

**Robust Output Feedback Disturbance Attenuation  
of Nonlinear Uncertain Dynamic Systems  
via State-Dependent Scaling <sup>¶§</sup>**

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Abstract: This paper presents a novel approach to the problem of output feedback stabilization with  $\mathcal{L}_2$  disturbance attenuation for nonlinear uncertain systems. A new method of state-dependent scaling is introduced into the output feedback design, which unifies treatment of nonlinear and linear gains. The effect of disturbance on the controlled output, which is allowed to be any function of measurement output, can be attenuated to an arbitrarily small level with global asymptotic stability if the plant belongs to a wide class of interconnected systems whose uncertain components unnecessarily have finite linear-gain. The uncertain dynamics is not limited to input-to-state stable systems either. The approach is not only a natural extension of popular approaches in robust linear control, but also advantageous to numerical computation. The design procedure proposed in this paper consists of novel recursive calculation of robust observer gain as well as feedback gain.

Keywords: State-dependent scaling, Nonlinear gain, Output feedback, Robust global stabilization, Almost disturbance decoupling, Robust control Lyapunov function

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# 1 Introduction

Recently, an idea of state-dependent scaling has been introduced into dissipativity-based robust control for nonlinear systems in [4, 5]. The primitive state-dependent scaling[4, 5, 6] aimed at nonlinear uncertain systems whose uncertainties were described as  $\mathcal{L}_2$ -gain balls, while the known part of the system was allowed to have infinite  $\mathcal{L}_2$ -gain. If we have more knowledge on the uncertainty such as nonlinear gain[12, 9, 13, 8] other than linear gain or if the uncertainty does not exhibit finite linear-gain properties, design results based on [4, 5, 6] might be too conservative. As for output feedback control based on state-dependent scaling, the design procedure presented in [6] did not address the existence of globally stabilizing controllers in the presence of dynamic uncertainties although the existence was proved for static uncertainties. The problem of disturbance attenuation was not addressed either. Although the recursive observer design proposed in [6] succeeded in extending the output-feedback form defined in [10] to a slightly wide class of systems, it would be possible to cover a more general class of systems by fully exploiting the unique nested structure of robust observer design.

In the last decade,  $\mathcal{L}_2$ -gain disturbance attenuation with global internal stabilization using full-state information has been extensively studied for linear and nonlinear systems(See [2] and references therein). In comparison, when only the output feedback is allowed, the problem is less understood. For essentially nonlinear systems, filtered transformation and backstepping technique were employed in [11] to solve a problem of output feedback tracking with almost disturbance decoupling. Another approach proposed in [1] resorts to solutions of Hamilton-Jacobi partial differential inequalities and a coupling condition. The recent work [7] has considered a more relaxed class of nonlinear systems than earlier work, and the plant is allowed to involve unmeasured dynamics which is input-to-state stable. The layout of interconnected uncertain systems for which this paper will give a new characterization of the output feedback disturbance attenuation and the existence of solutions is broader than setups considered in those earlier papers. This paper will also allow systems to have both static and dynamic uncertainties including systems which are not input-to-state stable. Nonlinearities allowed in the plant by this paper is more general than those in [10, 6, 7].

The purpose of this paper is to develop a new method of state-dependent(SD) scaling design in order to achieve the output feedback disturbance attenuation with global asymptotic stability for nonlinear systems described by interconnection of nonlinear-gain bounded systems. Thereby, the use of linear gain and nonlinear gain is unified. The design becomes a natural extension of popular techniques in linear robust control to nonlinear systems. The development is considered as a global robustification of the previous results[6] against dynamic uncertainties and its extension to  $\mathcal{L}_2$  disturbance attenuation. Dynamic and static uncertainties are treated in a unified way so that design formulas for the two types of uncertainties are identical. A difference only appears in classes from which scaling factors are chosen. The state-dependent scaling characterization does not require systems to fit in some geometric structure. For interconnected uncertain systems partially in an extended feedback form, the control laws can be systematically generated by selecting parameters of the observer and the feedback gain recursively. The recursive procedure proposed in this paper not only allows us to use nonlinear gains, but also brings in a unique way of

constructing robust observers which enable the output feedback to make the effect of disturbance arbitrarily small with respect to generalized equations of controlled output. The new procedure of observer design proposed in this paper is also unique in that it can be applied to a broader family of nonlinearities in the plant, compared with previous output feedback results in [10, 6, 7]. The design equations are obtained as affine algebraic inequalities with respect to the design parameters, so that the SD scaling approach is advantageous to systematic numerical computation as well as analytical computation.

## 2 System description

Consider the uncertain nonlinear system

$$\Sigma_0 : \begin{cases} \dot{x} = A(y)x + B(y)\bar{w} + G(y)u, & x(t) \in \mathcal{R}^n \\ \bar{z} = C(y)x, & \bar{z}(t), \bar{w}(t) \in \mathcal{R}^{p+q} \\ y = C_y x, & y(t) \in \mathcal{R} \end{cases} \quad (1)$$

The matrices  $A$ ,  $B$ ,  $G$ , and  $C$  are assumed to be  $\mathcal{C}^0$  functions of  $y$ , and  $C_y$  is a constant row vector. Scalars  $u(t)$  and  $y(t)$  are control input and measurement output, respectively. The signals  $\bar{w}$  and  $\bar{z}$  are partitioned as

$$\begin{aligned} w &= \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, & w_i(t), z_i(t) \in \mathcal{R}^{p_i} \\ & & p_i \geq 0 \\ & & p = \sum_{i=1}^m p_i \\ r &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}, & r_i(t), e_i(t) \in \mathcal{R}^{q_i} \\ & & q_i \geq 0 \\ & & q = \sum_{i=1}^m q_i \\ \bar{w}_i &= \begin{bmatrix} w_i \\ r_i \end{bmatrix}, \quad \bar{z}_i = \begin{bmatrix} z_i \\ e_i \end{bmatrix} \in \mathcal{R}^{p_i+q_i}, \quad \bar{w} = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_m \end{bmatrix}, \quad \bar{z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_m \end{bmatrix} \end{aligned}$$

Suppose that there is a system  $\Sigma_\Delta$  described by the following nonlinear mappings between  $z$  to  $w$ .

$$\Delta_i : z_i = \begin{bmatrix} z_{is} \\ z_{id} \end{bmatrix} \mapsto w_i = \begin{bmatrix} w_{is} \\ w_{id} \end{bmatrix}, \quad w_i = \begin{bmatrix} \Delta_{is} & 0 \\ 0 & \Delta_{id} \end{bmatrix} z_i. \quad (2)$$

Here,  $\Delta_{is}$  and  $\Delta_{id}$  represent a time-varying static system and a time-varying dynamic system, respectively. It is unnecessary for  $\Delta_i$  to have the both types. These systems are defined by

$$\Delta_{is} : w_{is} = h_{\Delta_{is}}(z_{is}, t) \quad (3)$$

$$\Delta_{id} : \begin{cases} \dot{x}_{\Delta_i} = f_{\Delta_{id}}(x_{\Delta_i}, z_{id}, t) \\ w_{id} = h_{\Delta_{id}}(x_{\Delta_i}, t) \end{cases} \quad (4)$$

Assume  $f_{\Delta_{id}}(0, 0, t) = 0$ ,  $h_{\Delta_{id}}(0, t) = 0$  and  $h_{\Delta_{is}}(0, t) = 0$  for all  $t \geq 0$ . Functions  $f_{\Delta_{id}}$ ,  $h_{\Delta_{i*}}$  (\* stands for  $s$  or  $d$ ) are locally Lipschitz in  $(x_{\Delta_i}, z_{i*})$  on  $\mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{i*}}$ , uniformly in  $t \in \mathcal{R}$ . The state variable of  $\Sigma_\Delta$  is  $x_\Delta = [x_{\Delta_1}^T, \dots, x_{\Delta_m}^T]^T \in \mathcal{R}^{n_\Delta}$ . The system  $\Sigma_\Delta$  represents uncertainty so that knowledge of  $f_{\Delta_{id}}$ ,  $h_{\Delta_{id}}$  and  $h_{\Delta_{is}}$  is unnecessary. We only assume that information about

nonlinear or linear gain is available in the sense described in Section 3. The interconnected system consisting of  $\Sigma_0$  and  $\Sigma_\Delta$  is denoted by  $\Sigma_P$ .

