_{i_a i_b}^{(j,l)}\right] \\
\hline
\left(\boldsymbol{c}^{(0)}\right)^{\mathrm{T}} & \left(\boldsymbol{c}^{(j)}\right)^{\mathrm{T}}
\end{array}
=
\begin{array}{cccc|ccccccc}
 & & & & & & & & & & \\
\frac{1}{2} & & & & 2-2\alpha_{31}^{(j,0)} & & & & & & \\
0 & \frac{1}{2} & & & \alpha_{31}^{(j,0)} & 0 & & & & & \\
0 & 0 & 1 & & 3\alpha_{31}^{(j,0)}-2 & 0 & 0 & & & & \\
 & & & & & & & & & & \\
1 & & & & \frac{2}{3} & & & & 0 & & \\
-\frac{9}{8} & \frac{9}{8} & & & \frac{1}{12} & \frac{1}{4} & & & \frac{1}{4} & \frac{3}{4} & \\
1 & 0 & 0 & & -\frac{5}{4} & \frac{1}{4} & 2 & & \frac{1}{4} & \frac{3}{4} & 0 \\
\hline
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & & &
\end{array},$$

$$\begin{array}{c}
\left[\alpha_{i_a i_b}^{(j,l,j,j)}\right] \\
\hline
\left(\boldsymbol{c}^{(j,l)}\right)^{\mathrm{T}}
\end{array}
=
\begin{array}{cccc}
 & & & \\
0 & & & \\
A_3^{(j,l,j,j)} - \alpha_{32}^{(j,l,j,j)} & \alpha_{32}^{(j,l,j,j)} & & \\
\alpha_{41}^{(j,l,j,j)} & \alpha_{42}^{(j,l,j,j)} & \alpha_{43}^{(j,l,j,j)} & \\
\hline
0 & \frac{1}{2A_3^{(j,l,j,j)}} & -\frac{1}{2A_3^{(j,l,j,j)}} & 0
\end{array}
\qquad (j < l),$$

$$\begin{array}{c}
\left[\alpha_{i_a i_b}^{(l,j,j,l)}\right] \\
\hline
\left(\boldsymbol{c}^{(l,j)}\right)^{\mathrm{T}}
\end{array}
=
\begin{array}{cccc}
 & & & \\
-A_3^{(l,j,j,l)} & & & \\
A_3^{(l,j,j,l)} - \alpha_{32}^{(l,j,j,l)} & \alpha_{32}^{(l,j,j,l)} & & \\
\alpha_{41}^{(l,j,j,l)} & \alpha_{42}^{(l,j,j,l)} & \alpha_{43}^{(l,j,j,l)} & \\
\hline
0 & -\frac{1}{4A_3^{(l,j,j,l)}} & \frac{1}{4A_3^{(l,j,j,l)}} & 0
\end{array}
\qquad (j < l)$$

as a solution of all the order conditions. As we can see from the process of calculations for $c_i^{(j)}$'s, $\alpha_{i_a i_b}^{(j,j)}$'s and $\alpha_{i_a i_b}^{(j,l)}$'s, the set of coefficients for them in the right-hand side is unique when we consider Cases I to V.

3.3 Numerical experiments

We show the results of numerical experiments to confirm that the explicit scheme in the previous subsection attains weak order 2 when $\alpha_{31}^{(j,0)} = \alpha_{32}^{(j,l,j,j)} = \alpha_{32}^{(l,j,j,l)} = 0$, $A_3^{(j,l,j,j)} = 1$,

$A_3^{(l,j,j,l)} = 1/2$ and $\alpha_{4i_b}^{(j,l,j,j)} = \alpha_{4i_b}^{(l,j,j,l)} = 0$ $(1 \leq i_b \leq 3)$ for $j < l$ and to compare it with Platen's scheme or a scheme for commuting SDEs, which is obtained by setting all $c_i^{(j,l)}$'s $(j \neq l)$, $\alpha_{i_a i_b}^{(j,l,j,j)}$'s and $\alpha_{i_a i_b}^{(l,j,j,l)}$'s $(j < l)$ at 0 in our scheme.

