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NOTE ON PARAMETER DEPENDENCE OF EIGENVALUES FOR A LINEARIZED EIGENVALUE PROBLEM

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1. Introduction and main theorems

Let us introduce the following problem of differential equations:

$$\begin{cases} \varepsilon^2 u_{xx}(x) + \lambda f(u(x)) = 0 & \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbf{R})$ and ε is a positive (bifurcation) parameter. In view of the bifurcation, let us consider *the bistable case* that f has exactly three zeros $u_- < 0 < u_+$ and they satisfy $f_u(0) > 0$ and $f_u(u_\pm) < 0$. Then the corresponding parabolic partial differential equation to (1.1), which is called as the Chafee-Infante problem (Allen-Cahn type equation, or Ginzburg-Landau equations), possesses two stable steady state $u = u_\pm$ and one unstable state $u = 0$. The solution set

$$\mathscr{S} := \{(\lambda, u) \in \mathbf{R}^+ \times C^2[0,1] \,|\, (\lambda, u(x)) \text{ solves (1.1).}\}$$

is given as follows. For any $n \in \mathbf{N}$, it appears the branch $\mathscr{S}_n^\pm$ of two *n-mode solutions* $u_n^\pm$ for $\lambda \in (n^2\pi^2/f_u(0), \infty)$, which is bifurcating from the line of trivial solution $u = 0$ at $\lambda_n^* = n^2\pi^2/f_u(0)$. For details, see for instance, [1], [3] and [4].

The problem (1.1) is the simplest bifurcation problem, and has been extensively studied by many authors since 1960's. Let $u(x)$ be any nontrivial solution of (1.1). The linearized eigenvalue problem associated with u is given by

$$\begin{cases} \varepsilon^2 \varphi_{xx}(x) + f_u(u(x))\varphi(x) + \mu\varphi(x) = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{1.2}$$

Problem (1.2) play an essential role to understand the dynamics of evolutionary problem to (1.1):

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(u), & (x,t) \in (0,1) \times (0,+\infty), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,+\infty), \\ u(x,0) = a(x), & x \in (0,1). \end{cases} \tag{1.3}$$

In several papers [7]–[11] the author and S. Yotsutani have investigated (1.1) and (1.2) in the two typical cases of bistable nonlinearity:

$$f(u) = \sin u$$

(with a restriction $-\pi < u < \pi$ due to the periodicity) and

$$f(u) = u - u^3.$$

In these cases of f above, it is shown that the precise information on all eigenpairs of (1.2) are obtained by using the elliptic integrals and Jacobi's elliptic function (for definitions of elliptic integrals and functions, see Section 2).

For simplicity, we take the first case $f(u) = \sin u$. For arbitrarily fixed $n \in \mathbf{N}$, exactly two n-mode solutions of (1.1) exist, and can be expressed with use of a new parameter $k \in (0,1)$: $(\varepsilon, u_n^{\pm}) = (\varepsilon_n(k), \pm u_n(x;k))$ where

$$u_n(x;k) := 2\sin^{-1}[k \cdot \mathrm{sn}(K(k)(1+2nx), k)],$$

and

$$\varepsilon(k) := \frac{1}{2nK(k)}$$

The linearized eigenvalue problem for $u_n(x;k)$ leads us to

$$\begin{cases} \varepsilon_n(k)^2 \varphi_{xx}(x) + (1 - 2k^2\,\mathrm{sn}^2(K(k)(1+2nx),k))\varphi(x) + \mu\varphi(x) = 0 \quad \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{1.4}$$

By $\mu_j^n(k)$ and $\varphi_j^n(x;k)$ $(j \in \mathbf{N} \cup \{0\})$, we denote the $(j+1)$-th eigenvalue of (1.4) and its corresponding eigenfunction.

It should be noted that the first equation of (1.4) is regarded as the Lamé equation through a suitable scaling (see Whittaker-Watson [12]). Moreover, in the case $f(u) = u - u^3$ we are led to the similar linearized eigenvalue problem as (1.4); it is reduced into the Lamé equation.

For the problem (1.4) the following two theorems are proved in [8].

THEOREM A ([8], Theorem 1). *Let $n \in \mathbf{N}$ and let $k \in (0,1)$. The linearized problem (1.4) has following pairs of eigenvalues and eigenfunctions.*

(i) $\mu_0^n(k) = (k^2 - 1)$, $\varphi_0^n(x;k) = \cos\dfrac{u_n(x;k)}{2} = \mathrm{dn}(K(k)(1+2nx),k)$.

(ii) $\mu_n^n(k) = k^2$, $\varphi_n^n(x;k) = \dfrac{1}{k}\sin\dfrac{u_n(x;k)}{2} = \mathrm{sn}(K(k)(1+2nx),k)$.