Since the state variable  $x$  is supposed to be unmeasurable, we employ the following observer to control the uncertain system  $\Sigma_P$ .

$$\begin{cases} \dot{\hat{x}} = A(y)\hat{x} + Y(y, \hat{x})(y - \hat{y}) + G(y)u, & \hat{x}(t) \in \mathcal{R}^n \\ \hat{y} = C_y \hat{x} & , \hat{y}(t) \in \mathcal{R} \end{cases} \quad (5)$$

In this paper, given an arbitrary number  $\tau > 0$ , we seek the output feedback control consisting of (5) and

$$u = K(y, \hat{x})\hat{x}. \quad (6)$$

which

- globally uniformly asymptotically stabilizes  $\Sigma_P$  when  $r \equiv 0$
- makes the mapping between  $r$  and  $e$  have  $\mathcal{L}_2$ -gain less than or equal to  $\tau$

The state variables  $x$  and  $x_\Delta$  are not measured for the feedback control. Functions  $Y$  and  $K$  are  $\mathcal{C}^0$  functions which have yet to be determined. The system  $\Sigma_P$  is said to be globally uniformly asymptotically stabilized if the equilibrium  $x_{cl} = [x^T, x_\Delta^T, \hat{x}^T]^T = 0$  is globally uniformly asymptotically stable. In this paper, the system  $\Sigma_P$  is said to have  $\mathcal{L}_2$ -gain less than or equal to  $\tau$  if there exists a storage function  $V(x_{cl})$  which is positive definite and radially unbounded such that for all initial states  $x_{cl}(0) \in \mathcal{R}^{2n+n_\Delta}$ , and all  $r \in \mathcal{L}_2[0, T]$ , the inequality

$$V(x_{cl}(T)) \leq V(x_{cl}(0)) + \int_0^T (\tau^2 \|r\|^2 - \|e\|^2) dt$$

holds for all  $T \geq 0$ .

### 3 Nonlinearly bounded uncertainty

In this paper, the uncertainty  $\Sigma_\Delta$  is supposed to belong to the following class of nonlinearly bounded systems.

**Assumption 1** For each  $i = 1, 2, \dots, m$ , the uncertain system  $\Sigma_\Delta$  satisfies the following.

(a) There exists a  $\mathcal{C}^0$  function  $\psi_{is} : [0, \infty) \rightarrow [0, \infty)$  such that

$$\|w_{is}\|^2 \leq \psi_{is}(\|z_{is}\|) \|z_{is}\|^2 \quad (7)$$

holds for all  $t \in [0, \infty)$ .

(b) There exists a  $\mathcal{C}^0$  function  $\psi_{id} : [0, \infty) \rightarrow [0, \infty)$  and a  $\mathcal{C}^1$  function  $W_{\Delta i} : [0, \infty) \times \mathcal{R}^{n_{\Delta i}} \rightarrow \mathcal{R}$  such that

$$\underline{\beta}_i(\|x_{\Delta i}\|) \leq W_{\Delta i}(t, x_{\Delta i}) \leq \bar{\beta}_i(\|x_{\Delta i}\|) \quad (8)$$

$$\frac{\partial W_{\Delta i}}{\partial t} + \frac{\partial W_{\Delta i}}{\partial x_{\Delta i}} f_{\Delta i} \leq -\beta_i(x_{\Delta i}) - \|w_{id}\|^2 + \psi_{id}(\|z_{id}\|) \|z_{id}\|^2 \quad (9)$$

hold for all  $(t, x_{\Delta_i}, z_{id}) \in [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{id}}$ , where  $\underline{\beta}_i$  and  $\bar{\beta}_i$  are class  $\mathcal{K}_\infty$  functions, and  $\beta_i$  is a positive definite  $\mathcal{C}^0$  function of  $x_{\Delta_i}$ .

A system  $\Sigma_\Delta$  is said to be admissible if Assumption 1 is true. The assumption does not require uncertain systems to have finite  $\mathcal{L}_2$ -gain. Instead, they are supposed to have finite nonlinear-gain. When  $\Delta_{is}$  ( $\Delta_{id}$ ) exhibits finite  $\mathcal{L}_2$ -gain, the parameter  $\psi_{is}$  ( $\psi_{id}$ , respectively) reduces to a positive constant. In such a case, we obtain  $\psi_{is} = \gamma_{is}^2$  and  $\psi_{id} = \gamma_{id}^2$ , where  $\gamma_{is}$  and  $\gamma_{id}$  are  $\mathcal{L}_2$ -gain. The new class of uncertain systems is broad and it includes input-to-state stable(ISS) systems[12] in the following sense.

**Lemma 1 (a)** *Suppose that a static system  $\Delta_{is}$  admits class  $\mathcal{K}_\infty$  functions  $\alpha_i$  and  $\sigma_i$  such that*

$$\alpha_i(\|w_{is}\|) \leq \sigma_i(\|z_{is}\|) \quad (10)$$

*holds for all  $t \in [0, \infty)$  and*

$$\lim_{s \rightarrow 0^+} \frac{\sigma_i(s)}{\alpha_i(s)} < +\infty \quad (11)$$

*holds. Then, there exists a  $\mathcal{C}^0$  function  $\psi_{is}$  such that (7) holds for all  $t \in [0, \infty)$ .*

**(b)** *Suppose that a dynamic system  $\Delta_{id}$  admits a  $\mathcal{C}^1$  function  $V_{\Delta_i} : [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \rightarrow \mathcal{R}$  such that*

$$\underline{\alpha}_i(\|x_{\Delta_i}\|) \leq V_{\Delta_i}(t, x_{\Delta_i}) \leq \bar{\alpha}_i(\|x_{\Delta_i}\|) \quad (12)$$

$$\frac{\partial V_{\Delta_i}}{\partial t} + \frac{\partial V_{\Delta_i}}{\partial x_{\Delta_i}} f_{\Delta_{id}} \leq -\alpha_i(\|x_{\Delta_i}\|) + \sigma_i(\|z_{id}\|) \quad (13)$$

*are satisfied for all  $(t, x_{\Delta_i}, z_{id}) \in [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{id}}$  where  $\underline{\alpha}_i$ ,  $\bar{\alpha}_i$  and  $\alpha_i$  are class  $\mathcal{K}_\infty$  functions and  $\sigma_i$  is a class  $\mathcal{K}$  function and they satisfy*

$$\lim_{\|x_{\Delta_i}\| \rightarrow 0} \frac{\|w_{id}\|^2}{\alpha_i(\|x_{\Delta_i}\|)} < +\infty, \quad \lim_{\|z_{id}\| \rightarrow 0} \frac{\sigma_i(\|z_{id}\|)}{\|z_{id}\|^2} < +\infty \quad (14)$$

*uniformly in  $t$ . Then, there exists a  $\mathcal{C}^0$  function  $\psi_{id}$ , a  $\mathcal{C}^1$  function  $W_{\Delta_i}$ , class  $\mathcal{K}_\infty$  functions  $\underline{\beta}_i$  and  $\bar{\beta}_i$  and a positive definite  $\mathcal{C}^0$  function  $\beta_i$  such that (8) and (9) hold for all  $(t, x_{\Delta_i}, z_{id}) \in [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{id}}$ .*

The functions  $\psi_{is}$  and  $\psi_{id}$  are obtained easily from  $\alpha_i$ ,  $\sigma_i$ ,  $\underline{\alpha}_i$  and  $\bar{\alpha}_i$  [3]. Note that  $\beta_i(x_{\Delta_i})$  can be always chosen as a class  $\mathcal{K}_\infty$  function of  $\|x_{\Delta_i}\|$  for ISS systems defined in Lemma 1(b). It is emphasized that Assumption 1 admits systems which are not ISS. An example of nonlinearly bounded static mappings which violate (11) is  $w_{is} = h_{\Delta_{is}}(z_{is}) = \sqrt{\|z_{is}\|}$  which is not Lipschitz at  $z_{is} = 0$ . Indeed, if  $h_{\Delta_{is}}(z_{is}, t)$  is Lipschitz at  $z_{is} = 0$  uniformly in  $t$  as assumed in Section 2, there always exists a class  $\mathcal{K}_\infty$  pair of  $\{\alpha(s), \sigma(s)\}$  satisfying (11) and (10). As for a dynamic nonlinear mapping  $\Delta_{id} : z_{id} \mapsto w_{id}$ , the condition (14) together with (13) is common in asymptotic analysis based on the nonlinear small-gain technique[7, 8, 9]. It is known that (14) is always satisfied for appropriate functions  $\alpha_i \in \mathcal{K}_\infty$  and  $\sigma_i \in \mathcal{K}$  if the Jacobian linearization of  $\Delta_{id}$  at  $x_{\Delta_i} = 0$  is uniformly asymptotically stable.