The following two SDEs are considered. The first one is

$$\begin{aligned} \mathrm{d}\boldsymbol{y}(t) &= \left(R - \frac{1}{2}\sum_{j=1}^{m} B_j^2\right)\boldsymbol{y}(t)\mathrm{d}t + \sum_{j=1}^{m} B_j \boldsymbol{y}(t) \circ \mathrm{d}W_j(t), \quad 0 \leq t \leq 1, \\ \boldsymbol{y}(0) &= \boldsymbol{x}_0. \end{aligned} \tag{3. 58}$$

This is non-commutative if $B_j B_l \neq B_l B_j$ $(j \neq l)$. The second one is

$$\begin{aligned} \mathrm{d}\boldsymbol{y}(t) &= \left(R\boldsymbol{y}(t) - \frac{1}{4}\sum_{j=1}^{2}\begin{bmatrix} b_{j1} \\ b_{j2} \end{bmatrix} \frac{\partial Q}{\partial \boldsymbol{y}}(\boldsymbol{y}(t)) \begin{bmatrix} b_{j1} \\ b_{j2} \end{bmatrix}\right)\mathrm{d}t \\ &\quad + \sqrt{Q(\boldsymbol{y}(t))}\sum_{j=1}^{2}\begin{bmatrix} b_{j1} \\ b_{j2} \end{bmatrix} \circ \mathrm{d}W_j(t), \quad 0 \leq t \leq 1, \\ \boldsymbol{y}(0) &= \boldsymbol{x}_0, \end{aligned} \tag{3. 59}$$

where $Q(\boldsymbol{y})$ is a non-negative function. This is non-commutative if $b_{11}b_{22} \neq b_{12}b_{21}$.

In (3. 58), we set $m = 2$,

$$R = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix}, \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(w.p.1)}.$$

Then, we sought $\boldsymbol{y}_M$ by means of the schemes, and calculated the arithmetic variances $\langle y_{M,i}^2 \rangle - \langle y_{M,i} \rangle^2$ of the ith element of $\boldsymbol{y}_M$ and $\langle y_{M,1} y_{M,2} \rangle$ as approximate values of variances $V[y_i(1)]$ $(i = 1, 2)$ and $E[y_1(1)y_2(1)]$, respectively. The notation $\langle \cdot \rangle$ stands for an arithmetic mean. On the other hand, their exact values were sought from $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = RE[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 2 & \frac{1}{16} \\ -3 & -\frac{33}{16} & 1 \\ \frac{1}{16} & -6 & -\frac{63}{16} \end{bmatrix}\begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix}.$$

In (3. 59), we set $b_{11} = 1/2$, $b_{12} = b_{21} = 1/4$, $b_{22} = -1$,

$$R = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \quad Q(\boldsymbol{y}) \stackrel{\text{def}}{=} y_1^2 - y_1 y_2 + y_2^2 + 1, \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(w.p.1)}.$$

The solution satisfies $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = RE[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{27}{16} & \frac{5}{16} \\ -\frac{25}{8} & -\frac{15}{8} & \frac{7}{8} \\ \frac{17}{16} & -\frac{113}{16} & -\frac{47}{16} \end{bmatrix}\begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} + \begin{bmatrix} \frac{5}{16} \\ -\frac{1}{8} \\ \frac{17}{16} \end{bmatrix}.$$

In both experiments, 1×10^6 sets of independent trajectories were simulated for each step. The results are indicated in Figures 3 and 4. The solid, dash or dotted line means our scheme, the scheme for commutating SDEs or Platen's scheme, respectively. The scheme for commuting SDEs is useful to see the influence of non-commutativity of SDEs. The figures illustrate that our scheme is of weak order 2. We can see the influence of non-commutativity in the relative errors of the approximations to $E[y_1(1)y_2(1)]$.