THEOREM B ([8], Theorem 2). *Let $n \in \mathbf{N}$ and let $k \in (0,1)$. Set $\Sigma := \Sigma_0 \cup \Sigma_1$ where*

$$\Sigma_0 := \{(k,\mu) \in (0,1) \times \mathbf{R} \mid k^2 - 1 < \mu < 0\},$$

$$\Sigma_1 := \{(k,\mu) \in (0,1) \times \mathbf{R} \mid \mu > k^2\},$$

and define for $(k, \mu) \in \Sigma$,

$$\mathscr{A}(k, \mu) := \sqrt{\frac{\mu(\mu - k^2 + 1)}{\mu - k^2}}\, \Pi\left(\frac{k^2}{\mu - k^2}, k\right). \tag{1.5}$$

Assume $j \neq 0, n$. *Then,* $\mu_j^n(k)$ *is the unique solution of*

$$\mathscr{A}(k, \mu) = \frac{j\pi}{2n}. \tag{1.6}$$

In particular, for each $k \in (0, 1)$, $\mu_j^n(k) \in (k^2 - 1, 0)$ *if* $0 < j < n$ *and* $\mu_j^n(k) \in (k^2, +\infty)$ *if* $j > n$. *Moreover,*

$$\varphi_j^n(x; k) = \sqrt{|R_n(x; k, \mu_j^n(k)))|}\cos(\theta_n(x; k, \mu_j^n(k))),$$

where

$$R_n(x; k, \mu) := k^2 - \mu - \sin^2 \frac{u_n(x; k)}{2},$$

$$\theta_n(x; k, \mu) := \frac{1}{\varepsilon_n(k)} \int_0^x \frac{\sqrt{\rho(k, \mu)}}{|R_n(\xi; k, \mu)|}\, d\xi,$$

and

$$\rho(k, \mu) := \mu(\mu - k^2)(\mu - k^2 + 1).$$

In Theorem B, the function $\mathscr{A}$ as in (1.5) is called as *the characteristeic function* and plays an essential role to determine all eigenvalues of (1.4) including $\mu_0^n(k)$ and $\mu_n^n(k)$. Through the characterization on the eigenvalue, the representation formulas for all eigenfunctions are given. We note that in the case $f(u) = u - u^3$ a similar types results on the representation formulas are obtained in [10].

The eigenvalues $\mu_j^n(k)$ $(0 \leq j < n)$ are negative; the corresponding eigenfunctions provide the local unstable manifold around $u_n(x; k)$ for the corresponding parabolic PDE problem to (1.1). Also, one can observe from [8] that

$$\lim_{k \to 0} \mu_j^n(k) = \frac{j^2 - n^2}{n^2} \qquad \text{and} \qquad \lim_{k \to 1} \mu_j^n(k) = 0.$$

To the linearized problems (1.4), we are interested in a parameter dependence of eigenvalues and eigenfunctions with respect to k. In the previous papers [8] and [9] Theorems A and B are applied to show asymptotic formulas of eigenvalues and eigenfunctions as $k \to 1$.

In this note we continue to discuss (1.4) and show the monotonicity of negative eigenvalues with respect to the parameter k. It is easy to see from Theorem A that $\mu_0^n(k)$ and $\mu_n^n(k)$ are increasing in k. Our main result is given by the following theorem.

THEOREM 1. *Let* $n \in \mathbf{N}$ *and let* $k \in (0,1)$. *Suppose* $0 \le j < n$. *Then,* $\mu_j^n(k)$ *is monotone increasing with respect to* k.

We have no information on $\mu_j^n(k)$ with $j > n$. However, the asymptotic formula of eigenvalues for $j > n$ suggest they are decreasing if k is sufficiently close to 1 ([8]). This fact implies some of eigenvalues does not admit monotonicity with respect to k.

Now we also consider the following quantity

$$\tilde{\mu}_j^n(k) := \frac{\mu_j^n(k)}{\varepsilon_n(k)^2} = 4n^2 \mu_j^n(k) K(k)^2$$

for $n \in \mathbf{N}$, $k \in (0,1)$ and $j \in \mathbf{N} \cup \{0\}$. They are eigenvalues of the following linearized eigenvalue problems

$$\begin{cases} \varepsilon_n(k)^2 \varphi_{xx}(x) + (1 - 2k^2 \operatorname{sn}^2(K(k)(1+2nx), k))\varphi(x) + \tilde{\mu}\varepsilon_n(k)^2 \varphi(x) = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0, \end{cases}$$

which appears from a rescaled parabolic PDE of (1.3):

$$\begin{cases} \varepsilon^2 u_t = \varepsilon^2 u_{xx} + f(u), & (x,t) \in (0,1) \times (0,+\infty), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,+\infty), \\ u(x,0) = a(x), & x \in (0,1). \end{cases}$$

For the monotonicity of $\tilde{\mu}_j^n(k)$ we obtain the following result.