**Example 1** *An example of admissible uncertain dynamics  $z_{id} \mapsto w_{id}$  is*

$$\begin{cases} \dot{x}_{\Delta_i} = -x_{\Delta_i}(1 - z_{id}^\rho) \\ w_{id} = \text{sat}(x_{\Delta_i}) \end{cases}$$

with  $\rho \geq 2$ . Indeed, it satisfies (9) for  $\psi_{id} = 2.2|z_{id}|^{\rho-2}$  and

$$W_{\Delta_i} = \int_0^{x_{\Delta_i}^2} \frac{1.1}{s} ds, \quad \beta_i = \frac{x_{\Delta_i}^2}{5(x_{\Delta_i}^2 + 1)}$$

Clearly, the system is not input-to-state stable although it is globally asymptotically stable when  $z_{id} \equiv 0$ .

**Example 2** The following input-to-state stable system

$$\begin{cases} \dot{x}_{\Delta_i} = -x_{\Delta_i}^3 + z_{id}^2 \\ w_{id} = x_{\Delta_i}^\rho \end{cases}$$

which is used in [7] with  $\rho \geq 2$  is also an admissible dynamic system. The functions  $\psi_{id}$  and  $\beta_i$  are obtained as

$$\begin{aligned} \psi_{id} &= \tau \pi^{\rho/2-1} 3^{\rho/2} 2^{-2\rho/3} |z_{id}|^{4\rho/3-2} \\ \beta_i &= (1 - \tau^{-1} - \pi^{-1}) \tau |x_{\Delta_i}|^{2\rho} \end{aligned}$$

for any  $\tau > 1$  and  $\pi > \tau/(\tau - 1)$ .

## 4 SD scaling characterization

This section derives a characterization of global robustness properties of the output feedback system described in Section 2 via a new concept of state-dependent (SD) scaling which incorporates the nonlinear gain.

First, a set of scaling factors associated with static uncertain components  $\Delta_{is}$  is defined by

$$\mathbf{\Phi}_{is} = \left\{ \Phi_{is}(y, \hat{x}) = \phi_{is}(y, \hat{x}) I : \begin{array}{l} \phi_{is}(\cdot) \in \mathcal{C}^0 \\ \phi_{is}(y, \hat{x}) > 0, \forall (y, \hat{x}) \in \mathcal{R}^{n+1} \end{array} \right\} \quad (15)$$

The identity matrix  $I$  is compatible in size with  $z_{is}$ . The scaling factors are functions of the output and the state estimate. For dynamic uncertain components  $\Delta_{id}$ , a set of scaling factors is defined by

$$\mathbf{\Phi}_{id} = \left\{ \Phi_{id} = \phi_d \begin{bmatrix} \check{\phi}_{id} I & 0 \\ 0 & I \end{bmatrix} : \phi_d, \check{\phi}_{id} > 0 \right\} \quad (16)$$

The block partition of  $\mathbf{\Phi}_{id}$  is compatible in size with that of  $[z_{id}^T, e_i^T]^T$ . All sets  $\mathbf{\Phi}_{id}$ ,  $i = 1, 2, \dots, m$  are defined with a common constant  $\phi_d$ . For  $i = 1, 2, \dots, m$ , define  $\mathbf{\Phi}_i(x)$  as

$$\mathbf{\Phi}_i = \left\{ \Phi_i(y, \hat{x}) = \begin{bmatrix} \Phi_{is}(y, \hat{x}) & 0 \\ 0 & \Phi_{id} \end{bmatrix} : \begin{array}{l} \Phi_{is} \in \mathbf{\Phi}_{is} \\ \Phi_{id} \in \mathbf{\Phi}_{id} \end{array} \right\}$$

Using  $\mathcal{C}^0$  functions  $\psi_{id}, \psi_{is} : [0, \infty) \rightarrow [0, \infty)$  in Assumption 1, define  $\bar{\Psi}(x)$  as

$$\bar{\Psi}(x) = \text{block-diag}_{i=1}^m \bar{\Psi}_i(x) \quad (17)$$

$$\bar{\Psi}_i(x) = \begin{bmatrix} \psi_{is}(\|z_{is}\|)^{1/2} I & 0 & 0 \\ 0 & \psi_{id}(\|z_{id}\|)^{1/2} I & 0 \\ 0 & 0 & \tau^{-1} I \end{bmatrix} \quad (18)$$

The block diagonal structure of  $\bar{\Psi}_i$  is conformable in size to the partition  $\bar{z}_i = [z_{is}^T, z_{id}^T, e_i^T]^T$ . The scalar  $\tau$  is a positive number to describe the level of disturbance attenuation. We are now ready to define three sets of SD scaling matrices by

$$\Phi = \left\{ \Phi(y, \hat{x}) = \text{block-diag}_{i=1}^m \Phi_i(y, \hat{x}), \Phi_i \in \Phi_i \right\} \quad (19)$$

$$\Theta = \left\{ \Theta(y, \hat{x}) : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^{(p+q) \times (p+q)}, \Theta(\cdot) \in \mathcal{C}^0 \right. \\ \left. \Theta(y, \hat{x}) > 0 \forall (y, \hat{x}) \in \mathcal{R}^{n+1} \right\} \quad (20)$$

$$\Psi = \left\{ \Psi(x, \hat{x}) : \mathcal{R}^{2n} \rightarrow \mathcal{R}^{(p+q) \times (p+q)}, \Psi(\cdot) \in \mathcal{C}^0 \right. \\ \left. \Psi(x, \hat{x}) \geq 0 \forall (x, \hat{x}) \in \mathcal{R}^{2n} \right\} \quad (21)$$

All scaling matrices  $\Phi$ ,  $\Theta$  and  $\Psi$  are ‘state-dependent’.

Based on the triplet of these scaling matrices, we shall characterize stability and  $\mathcal{L}_2$  disturbance attenuation of  $\Sigma_P$ . Consider a global diffeomorphism between  $[\hat{x}^T, \hat{x}^T - x^T]^T \in \mathcal{R}^{2n}$  and  $[\hat{\chi}^T, \eta^T]^T \in \mathcal{R}^{2n}$  as follows:

$$\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} = \begin{bmatrix} S(y, \hat{x}) & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix} \quad (22)$$

where  $W$  is a constant matrix. The time-derivative of  $\hat{\chi}$  is obtained as

$$\begin{aligned} \dot{\hat{\chi}} &= \frac{\partial S}{\partial y} \hat{x} C_y \dot{x} + \left[ \frac{\partial S}{\partial \hat{x}_1} \hat{x}, \frac{\partial S}{\partial \hat{x}_2} \hat{x}, \dots, \frac{\partial S}{\partial \hat{x}_n} \hat{x} \right] \dot{\hat{x}} + S(y, \hat{x}) \dot{\hat{x}} \\ &= X(y, \hat{x}) \dot{x} + T(y, \hat{x}) \dot{\hat{x}} \end{aligned}$$

Define

$$\bar{A} = \begin{bmatrix} C_y^T & A^T \end{bmatrix}, \hat{W} = \begin{bmatrix} -Y^T W^T \\ W^T \end{bmatrix}, \hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \hat{A} = \begin{bmatrix} A & G \end{bmatrix}$$

Then, we obtain the following theorem.

**Theorem 1** *If there exist  $P > 0$ ,  $\tilde{P} > 0$  and scaling matrices  $\Phi \in \Phi$ ,  $\Theta \in \Theta$ ,  $\Psi \in \Psi$  such that*

$$M(y, \hat{x}) = \begin{bmatrix} \hat{S}^T \hat{A}^T (X + T)^T P + P (X + T) \hat{A} \hat{S} & P X B \\ B^T X^T P & -\Theta \\ \Phi \Psi C S^{-1} & 0 \\ -W^{-T} (X A + T Y C_y)^T P & -\tilde{P} W B \\ S^{-T} C^T \Psi \Phi & -P (X A + T Y C_y) W^{-1} \\ 0 & -B^T W^T \tilde{P} \\ -\Phi & -\Phi \Psi C W^{-1} \\ -W^{-T} C^T \Psi \Phi & W^{-T} \bar{A} \hat{W} \tilde{P} + \tilde{P} \hat{W}^T \bar{A}^T W^{-1} \end{bmatrix} < 0 \quad (23)$$

$$\Theta \leq \Phi \quad (24)$$

$$\bar{\Psi} \leq \Psi \quad (25)$$

hold for all  $(x, \hat{x}) \in \mathcal{R}^{2n}$ , the output-feedback law (5-6) globally uniformly asymptotically stabilizes  $\Sigma_P$  for all admissible uncertainties and  $\Sigma_P$  has  $\mathcal{L}_2$ -gain less than or equal to  $\tau$ .

In the case of  $\{q = 0, \phi_d = 1, \Theta = \Phi, \Psi = I\}$ , Theorem 1 reduces to the primitive result[6].