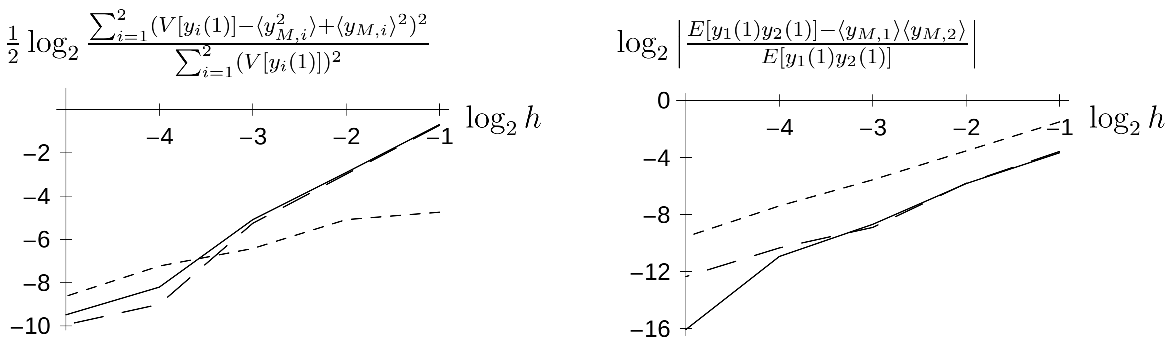


Figure 3: Relative errors in (3. 58)

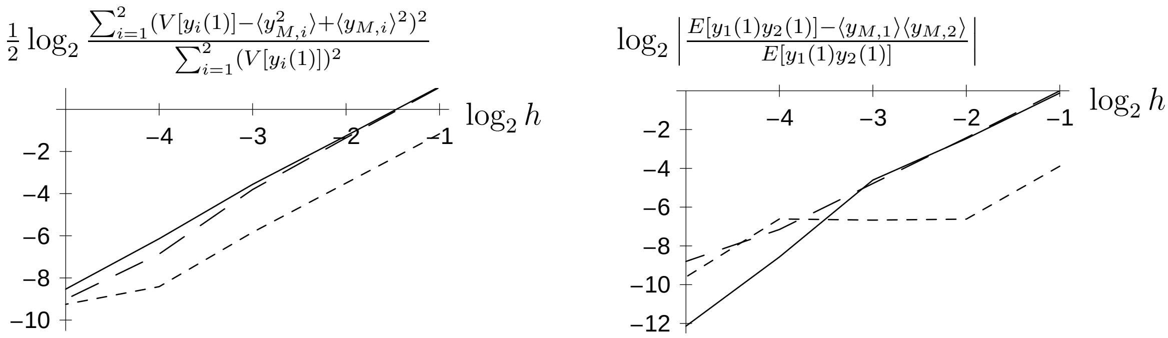


Figure 4: Relative errors in (3. 59)

4 Summary and remarks

First, we have introduced our stochastic Runge-Kutta family and the way of seeking order conditions for it with MRTs. Second, after introducing the ingenious simplifying conditions for the non-commutative case, we have found a solution of all the order conditions. Third, we have performed the numerical experiments and shown the explicit stochastic Runge-Kutta scheme with 4 stages is of weak order 2.

The scheme has the following three features.

- It needs random variables less than Platen's scheme does because it has only $m-1$ random variables ($\triangle\tilde{W}_j$'s) except $\triangle W_j$'s.
- It is of order 4 for ODEs. For this, it can be expected to show better performance in the case of small noise.
- It is directly applicable to non-commuting Stratonovich SDEs, whereas Platen's scheme is for non-commuting Itô SDEs.

Appendix

Expectations of elementary numerical weights

Noting the statement in Subsection 3.2, we show only the expectations that do not vanish, of elementary numerical weights or the products of them for weak order 2. These are obtained directly from a diagrams for MRTL's. (See [7] for details.) For ease of notation, we omit all indices and the range of values of all indices in all summations.