THEOREM 2. *Let* $n \in \mathbf{N}$ *and let* $k \in (0,1)$. *Suppose* $0 < j < n$. *Then,* $\tilde{\mu}_j^n(k)$ *is monotone increasing with respect to* k.

We will prove the main theorems by analyzing of the characteristic equations (1.6) with use of formulas on elliptic integrals. In this decade, the approach by elliptic integral has been developed by Kosugi-Morita-Yotsutani [5], [6], and [8]–[11], et al. One of the difficulty is that we have to treat the complete elliptic integral of the third kind Π; a few results are shown in literature. In this note we introduce a slightly modified form of Π, and improve some technical calculations on elliptic integrals. The modified elliptic integral has been already found in the characterisic functions appearing in (1.2) for the both cases $f(u) = \sin u$ and $f(u) = u - u^3$. We expect that the modified form of elliptic integral gives a new perspective in the theory of differential equations.

The organization on this note is as follows. In Section 2 we introduce standard complete elliptic integrals and modified form of the elliptic integral Π. The fundamental formulas of the modified elliptic integral will be given. In Section 3 we give proofs of Theorems 1 and 2.

2. Elliptic integrals and modified form

We begin with definitions and fundamental properties of the complete elliptic integrals. We refer to a handbook by Byrd and Friedman [2], and also, the appendix of [10]. Let $k \in (0,1)$. The complete elliptic integrals of the first and second are defined by

$$K(k) := \int_0^1 \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}}\,ds \qquad \text{and} \qquad E(k) := \int_0^1 \frac{\sqrt{1-k^2s^2}}{\sqrt{(1-s^2)}}\,ds,$$

respectively. One can easily see that K is monotone increasing in k and

$$K(0) = \frac{\pi}{2} \qquad \text{and} \qquad \lim_{k\to 1} K(k) = +\infty;$$

that E is decreasing in k,

$$E(0) = \frac{\pi}{2} \qquad \text{and} \qquad \lim_{k\to 1} E(k) = 1;$$

and that for every $k \in (0,1)$

$$K(k) > E(k).$$

Also, $\mathrm{sn}(\cdot,k)$ is defined by

$$x = \int_0^{\mathrm{sn}(x,k)} \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}}\,ds$$

for $x \in [0, K(k)]$, and it is extended to a smooth and periodic function on $\mathbf{R}$ in the standard way.

Let $k \in (0,1)$ and $\nu \in \mathbf{C}\backslash(-\infty,-1]$. The complete elliptic integral of the third kind is given by

$$\Pi(\nu,k) := \int_0^1 \frac{1}{(1+\nu s^2)\sqrt{(1-s^2)(1-k^2s^2)}}\,ds.$$

We summarize the derivative formulas on K, E and Π below without proofs.

Proposition 2.1. *Let $k \in (0,1)$ and $\nu \neq 0, -1, -k^2$. Then*

(i) $\dfrac{dE}{dk}(k) = \dfrac{E(k) - K(k)}{k}$,

(ii) $\dfrac{dK}{dk}(k) = \dfrac{E(k) - (1-k^2)K(k)}{k(1-k^2)}$,

(iii) $\dfrac{\partial \Pi}{\partial k}(\nu, k) = \dfrac{k(E(k) - (1-k^2)\Pi(\nu,k))}{(k^2+\nu)(1-k^2)}$,

(iv) $\dfrac{\partial \Pi}{\partial \nu}(\nu, k) = -\dfrac{K(k)}{2\nu(1+\nu)} + \dfrac{E(k)}{2(1+\nu)(k^2+\nu)} + \dfrac{(k^2-\nu^2)\Pi(\nu,k)}{2\nu(1+\nu)(k^2+\nu)}$.

Now we introduce the modified form of Π for $\mathscr{D}$

$$\mathscr{D} := \{(k,\mu) \in (0,1) \times \mathbf{C} \mid \mu \notin (-\infty,-1] \cup [-k^2, 0]\}$$

by

$$\mathscr{M}(\nu,k) := m(\mu,k)\Pi(\nu,k). \tag{2.1}$$

where

$$m(\mu,k) := \sqrt{\frac{(1+\nu)(k^2+\nu)}{\nu}}. \tag{2.2}$$

The following proposition is derived from Lemma 2.1 with standard calculation.

PROPOSITION 2.2. *Let* $(k,\mu) \in \mathscr{D}$. *Then*

(i) $\dfrac{\partial \mathscr{M}}{\partial k}(\nu,k) = \sqrt{\dfrac{(1+\nu)(k^2+\nu)}{\nu}}\,\dfrac{kE(k)}{(k^2+\nu)(1-k^2)}$,

(ii) $\dfrac{\partial \mathscr{M}}{\partial \nu}(\nu,k) = \sqrt{\dfrac{(1+\nu)(k^2+\nu)}{\nu}}\left[-\dfrac{K(k)}{2\nu(1+\nu)} + \dfrac{E(k)}{2(1+\nu)(k^2+\nu)}\right]$.