## 5 Recursive design

This section defines a class of systems  $\Sigma_P$  and presents a recursive procedure of the output feedback control design for this class. Suppose that  $\Sigma_0$  is in the following triangular structure.

$$y = x_1, \quad C_y = [1 \ 0 \ \cdots \ 0] \quad (26)$$

$$A(y) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad G(y) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \quad (27)$$

$$a_{i,i+1}(y) \neq 0, \quad 1 \leq i \leq n, \quad \forall y \in \mathcal{R} \quad (28)$$

$$B(y) = \begin{bmatrix} B_{11} \\ \vdots \\ B_{n1} \end{bmatrix}, \quad C(y) = [C_{11} \ 0 \ \cdots \ 0] \quad (29)$$

where  $B_{i1}(y) \in \mathcal{R}^{1 \times (p_1+q_1)}$ ,  $C_{11}(y) \in \mathcal{R}^{(p_1+q_1) \times 1}$ ,  $m = 2n$ ,  $p_1 = p$ ,  $q_1 = q$ . In addition, we assume that

$$\sup_{y \in \mathcal{R}} \left| \frac{a_{ij}^2(y)}{a_{k-1,k}(y)a_{l,l+1}(y)} \right| < +\infty, \quad \begin{array}{l} 2 \leq k \leq n-1 \\ k \leq l \leq n-1 \\ k \leq i \leq n, \ k \leq j \leq n \end{array} \quad (30)$$

$$\sup_{y \in \mathcal{R}} \frac{a_{nn}(y)}{|a_{n-1,n}(y)|} < +\infty \quad (31)$$

The conditions (30) and (31) will be used for ensuring the existence of global solutions to the observer design problem described later. In this paper, a system  $\Sigma_P$  consisting of  $\Sigma_0$  and  $\Sigma_\Delta$  which fulfill these structural assumptions (26-31) and Assumption 1, respectively, is said to be in the generalized robust output-feedback form. Compared with a standard output-feedback form defined in [10], the generalized robust output-feedback form not only allows for disturbance signals and dynamic uncertain components which unnecessarily have finite linear-gain, but also nonlinearity is not restricted to  $A(y)x = A_0x + A_1(y)$  where  $A_0$  is a constant matrix. The class of generalized robust output-feedback form is also broader than an extended class considered in [6]. When the nonlinearity is limited to  $A(y)x = A_0x + A_1(y) + A_2(y)x_2$  with a constant matrix  $A_0$ , the conditions (30) and (31) reduces to

$$\begin{aligned} \sup_{y \in \mathcal{R}} \left| a_{i2}^2(y)/a_{12}(y) \right| < +\infty, \quad 2 \leq i \leq n \\ \inf_{y \in \mathcal{R}} |a_{12}(y)| \neq 0 \end{aligned}$$

which are assumptions employed in [6].

In order to solve the disturbance attenuation problem with robust stability for the above class of systems, we first pick any constant matrices  $P$  and  $\tilde{P}$  of the form

$$P = \text{diag}_{i=1}^n P_i > 0, \quad \tilde{P} = \text{diag}_{i=1}^n \tilde{P}_i > 0 \quad (32)$$

Define

$$S^{-1}(x_1, \hat{x}_{[n-2]}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 1 \end{bmatrix} \quad (33)$$



$$u = s_n(x_1, \hat{x}_{[n-1]}) \hat{\chi}_n \quad (34)$$

Here,  $s_1(x_1)$ ,  $s_2(x_1, \hat{x}_1)$ , ...,  $s_n(x_1, \hat{x}_{[n-1]})$  are smooth functions to be determined. The notation

$$\hat{x}_{[k]} = [\hat{x}_1 \ \hat{x}_2 \ \cdots \ \hat{x}_k]^T$$

is used. Let  $W$  be

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ w_2 & 1 & 0 & \cdots & 0 \\ 0 & w_3 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & w_n & 1 \end{bmatrix} \quad (35)$$

whose components  $w_i$  for  $2 \leq i \leq n$  are constant. Let the observer gain  $Y$  be of the form

$$Y(x_1) = -W^{-1} \begin{bmatrix} w_1(x_1) \\ 0 \end{bmatrix} \quad (36)$$

where  $w_1$  is a smooth scalar function of  $x_1$ . The parameters  $w_1, \dots, w_n$  have yet to be determined. Candidates of state-dependent scaling matrices are parameterized as follows:

$$\Phi = \{\Phi = \phi_1(x_1) I_{p+q} : \phi_1(x_1) > 0, \forall x_1 \in \mathcal{R}\} \quad (37)$$

$$\Theta = \{\Theta = \phi_1(x_1) I_{p+q} : \phi_1(x_1) > 0, \forall x_1 \in \mathcal{R}\} \quad (38)$$

The scalar function  $\phi_1$  has yet to be determined. Choose a matrix  $\Psi$  from  $\Psi$  so that (25) holds and  $\Psi$  depends only on  $x_1$ . Such a SD scaling matrix  $\Psi$  exists due to the definition of  $C$ . A simple choice is  $\Psi = \bar{\Psi}$ . Extract  $M_{[k]}$  from  $M$  as

$$\begin{aligned} M_{[k]} &= \begin{bmatrix} Q_k^T & 0 \\ 0 & I_n \end{bmatrix} M \begin{bmatrix} Q_k & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} M_{[k]11} & M_{[k]12} \\ M_{[k]21} & H \end{bmatrix} \\ H &= W^{-T} \bar{A} \bar{W} \bar{P} + \bar{P} \bar{W}^T \bar{A}^T W^{-1} \\ Q_k &= \begin{bmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{p+q} & 0 \\ 0 & 0 & I_{p+q} \end{bmatrix}, \quad Q_n = I_{n+2(p+q)} \end{aligned}$$

where  $I_k$  is a  $k \times k$  identity matrix. This matrix  $M_{[k]}$  has the following properties.

- (a-i)  $M_{[k]}$  is independent of  $\{\hat{x}_k, \hat{x}_{k+1}, \dots, \hat{x}_n\}$ .
- (a-ii)  $M_{[k]}$  does not include  $\{s_{k+1}, \dots, s_{n-1}, s_n\}$ .
- (a-iii)  $M_{[k]} < 0$  implies  $M_{[k-1]} < 0$ .
- (a-iv)  $M_{[n]} = M$
- (a-v)  $M_{[1]}$  is jointly affine in  $\{s_1, \phi_1\}$ . For  $k \geq 2$ ,  $M_{[k]}$  is affine in  $s_k$ .
- (a-vi)  $M_{[k]} < 0$  implies  $H < 0$ .

For achieving  $M < 0$ , the properties suggests a recursive procedure in which

$$M_{[k]}(x_1, \hat{x}_{[k-1]}) < 0, \quad \forall (x_1, \hat{x}_{[k-1]}) \in \mathcal{R} \times \mathcal{R}^{k-1} \quad (39)$$

is solved for  $s_k$ , and  $\phi_1$  (when  $k = 1$ ), recursively from  $k = 1$  through  $k = n$ . The property (a-vi) claims that  $H < 0$  should be secured beforehand. In order to obtain  $H < 0$ , employing an idea

which is analogous to recursive design of robust observers [6], the parameters  $\{w_1(x_1), w_2, \dots, w_n\}$  are selected by solving

$$H_{\langle k \rangle}(x_1) < -\Gamma_k(x_1)^{-1}, \quad \forall x_1 \in \mathcal{R} \quad (40)$$

for  $w_k$  recursively from  $k = n$  down to  $k = 1$ . The notation  $H_{\langle k \rangle}$  is defined as

$$H_{\langle k \rangle} = \begin{bmatrix} H_{k,k} & H_{k,*} \\ H_{*,k} & H_{\langle k+1 \rangle} \end{bmatrix}, \quad H_{\langle n \rangle} = H_{n,n}$$

The next section shows how to determine appropriate matrices  $\Gamma_k(x_1) > 0$  of  $\mathcal{C}^0$  functions which guarantee the existence of solutions to (39) for all  $k = 1, 2, \dots, n$ . The matrix  $H_{\langle k \rangle}$  also satisfies the following.

- (b-i)  $H_{\langle k \rangle}$  does not include  $\{w_{k-1}, \dots, w_2, w_1\}$ .
- (b-ii)  $H_{\langle k \rangle} < -\Gamma_k^{-1}$  implies  $H_{\langle k+1 \rangle} < -[\Gamma_k^{-1}]_{\langle k+1 \rangle}$
- (b-iii)  $H_{\langle 1 \rangle} = H$
- (b-iv)  $H_{\langle k \rangle}$  is affine in  $w_k$ .