Table 1: Expectations of elementary numerical weights or the products of them

(a) For $t \in \mathcal{T}_{A^{(0)}}$ such that $\rho(\hat{t}) + r(\hat{t}) = 4$

t	$E[\bar{\Phi}_{(m+1)s+1}(t)]$
$[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,0)} \tilde{\alpha}_{i_1i_2}^{(j'_1,0,j'_2,0)} E\left[\tilde{\eta}_{i_1}^{(j'_1,0)} \tilde{\eta}_{i_2}^{(j'_2,0)}\right]$
$[[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,j)} \tilde{\alpha}_{i_2i_3}^{(j'_2,j,j'_3,0)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,0)}\right]$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,0)} \tilde{\alpha}_{i_1i_2}^{(j'_1,0,j'_2,j)} \tilde{\alpha}_{i_2i_3}^{(j'_2,j,j'_3,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,0)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,j)}\right]$
$[[\tau^{(j)}_{A^{(0)}}]^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,0)} \tilde{\alpha}_{i_2i_3}^{(j'_2,0,j'_3,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,0)} \tilde{\eta}_{i_3}^{(j'_3,j)}\right]$
$[\tau^{(j)}_{A^{(0)}}, \tau^{(j)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,0)} \alpha_{i_1i_2}^{(j'_1,0,j'_2,j)} \alpha_{i_1i_3}^{(j'_1,0,j'_3,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,0)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,j)}\right]$
$[\tau^{(0)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,0)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,0)} \tilde{\eta}_{i_3}^{(j'_3,j)}\right]$
$[[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,j)} \tilde{\alpha}_{i_2i_3}^{(j'_2,j,j'_3,j)} \tilde{\alpha}_{i_3i_4}^{(j'_3,j,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,j)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[[[\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,j)} \tilde{\alpha}_{i_2i_3}^{(j'_2,j,j'_3,l)} \tilde{\alpha}_{i_3i_4}^{(j'_3,l,j'_4,l)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,l)}\right]$
$[[[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,l)} \tilde{\alpha}_{i_2i_3}^{(j'_2,l,j'_3,j)} \tilde{\alpha}_{i_3i_4}^{(j'_3,j,j'_4,l)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,l)} \tilde{\eta}_{i_3}^{(j'_3,j)} \tilde{\eta}_{i_4}^{(j'_4,l)}\right]$
$[[[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,l)} \tilde{\alpha}_{i_2i_3}^{(j'_2,l,j'_3,l)} \tilde{\alpha}_{i_3i_4}^{(j'_3,l,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,l)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,j)} \alpha_{i_2i_3}^{(j'_2,j,j'_3,j)} \alpha_{i_2i_4}^{(j'_2,j,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,j)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[[\tau^{(l)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,j)} \alpha_{i_2i_3}^{(j'_2,j,j'_3,l)} \alpha_{i_2i_4}^{(j'_2,j,j'_4,l)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,l)}\right]$
$[[\tau^{(l)}_{A^{(l)}}, \tau^{(j)}_{A^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \tilde{\alpha}_{i_1i_2}^{(j'_1,j,j'_2,l)} \alpha_{i_2i_3}^{(j'_2,l,j'_3,l)} \alpha_{i_2i_4}^{(j'_2,l,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,l)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,j)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,j)} \tilde{\alpha}_{i_3i_4}^{(j'_3,j,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,j)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[\tau^{(j)}_{A^{(j)}}, [\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,j)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,l)} \tilde{\alpha}_{i_3i_4}^{(j'_3,l,j'_4,l)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,l)}\right]$
$[\tau^{(l)}_{A^{(j)}}, [\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,l)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,j)} \tilde{\alpha}_{i_3i_4}^{(j'_3,j,j'_4,l)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,l)} \tilde{\eta}_{i_3}^{(j'_3,j)} \tilde{\eta}_{i_4}^{(j'_4,l)}\right]$
$[\tau^{(l)}_{A^{(j)}}, [\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,l)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,l)} \tilde{\alpha}_{i_3i_4}^{(j'_3,l,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,l)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,j)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,j)} \alpha_{i_1i_4}^{(j'_1,j,j'_4,j)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,j)} \tilde{\eta}_{i_4}^{(j'_4,j)}\right]$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c_{i_1}^{(j'_1,j)} \alpha_{i_1i_2}^{(j'_1,j,j'_2,j)} \alpha_{i_1i_3}^{(j'_1,j,j'_3,l)} \alpha_{i_1i_4}^{(j'_1,j,j'_4,l)} E\left[\tilde{\eta}_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_2}^{(j'_2,j)} \tilde{\eta}_{i_3}^{(j'_3,l)} \tilde{\eta}_{i_4}^{(j'_4,l)}\right]$