The properties (a-v) and (b-iv) are advantageous to numerical computation of (39) and (40).

## 6 Existence of solution

Let  $[H^{-1}]_{11}$  denote the (1,1)-component of the matrix  $H^{-1}$ . The following can be obtained by modifying a result in [6] properly.

**Lemma 2** *Suppose that  $H(x_1) < 0$  is satisfied for all  $x_1 \in \mathcal{R}$ .*

**(i) Case  $k = 1$  :** *There exist smooth functions  $\{s_1(x_1), \phi_1(x_1)\}$  such that (39) is satisfied if*

$$-[H^{-1}]_{11} \lambda_{\max}(-B^T W^T \tilde{P} H^{-1} \tilde{P} W B) \lambda_{\max}(\Psi C_{11} C_{11}^T \Psi) < \frac{1}{4} \quad (41)$$

*holds for all  $x_1 \in \mathcal{R}$ .*

**(ii) Case  $k \geq 2$  :** *Assume that  $M_{[k-1]} < 0$  holds for all  $(x_1, \hat{x}_{[k-2]}) \in \mathcal{R} \times \mathcal{R}^{k-2}$ . Then, there exists a smooth function  $s_k(x_1, \hat{x}_{[k-1]})$  such that (39) is satisfied.*

Here,  $\lambda_{\max}(\cdot)$  denotes the maximum eigenvalue of a matrix. The inequality (39) is solvable recursively from  $k = 1$  through  $k = n$  if  $H(x_1) < 0$  and (41) are satisfied for all  $x_1$ . Let

$$\Gamma_k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \nu_k \Gamma_{k+1} \end{bmatrix}, \quad k = 1, 2, \dots, n-1, \quad \Gamma_n = \gamma_n$$

where  $\gamma_i, i = 1, 2, \dots, n$  and  $\nu_i, i = 1, 2, \dots, n-1$  are real scalars. In order to achieve (40) and (41) simultaneously, the following lemma is useful, which successfully extends the previous result of the robust observer design [6] to much more general systems defined with the relaxed assumptions (30-31).

**Lemma 3** Let  $\{\gamma_1(x_1), \gamma_2(x_1), \dots, \gamma_n(x_1)\}$  be any  $\mathcal{C}^0$  functions satisfying

$$0 < \gamma_i(x_1), \quad \forall x_1 \in \mathcal{R}, \quad i = 1, 2, \dots, n \quad (42)$$

$$0 < \inf_{x_1 \in \mathcal{R}} \{\gamma_i(x_1)|a_{i-1,i}(x_1)|\}, \quad i = 2, 3, \dots, n \quad (43)$$

$$\sup_{x_1 \in \mathcal{R}} \{\gamma_i(x_1)|a_{i-1,i}(x_1)|\} < +\infty, \quad i = 3, 4, \dots, n \quad (44)$$

Let  $\{\nu_1, \nu_2, \dots, \nu_{n-1}\}$  be any constants satisfying

$$\nu_1 \geq 1, \quad \nu_i > 1, \quad i = 2, 3, \dots, n-1 \quad (45)$$

Then, there exist a smooth function  $w_1(x_1)$  and constants  $w_2, w_3, \dots, w_n$  which solve (40) sequentially in descending order of  $k$ , where the existence of  $w_k$  is independent of  $\{\gamma_1, \gamma_2, \dots, \gamma_{k-1}\}$  and  $\{w_1, w_2, \dots, w_{k-1}\}$ .

Pick a  $\mathcal{C}^0$  function  $\gamma_1(x_1)$  such that

$$\gamma_1 \lambda_{\max}(B^T W^T \tilde{P} \Gamma_1 \tilde{P} W B) \lambda_{\max}(\Psi C_{11} C_{11}^T \Psi) < \frac{1}{4} \quad (46)$$

If  $-H^{-1} < \Gamma_1$  is achieved, this inequality implies (41). Thus, if we select  $\gamma_1, \dots, \gamma_n$  as (46), (42-44), Lemma 2 proves that  $s_k$  and  $\phi_1$  solving  $M < 0$  can be constructed recursively from  $k = 1$  up to  $k = n$ .

According to the proof of Lemma 2, any  $\mathcal{C}^0$  function satisfying

$$\bar{e}_- < \phi_1(x_1) < \bar{e}_+, \quad x_1 \in \mathcal{R} \quad (47)$$

is a solution of  $M_{[1]} < 0$ . The real numbers  $\bar{e}_-$  and  $\bar{e}_+$  are given by

$$\bar{e}_{\pm} = \frac{1 + \bar{a}\bar{c} - \bar{b} \pm \sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}}}{2\bar{c}} \quad (48)$$

where

$$\begin{aligned} \bar{a} &= \lambda_{\max}(-B^T W^T \tilde{P} H^{-1} \tilde{P} W B) \\ \bar{b} &= \lambda_{\max}(Z_b^T Z_b), \quad Z_b = -B^T W^T \tilde{P} H^{-1} C^T \Psi \\ \bar{c} &= \lambda_{\max}(-\Psi C_{11} [H^{-1}]_{11} C_{11}^T \Psi) \end{aligned}$$

Set  $\phi_{1s} = \phi_1$ . Then, according to Theorem 1, we achieve the condition (23) for global asymptotic stability when  $\Sigma_P$  has neither dynamic uncertain components nor exogenous disturbances. However, Theorem 1 requires a constant  $\phi_1$  when either dynamic uncertain components or exogenous disturbances is involved. Lemma 2 does not guarantee that the set  $(\bar{e}_-, \bar{e}_+)$  admits a constant solution  $\phi_1$  globally in  $x_1$ . The following new result is the key to the existence of constant  $\phi_1$ .

**Lemma 4** Let  $\{\nu_1, \nu_2, \dots, \nu_{n-1}\}$  be any positive real constants. Suppose that  $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$  satisfy (42) and

$$\sup_{x_1 \in \mathcal{R}} \{\gamma_i(x_1)|a_{i-1,i}(x_1)|\} < +\infty, \quad i = 2, 3, \dots, n \quad (49)$$

(a) If  $B(x)$  satisfies

$$\sup_{x_1 \in \mathcal{R}} \frac{\lambda_{\max}(B_{i1}(x_1)B_{i1}^T(x_1))}{|a_{i,i+1}(x_1)|} < +\infty, \quad i = 1, 2, \dots, n-1 \quad (50)$$

$$\sup_{x_1 \in \mathcal{R}} \frac{\lambda_{\max}(B_{n1}(x_1)B_{n1}^T(x_1))}{|a_{n-1,n}(x_1)|} < +\infty \quad (51)$$

then, there exists a  $\mathcal{C}^0$  function  $\gamma_1(x_1)$  such that (42) and

$$\lambda_{\max}(B^T W^T \tilde{P} \Gamma_1 \tilde{P} W B) \leq \alpha, \quad \forall x_1 \in \mathcal{R} \quad (52)$$

$$\gamma_1 \lambda_{\max}(\Psi C_{11} C_{11}^T \Psi) < \frac{1}{4\alpha}, \quad \forall x_1 \in \mathcal{R} \quad (53)$$

hold for a finite constant  $\alpha > 0$ .

(b) If  $\gamma_1(x_1)$  satisfies (42), (52) and (53) and  $H < -\Gamma_1^{-1}$  holds, there exists a positive constant  $\phi_1$  such that (47) holds.

According to the proof, a constant solution  $\phi_1$  fulfilling (47) for ‘all’  $x_1 \in \mathcal{R}$  is any real number belonging to  $(\hat{e}_-, \hat{e}_+)$ , where constants  $\hat{e}_-$  and  $\hat{e}_+$  are given by

$$\hat{e}_{\pm} = \frac{1 + \hat{a}\hat{c} - \hat{b} \pm \sqrt{(1 + \hat{a}\hat{c} - \hat{b})^2 - 4\hat{a}\hat{c}}}{2\hat{c}} \quad (54)$$

$$\hat{a} = \sup_{x_1 \in \mathcal{R}} \bar{a}(x_1), \quad \hat{b} = \sup_{x_1 \in \mathcal{R}} \bar{b}(x_1), \quad \hat{c} = \sup_{x_1 \in \mathcal{R}} \bar{c}(x_1)$$

Here,  $\hat{a}$ ,  $\hat{b}$  and  $\hat{c}$  are guaranteed to exist. Thanks to Lemma 4, the recursive design of output feedback controllers which accommodate nonlinear dynamic uncertainties proceeds as follows.

- 1) solve (40) for  $w_k$  recursively in descending order of  $k$  with  $\gamma_n, \dots, \gamma_2, \gamma_1$  given in (49) (43), (42), (52), (53) and (45).
- 2) solve (39) for  $s_k$  and a constant  $\phi_1$  (when  $k = 1$ ) recursively in ascending order of  $k$ .