(Continuation of Table 1)

(b) For $t \in \mathcal{T}_{A^{(0)}}$ such that $\rho(\hat{t}) + r(\hat{t}) = 3$

t	t_1	$E[\bar{\Phi}_{(m+1)s+1}(t)\bar{\Phi}_{(m+1)s+1}(t_1)]$
$[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,0)}_{i_1 i_2} c^{(j'_3,j)}_{i_3} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,0)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3}\right]$
$[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,0)}_{i_1} \tilde{\alpha}^{(j'_1,0,j'_2,j)}_{i_1 i_2} c^{(j'_3,j)}_{i_3} E\left[\tilde{\eta}^{(j'_1,0)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3}\right]$
$[[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,j,j'_3,j)}_{i_2 i_3} c^{(j'_4,j)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,j,j'_3,l)}_{i_2 i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[[\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,l,j'_3,l)}_{i_2 i_3} c^{(j'_4,j)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,l,j'_3,j)}_{i_2 i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(j)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \alpha^{(j'_1,j,j'_2,j)}_{i_1 i_2} \alpha^{(j'_1,j,j'_3,j)}_{i_1 i_3} c^{(j'_4,j)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[\tau^{(j)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \alpha^{(j'_1,j,j'_2,j)}_{i_1 i_2} \alpha^{(j'_1,j,j'_3,l)}_{i_1 i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[\tau^{(l)}_{A^{(j)}}, \tau^{(l)}_{A^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \alpha^{(j'_1,j,j'_2,l)}_{i_1 i_2} \alpha^{(j'_1,j,j'_3,l)}_{i_1 i_3} c^{(j'_4,j)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$

(c) For $t \in \mathcal{T}_{A^{(0)}}$ such that $\rho(\hat{t}) + r(\hat{t}) = 2$

t	$E[\bar{\Phi}_{(m+1)s+1}(t)]$
$[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2}\right]$
$\tau^{(0)}_{A^{(0)}}$	$\sum c^{(j'_1,0)}_{i_1} E\left[\tilde{\eta}^{(j'_1,0)}_{i_1}\right]$

t	t_1	$E[\bar{\Phi}_{(m+1)s+1}(t)\bar{\Phi}_{(m+1)s+1}(t_1)]$
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(0)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} c^{(j'_3,0)}_{i_3} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,0)}_{i_3}\right]$
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} c^{(j'_3,j)}_{i_3} \tilde{\alpha}^{(j'_3,j,j'_4,j)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$[\tau^{(l)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} c^{(j'_3,l)}_{i_3} \tilde{\alpha}^{(j'_3,l,j'_4,l)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} c^{(j'_3,j)}_{i_3} \tilde{\alpha}^{(j'_3,j,j'_4,l)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$[\tau^{(j)}_{\tilde{A}^{(l)}}]^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} c^{(j'_3,l)}_{i_3} \tilde{\alpha}^{(j'_3,l,j'_4,j)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$\tau^{(0)}_{A^{(0)}}$	$\tau^{(0)}_{A^{(0)}}$	$\sum c^{(j'_1,0)}_{i_1} c^{(j'_2,0)}_{i_2} E\left[\tilde{\eta}^{(j'_1,0)}_{i_1} \tilde{\eta}^{(j'_2,0)}_{i_2}\right]$