Using Schur complements formula, design equations (39) and (40) in each step  $k$  reduce to scalar inequalities which are affine in the decision variables. We thereby arrive at the main result by setting  $\phi_{1s} = \phi_d = \phi_1$  and  $\check{\phi}_{1d} = 1$ .

**Theorem 2** Suppose that the system  $\Sigma_P$  is in the generalized robust output-feedback form and satisfies (50-51). Then, the system  $\Sigma_P$  can be globally uniformly asymptotically stabilized, and the  $\mathcal{L}_2$ -gain from  $r$  to  $e$  can be rendered less than or equal to  $\tau$  for all admissible uncertainties  $\Sigma_{\Delta}$  by the output-feedback law (5-6).

It is emphasized that the disturbance attenuation level  $\tau$  can be made arbitrarily small. The scaling factor  $\phi_1$  can be chosen as either a constant or a function of  $x_1$ . The uncertainty  $\Sigma_{\Delta}$  is not allowed to be dynamic unless  $\phi_1$  is a constant. Non-constant  $\phi_1$  may often lead us to a less complicated feedback gain whose growth order and local gain are not very large.

It is worth noting that Theorem 2 does not restrict  $\Sigma_{\Delta}$  to ISS dynamic systems. When we restrict  $\Sigma_{\Delta}$  to ISS dynamics considered in [7], the outcome of Theorem 2 is similar to Corollary 1 of [7].

However, it is emphasized that this paper allows more general equations of the regulated output  $e(t)$  than [7] as well as matrices  $A$ ,  $B$  and  $G$ . This successful generalization of the functions  $C$  and  $B$  is mainly due to the unique idea of robust observers employed in this paper. Indeed, it is observed from (53), (52) and (40). Robust observers are systematically constructed in a recursive manner. The result of this paper is also applicable to  $\Sigma_0$  whose parameter  $B(y)$  is replaced by  $B(t, x_\Delta, y)$  if  $B(t, x_\Delta, y)$  is uniformly bounded in  $t$  and  $x_\Delta$ . In addition, under the assumption (7) of [7], the methodology proposed in this paper can lead us to a result corresponding to Theorem 1 of [7] with ‘nonlinear gain’ disturbance attenuation. In contrast to other constructive nonlinear design techniques in the literature, this paper does not rely on completion of the squares. Instead, this paper employs Schur complements formula and scaling functions which not only provide us with appropriate measures for robustification against static and dynamic uncertainties, but also incorporates nonlinear gains in the design. Schur complements formula is usually less conservative than completion of the squares when they are applied to vectors[6]. Combination of Schur complements formula and state-dependent scaling also allows the nonlinear design to locally fall in with LMI-based designs for robust linear control[3].

## 7 An example

This subsection presents an example to illustrate the output-feedback design proposed in this paper briefly. Consider the system  $\Sigma_0$  given by

$$\begin{aligned}\dot{x}_1 &= (1 + x_1^2)x_2 \\ \dot{x}_2 &= -x_1x_2 + (1 + |x_1|)x_3 - x_1w \\ \dot{x}_3 &= x_3 - 2x_1w + r + u \\ z &= x_1, \quad e = x_1, \quad y = x_1\end{aligned}$$

and the uncertain system  $\Sigma_\Delta$  between  $z$  and  $w$  in the form of

$$\dot{x}_\Delta = f_\Delta(x_\Delta, z, t), \quad w = h_\Delta(x_\Delta, t)$$

This uncertain component is supposed to be admissible in the sense of Assumption 1 with

$$\psi_d = 2.2z^2$$

Thus, the system  $\Sigma_\Delta$  has zero  $\mathcal{L}_2$ -gain locally although it is globally bounded only in nonlinear gain. The objective is to find an output-feedback controller which globally uniformly asymptotically stabilizes  $\Sigma_P$  and achieves the level  $\tau = 0.5$  of disturbance attenuation between  $r$  and  $e$ . An observer gain we can obtain using formulas of the recursive design procedure is

$$Y = [-w_1 \quad w_1w_2 \quad -w_1w_2w_3]^T$$

where calculated parameters are

$$w_1 = -15 - 12x_1^2, \quad w_2 = -13, \quad w_3 = -2$$

$$\tilde{P} = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 18x_1^2 + 33 & 0 & 0 \\ 0 & 1 + x_1^2 & 0 \\ 0 & 0 & (1 + |x_1|)/2 \end{bmatrix}^{-1}$$

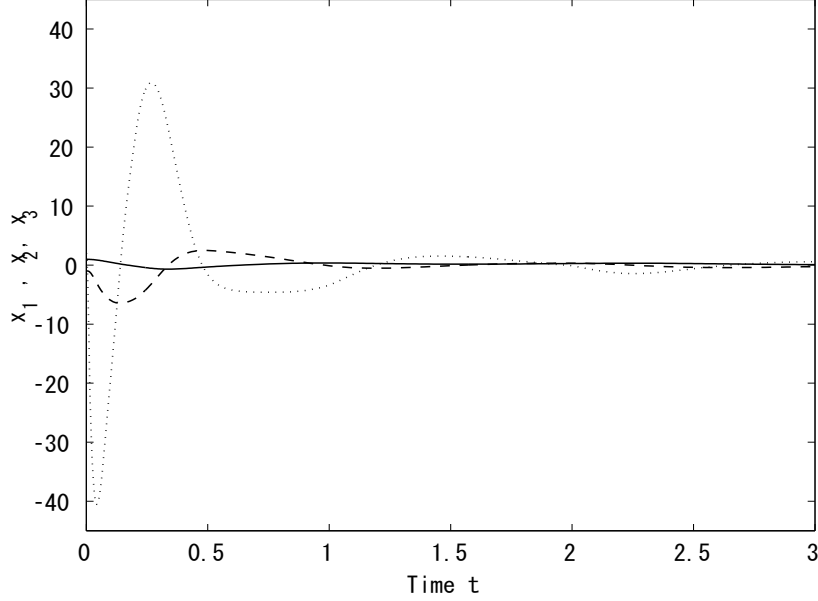


Figure 1: State transition of output feedback control.

The scaling factor and the feedback gain are obtained as

$$\begin{aligned}
\phi_1 &= 1 \\
u &= s_3 [s_2 s_1 \ -s_1 \ 1] x \\
s_1 &= -3, \quad s_2 = -4\sqrt{x_1^2 + 1} \\
s_3 &= -12(x_1 - 0.2)^4 - 95 - \frac{16x_1^2(0.3x_1^2 + 2)(3\hat{x}_1 + \hat{x}_2)^2}{x_1^2 + 1}
\end{aligned}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}$$

is used. One of admissible uncertain components  $z \mapsto w$  is

$$\begin{cases} \dot{x}_\Delta = -x_\Delta(1 - z^4) \\ w = \text{sat}(x_\Delta) \end{cases} \quad (55)$$

It satisfies (9) for  $\psi_d = 2.2z^2$  as described in Section 3. The system (55) is not input-to-state stable although it is globally asymptotically stable when  $z_3 \equiv 0$ . Figure 1 shows state transition of  $\Sigma_0$  in the presence of (55) and the disturbance

$$r(t) = \begin{cases} 5, & 1 \leq t < 2 \\ 0, & \text{otherwise} \end{cases}$$

for the initial condition  $x(0) = [1, -1, 2]^T$ ,  $x_\Delta(0) = 1$  and  $\hat{x}(0) = 0$ . The damped response  $x_1(t)$  shown by the solid line demonstrates clearly that the output-feedback controller attenuate the effect of the disturbance on  $e = x_1$  substantially.

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## Appendix

### Proof of Theorem 1

Let  $x_{cl}$  denote  $x_{cl} = [x^T, \hat{x}^T, x_{\Delta}^T]^T$ . Define

$$V(t, x_{cl}) = \hat{\chi}^T P \hat{\chi} + \eta^T \tilde{P} \eta + \sum_{i=1}^m \check{\phi}_{id} W_{\Delta i}(t, x_{\Delta i})$$

which is a  $\mathcal{C}^1$  function of  $(t, x_{cl})$ , and there exist class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_{cl}$  and  $\bar{\alpha}_{cl}$  such that

$$\underline{\alpha}_{cl}(\|x_{cl}\|) \leq V(t, x_{cl}) \leq \bar{\alpha}_{cl}(\|x_{cl}\|)$$

The time-derivative of  $V$  along the trajectories of the output feedback system satisfies

$$\frac{d}{dt}V \leq \begin{bmatrix} \hat{\chi} \\ w \\ \eta \end{bmatrix}^T Q \begin{bmatrix} \hat{\chi} \\ w \\ \eta \end{bmatrix} + r^T r - \tau^{-2} e^T e - \sum_{i=1}^m \check{\phi}_{id} \beta_i(x_{\Delta i}) \quad (56)$$

Here, (24), (25) and

$$0 \leq \begin{bmatrix} w_{is} \\ z_{is} \end{bmatrix}^T \begin{bmatrix} -\Phi_{is} & 0 \\ 0 & \psi_{is} \Phi_{is} \end{bmatrix} \begin{bmatrix} w_{is} \\ z_{is} \end{bmatrix}$$

are used. The matrix  $Q(x, \hat{x})$  is obtained as

$$Q = \begin{bmatrix} \left( \begin{array}{c} \hat{S}^T \hat{A}^T (X+T)^T P + \\ P(X+T) \hat{A} \hat{S} \\ B^T X^T P \end{array} \right) & P X B & -P(XA + T Y C_y) W^{-1} \\ & -\Theta & -B^T W^T \tilde{P} \\ -W^{-T} (XA + T Y C_y)^T P & -\tilde{P} W B & \left( \begin{array}{c} W^{-T} \bar{A} \hat{W} \tilde{P} + \\ \tilde{P} \hat{W}^T \bar{A}^T W^{-1} \end{array} \right) \end{bmatrix} \\ + \begin{bmatrix} S^{-T} C^T \Psi \Phi \\ 0 \\ -W^{-T} C^T \Psi \Phi \end{bmatrix} \Phi^{-1} \begin{bmatrix} \Phi \Psi C S^{-1} & 0 & -\Phi \Psi C W^{-1} \end{bmatrix}$$

The inequality (23) is equivalent to the pair of  $Q < 0$  and  $\Phi > 0$ . Thus, under the condition (23), the global uniform asymptotic stability of the output-feedback system follows from (56) with  $r \equiv 0$ . Finally, integrating (56) from  $t = 0$  to  $t = T > 0$ , we obtain

$$V(t, x_{cl}(T)) - V(t, x_{cl}(0)) \leq \int_0^T (r^T r - \tau^{-2} e^T e) dt$$

This proves that  $\Sigma_P$  has  $\mathcal{L}_2$ -gain less than or equal to  $\tau$ .

### Proof of Lemma 2:

Define the following matrix.

$$\bar{M}_{[k]} = M_{[k]11} - M_{[k]12} H^{-1} M_{[k]21}$$

Let this matrix be partitioned as

$$\begin{bmatrix} J_k & E_k \\ E_k^T & F_k \end{bmatrix} = \bar{Q}_k^T \bar{M}_{[k]} \bar{Q}_k \\ \bar{Q}_k = \begin{bmatrix} 0 & I_{k-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{2(p+q)} \end{bmatrix} \in \mathcal{R}^{[k+2(p+q)] \times [k+2(p+q)]}$$

The scalar  $J_k$  and the matrix  $F_k$  is given by

$$\begin{aligned} J_k &= 2P_k a_{k,k+1} s_k + \tilde{J}_k \\ F_1 &= \begin{bmatrix} -\Phi & 0 \\ 0 & -\Phi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \Phi \end{bmatrix} Z_0 \begin{bmatrix} I & 0 \\ 0 & \Phi \end{bmatrix} \\ F_k &= \bar{M}_{[k-1]}, \quad \text{for } 2 \leq k \leq n \end{aligned}$$



where  $\tilde{J}_k$ ,  $E_k$  and  $F_k$  are independent of  $\{s_k, s_{k+1}, \dots, s_n\}$ . The matrix  $Z_0$  is obtained as

$$Z_0 = - \begin{bmatrix} B^T W^T \tilde{P} \\ \Psi C \end{bmatrix} H^{-1} \begin{bmatrix} \tilde{P} W B & C^T \Psi \end{bmatrix}$$

The assumption  $H < 0$  implies  $Z_0 \geq 0$ . Let  $\bar{Z}_0$  be defined by

$$\bar{Z}_0 = \begin{bmatrix} \bar{a}I_{p+q} & Z_b \\ Z_b^T & \bar{c}I_{p+q} \end{bmatrix}$$

From  $\bar{Z}_0 \geq Z_0 \geq 0$  we obtain  $0 \leq \bar{b} \leq \bar{a}\bar{c}$ .

(i) Due to  $\bar{Z}_0 \geq Z_0$ ,  $F_1 < 0$  is achieved if  $\Phi = \phi_1 I$  satisfies

$$\begin{bmatrix} -\Phi & 0 \\ 0 & -\Phi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \Phi \end{bmatrix} \bar{Z}_0 \begin{bmatrix} I & 0 \\ 0 & \Phi \end{bmatrix} < 0$$

which is equivalent to

$$\phi_1 > \bar{a} \tag{57}$$

$$\bar{c}\phi_1^2 + (\bar{b} - \bar{a}\bar{c} - 1)\phi_1 + \bar{a} < 0 \tag{58}$$

Under the assumption that  $H < 0$ , the inequality (41) implies  $4\bar{a}\bar{c} < 1$ . From  $\bar{b} \leq \bar{a}\bar{c}$  we obtain  $1 + \bar{a}\bar{c} - \bar{b} > 2\sqrt{\bar{a}\bar{c}}$ . Thus, the quadratic inequality (58) admits real solutions  $\phi_1$ , and the solutions are given by  $(\bar{e}_-, \bar{e}_+)$  defined with (48). Here,  $\bar{e}_\pm$  are real numbers satisfying  $\bar{e}_- < \bar{e}_+$ . From  $\bar{b} \leq \bar{a}\bar{c}$  and  $4\bar{a}\bar{c} < 1$  it follows that  $1 - 3\bar{a}\bar{c} - \bar{b} > 0$ . Then, it is seen that

$$\begin{aligned} 2\bar{c}(\bar{e}_- - \bar{a}) &= 1 - \bar{a}\bar{c} - \bar{b} - \sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}} \\ &= \sqrt{(1 - \bar{a}\bar{c} - \bar{b})^2} - \sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}} \\ &> 0 \end{aligned}$$

The last inequality implies  $0 \leq \bar{a} \leq \bar{e}_-$ . Therefore, any real number  $\phi_1$  belonging to  $(\bar{e}_-, \bar{e}_+)$  fulfills (57) automatically. Hence, there exist smooth functions  $\phi_1(x_1) > 0$  satisfying  $F_1(x_1) < 0$  for all  $x_1 \in \mathcal{R}$ . Now, suppose that  $F_1 < 0$  has been achieved by a function  $\phi_1$ . According to Schur complements formula,  $\bar{M}_{[1]} < 0$  is equivalent to a scalar inequality  $J_1 - E_1 F_1^{-1} E_1^T < 0$ . Since the left hand side is affine in  $s_1$ , the assumption (28) assures the existence of a smooth function  $s_1(x_1)$  which fulfills  $\bar{M}_{[1]} < 0$ . Applying Schur complements formula to  $M_{[1]} < 0$ , the condition  $\bar{M}_{[1]} < 0$  is equivalent to  $M_{[1]} < 0$  on the assumption of  $H < 0$ .

(ii) Since  $M_{[k-1]} < 0$  and  $H < 0$  are assumed, the application of Schur complements formula leads us to  $\bar{M}_{[k-1]} < 0$ . Since  $J_k - E_k F_k^{-1} E_k^T < 0$  is affine in  $s_k$ , the assumption (28) guarantees the existence of a smooth function  $s_k(x_1, \hat{x}_{[k-1]})$  solving  $\bar{M}_{[k]} < 0$  which is identical with  $M_{[k]} < 0$ .