t	t_1	t_2	$E[\bar{\Phi}_{(m+1)s+1}(t)\bar{\Phi}_{(m+1)s+1}(t_1)\bar{\Phi}_{(m+1)s+1}(t_2)]$
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} c^{(j'_3,j)}_{i_3} c^{(j'_4,j)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,j)}_{i_1 i_2} c^{(j'_3,l)}_{i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} c^{(j'_3,j)}_{i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$\tau^{(0)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,0)}_{i_1} c^{(j'_2,j)}_{i_2} c^{(j'_3,j)}_{i_3} E\left[\tilde{\eta}^{(j'_1,0)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3}\right]$

(d) For $t \in \mathcal{T}_{A^{(0)}}$ such that $\rho(\hat{t}) + r(\hat{t}) = 1$

t	$E[\{\bar{\Phi}_{(m+1)s+1}(t)\}^2]$
$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} c^{(j'_2,j)}_{i_2} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2}\right]$

t	$E[\{\bar{\Phi}_{(m+1)s+1}(t)\}^4]$
$\tau^{(j)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} c^{(j'_2,j)}_{i_2} c^{(j'_3,j)}_{i_3} c^{(j'_4,j)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$

t	t_1	$E[\{\bar{\Phi}_{(m+1)s+1}(t)\}^2\{\bar{\Phi}_{(m+1)s+1}(t_1)\}^2]$
$\tau^{(j)}_{A^{(0)}}$	$\tau^{(l)}_{A^{(0)}}$	$\sum c^{(j'_1,j)}_{i_1} c^{(j'_2,j)}_{i_2} c^{(j'_3,l)}_{i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,j)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$

References

[1] K. Burrage and P.M. Burrage. General order conditions for stochastic Runge-Kutta methods for both commuting and non-commuting stochastic ordinary differential equation systems. *Appl. Numer. Math.*, **28**:161–177, 1998.

[2] K. Burrage and P.M. Burrage. Order conditions of stochastic Runge-Kutta methods by B-series. *SIAM J. Numer. Anal.*, **38**(**5**):1626–1646, 2000.

[3] K. Burrage, P.M. Burrage, and T. Tian. Numerical methods for strong solutions of stochastic differential equations: an overview. *Proc. R. Soc. Lond. A*, **460**:373–402, 2004.

[4] J.C. Butcher. *Numerical Methods for Ordinary Differential Equations*. John Wiley & Sons, Chichester, 2003.

[5] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Systems*. Springer-Verlag, Berlin, 1996.

[6] P.E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag, New York, 1992.

[7] Y. Komori. Multi-colored rooted tree analysis of the weak order conditions of a stochastic Runge-Kutta family. (submitted to *Appl. Numer. Math.*).

[8] Y. Komori. Weak order stochastic Runge-Kutta methods for commuting stochastic differential equations. (submitted to *J. Comput. Appl. Math.*).

[9] Y. Komori and T. Mitsui. Stable ROW-type weak scheme for stochastic differential equations. *Monte Carlo Methods and Appl.*, **1**(**4**):279–300, 1995.

[10] E. Platen. Higher-order weak approximation of Ito diffusions by Markov chains. *Prob. in the Engineering and Informational Sciences*, **6**:391–408, 1992.

[11] A. Rößler. *Runge-Kutta methods for the numerical solution of stochastic differential equations*. PhD thesis, Darmstadt University of Technology, Germany, 2003.

[12] A. Rößler. Runge-Kutta methods for Stratonovich stochastic differential equation systems with commutative noise. *J. Comput. Appl. Math.*, **164–165**:613–627, 2004.

[13] D. Talay. Simulation of stochastic differential systems. *Probabilistic Methods in Applied Physics*, pages 54–96, 1995.

[14] D. Talay and L. Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Anal. and Appl.*, **8**(4):483–509, 1990.

[15] A. Tocino and J. Vigo-Aguiar. Weak second order conditions for stochastic Runge-Kutta methods. *SIAM J. Sci. Comput.*, **24**(2):507–523, 2002.