### Proof of Lemma 3:

Define the following set of integers.

$$\Omega = \left\{ (i, j, \rho, \iota) : \begin{array}{l} k \leq i \leq n, \quad k \leq j \leq n \\ \rho = i, i+1 < n, \quad k \leq \iota \leq n \end{array} \right\}$$

The assumption (30) implies

$$\sup_{y \in \mathcal{R}} \left| \frac{a_{ij} a_{\rho\iota}}{a_{k-1,k}(y) a_{l,l+1}(y)} \right| < +\infty, \quad \begin{array}{l} 2 \leq k \leq n-1 \\ k \leq l \leq n-1 \\ (i, j, \rho, \iota) \in \Omega \end{array} \tag{59}$$

$$\sup_{y \in \mathcal{R}} \left| \frac{a_{k,k}(y)}{a_{k-1,k}(y)} \right| < +\infty, \quad 2 \leq k \leq n-1 \quad (60)$$

$$\sup_{y \in \mathcal{R}} \left| \frac{a_{k,k+1}(y)}{a_{k-1,k}(y)} \right| < +\infty, \quad 2 \leq k \leq n-1 \quad (61)$$

For systems satisfying (26-27) and (32), (35) and (36), the inequality (40) is equivalent to

$$H_{\langle k+1 \rangle} < -\Gamma_{k+1}^{-1} \quad (62)$$

$$2\tilde{P}_k a_{k-1,k} w_k < -2\tilde{P}_k a_{k,k} + 2\tilde{P}_k a_{k,k+1} w_{k+1} - \gamma_k^{-1} + e_k^T \left( H_{\langle k+1 \rangle} + \nu_k^{-1} \Gamma_{k+1}^{-1} \right)^{-1} e_k \quad (63)$$

in the case of  $1 \leq k \leq n-1$ . where  $a_{0,1} = 1$ . For  $K = n$ , the above two inequalities are replaced by a single inequality

$$2\tilde{P}_n a_{n-1,n} w_n < -2\tilde{P}_n a_{n,n} - \gamma_n^{-1} \quad (64)$$

Here,  $e_k$  is a vector satisfying

$$e_k^T e_k \leq \sum_{(i,j,\rho,\iota) \in \Omega} c_k(i,j,\rho,\iota) |a_{ij} a_{\rho\iota}|, \quad 1 \leq k \leq n-1 \quad (65)$$

for some finite non-negative constants  $c_k(\cdot, \cdot, \cdot, \cdot)$  since  $w_2, w_3, \dots, w_n$  are constants. Here,  $c_k(\cdot, \cdot, \cdot, \cdot)$  are independent of  $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$  and  $\{w_1, w_2, \dots, w_k\}$ . Combining (65) and (44), we obtain

$$\frac{e_k^T e_k \gamma_{l+1}}{|a_{k-1,k}|} \leq \frac{\sum_{(i,j,\rho,\iota) \in \Omega} d_k(i,j,\rho,\iota) |a_{ij} a_{\rho\iota}|}{|a_{k-1,k} a_{l,l+1}|}, \quad \begin{matrix} 2 \leq k \leq n-1 \\ k \leq l \leq n-1 \end{matrix}$$

for some finite non-negative constants  $d_k(\cdot, \cdot, \cdot, \cdot)$ . Thus,

$$\sup_{y \in \mathcal{R}} \frac{e_k^T e_k \gamma_{l+1}}{|a_{k-1,k}|} < +\infty, \quad 2 \leq k \leq n-1, \quad k \leq l \leq n-1 \quad (66)$$

follows from (59). Now, we choose  $w_k$  from  $k = n$  down to  $k = 1$  recursively as follows.

**(i) Case  $k = n$**  : Let  $\gamma_n(x_1)$  be a  $\mathcal{C}^0$  function fulfilling (42-44). Due to (43), (31) and (28), there exists a constant  $w_n$  such that (64) is satisfied for all  $x_1 \in \mathcal{R}$ . Then, the inequality (40) is achieved for  $k = n$ . This process does not involve  $\{\gamma_1, \gamma_2, \dots, \gamma_{n-1}\}$  and  $\{w_1, w_2, \dots, w_{n-1}\}$ .

**(ii) Case  $2 \leq k \leq n-1$**  : Suppose that the set of constants  $\{w_{k+1}, w_{k+2}, \dots, w_n\}$  satisfying (62) are given. Let a constant  $\nu_k$  be chosen as (45). It is verified that

$$-\left( H_{\langle k+1 \rangle} + \nu_k^{-1} \Gamma_{k+1}^{-1} \right)^{-1} < \frac{\nu_k}{\nu_k - 1} \Gamma_{k+1} \quad (67)$$

Let  $\gamma_k(x_1)$  be a  $\mathcal{C}^0$  function which fulfills (42-44). Due to (43), (60), (61) (66), (67) and (28), there exists a constant  $w_k$  such that (63) is satisfied for all  $x_1 \in \mathcal{R}$ . Again, this selection of  $w_k$  does not depends on  $\{\gamma_1, \gamma_2, \dots, \gamma_{k-1}\}$  and  $\{w_1, w_2, \dots, w_{k-1}\}$ . The inequality (40) is achieved for  $k$ .

**(iii) Case  $k = 1$**  : Suppose that a given set of constants  $\{w_2, w_3, \dots, w_n\}$  satisfies (62) with  $k = 1$ . Pick a constant  $\nu_1$  as (45). Let  $\gamma_1(x_1)$  be any  $\mathcal{C}^0$  function fulfilling (42). Due to (28), there

exists a smooth function  $w_1(x_1)$  such that (63) with  $k = 1$  is satisfied for all  $x_1 \in \mathcal{R}$ . Hence, the inequality (40) is achieved for  $k = 1$ .

**Proof of Lemma 4:**

(a) Let  $B_{\langle k \rangle}$  denote

$$B_{\langle k \rangle} = \begin{bmatrix} B_{k1} \\ \vdots \\ B_{n1} \end{bmatrix}$$

The following holds.

$$\begin{aligned} B_{\langle k \rangle}^T W_{\langle k \rangle}^T \tilde{P}_{\langle k \rangle} \Gamma_{1\langle k \rangle} \tilde{P}_{\langle k \rangle} W_{\langle k \rangle} B_{\langle k \rangle} &= \left( \prod_{j=1}^{k-1} \nu_j \right) \gamma_k \tilde{P}_k^2 B_{k1}^T B_{k1} + \\ &\left( \prod_{j=1}^k \nu_j \right) \gamma_{k+1} w_{k+1} \tilde{P}_{k+1}^2 \left( w_{k+1} B_{k1}^T B_{k1} + B_{\langle k+1 \rangle}^T B_{k1} + B_{k1}^T B_{\langle k+1 \rangle} \right) + \\ &B_{\langle k+1 \rangle}^T W_{\langle k+1 \rangle}^T \tilde{P}_{\langle k+1 \rangle} \Gamma_{1\langle k+1 \rangle} \tilde{P}_{\langle k+1 \rangle} W_{\langle k+1 \rangle} B_{\langle k+1 \rangle} \end{aligned}$$

for  $1 \leq k \leq n-1$ . In the case of  $k = n$ ,

$$B_{\langle n \rangle}^T W_{\langle n \rangle}^T \tilde{P}_{\langle n \rangle} \Gamma_{1\langle n \rangle} \tilde{P}_{\langle n \rangle} W_{\langle n \rangle} B_{\langle n \rangle} = \left( \prod_{j=1}^{n-1} \nu_j \right) \gamma_n \tilde{P}_n^2 B_{n1}^T B_{n1}$$

Thus, under the assumption of (49), there exist a  $\mathcal{C}^0$  function  $\gamma_1(x_1)$  and a constant  $\alpha > 0$  such that (42), (52) and (53) hold if (50) and

$$\sup_{x_1 \in \mathcal{R}} \frac{\lambda_{\max}(B_{i+1,1}(x_1) B_{i+1,1}^T(x_1))}{|a_{i,i+1}(x_1)|} < +\infty, \quad i = 1, 2, \dots, n-1 \quad (68)$$

Since (61) follows from (30), (68) is equivalent to (51).

(b) The inequality (52-53) imply  $4\hat{a}\hat{c} < 1$  due to the assumption that  $0 < -H^{-1} < \Gamma_1$ . The assumptions (52) and (53) also guarantee boundedness of  $\hat{a}$  and  $\hat{c}$ . Recall that  $H < 0$  implies  $\bar{b} \leq \bar{a}\bar{c}$  for all  $x_1 \in \mathcal{R}$ . It is obvious that  $\hat{b} \leq \hat{a}\hat{c}$  holds and  $\hat{b}$  is bounded. Let  $\hat{Z}_0$  be defined by

$$\hat{Z}_0 = \begin{bmatrix} \hat{a}I_{p+q} & Z_b \\ Z_b^T & \hat{c}I_{p+q} \end{bmatrix}$$

Since  $\hat{Z}_0 \geq Z_0 \geq 0$  holds for all  $x_1 \in \mathcal{R}$ ,  $F_1 < 0$  is implied for all  $x_1 \in \mathcal{R}$  by

$$\begin{bmatrix} -\Phi & 0 \\ 0 & -\Phi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \Phi \end{bmatrix} \hat{Z}_0 \begin{bmatrix} I & 0 \\ 0 & \Phi \end{bmatrix} < 0 \quad (69)$$

Using the same argument as the proof of Lemma 2, it is shown that the inequality (69) is achieved by any real constant  $\phi_1$  belonging to  $(\hat{e}_-, \hat{e}_+)$  defined with (54).