PROOF. We first see that the function m defined as above satisfies

$$\frac{m_k}{m} = \frac{k}{k^2+\nu}$$

and

$$\begin{aligned}\frac{m_\nu}{m} &= \frac{1}{2}\left(\frac{1}{1+\nu} + \frac{1}{k^2+\nu} - \frac{1}{\nu}\right)\\ &= \frac{\nu(k^2+\nu) + \nu(1+\nu) - (1+\nu)(k^2+\nu)}{2\nu(1+\nu)(k^2+\nu)}\\ &= \frac{\nu^2-k^2}{2\nu(1+\nu)(k^2+\nu)}.\end{aligned}$$

Therefore, we obtain a desired formulas by combining these facts with Proposition 2.1. □

REMARK 2.1. In the same way as Proposition 2.2, the several limit formulas on Π, which are used to understand the chracteristic functions ([8] and [10]), leads us to the following formulas on $\mathscr{M}$:

(i) $\lim_{\nu \to -1} \mathscr{M}(\nu, k) = \dfrac{\pi}{2}$,

(ii) $\lim_{\nu \to -k^2} \mathscr{M}(\nu, k) = 0$,

(iii) $\lim_{\nu \to 0} \mathscr{M}(\nu, k) = \infty$,

(iv) $\lim_{\nu \to \infty} \mathscr{M}(\nu, k) = \dfrac{\pi}{2}$.

See also Fig. 1.

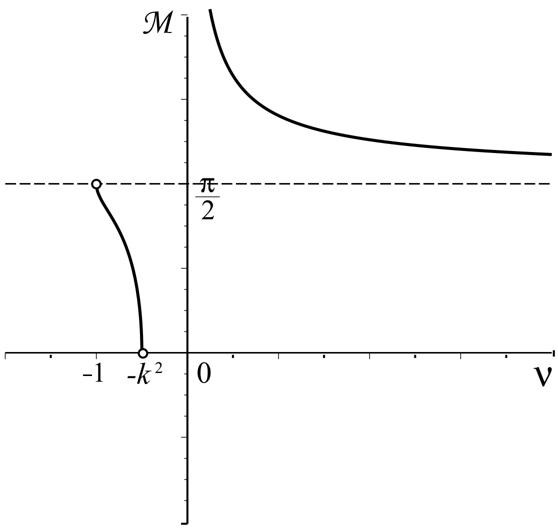

Figure 1. A Graph of $\mathscr{M}(k, \mu)$ $(k = 1/\sqrt{2})$.

3. Proof of main theorems

In this section we prove Theorems 1 and 2. We first prepare fundamental lemmas on $\mathscr{A}$ as in (1.5). Define a function $\nu : \Sigma \to \mathbf{R}$ by

$$\nu(k, \mu) := \frac{k^2}{\mu - k^2}. \tag{3.1}$$

We note that if $(k, \mu) \in \Sigma$ then $(k, \nu(k, \mu)) \in \mathscr{D}$; in particular, if $(k, \mu) \in \Sigma_0$ then $\nu(k, \mu) \in (-1, -k^2)$, and if $(k, \mu) \in \Sigma_1$ then $\nu(k, \mu) \in (0, +\infty)$. In addition,

$$\mu = \frac{k^2(1 + \nu(k, \mu))}{\nu(k, \mu)}, \qquad \mu - k^2 = \frac{k^2}{\nu(k, \mu)} \qquad \text{and} \qquad \mu - k^2 + 1 = \frac{k^2 + \nu(k, \mu)}{\nu(k, \mu)}.$$

Hence we obtain, for $(k, \mu) \in \Sigma$,

$$\mathscr{A}(k, \mu) = \mathscr{M}(\nu(k, \mu), k). \tag{3.2}$$

LEMMA 3.1. *Let Σ and $\mathscr{A}$ as in Theorem B. Let $\mathscr{M}(k,\mu)$, $m(k,\mu)$ and $v(k,\mu)$ be given by* (2.1), (2.2) *and* (3.1). *Then, for $(k,\mu) \in \Sigma$, the following* (i) *and* (ii) *hold:*

(i) $\displaystyle \mathscr{A}_\mu(k,\mu) = \frac{1}{2} m(v(k,\mu),k)^{-1} \cdot \frac{(k^2+v(k,\mu))K(k) - v(k,\mu)E(k)}{k^2},$

(ii)

$$\mathscr{A}_k(k,\mu) = -m(k,v(k,\mu)) \frac{(1-k^2)(k^2+v(k,\mu))K(k) - [k^2+(1-k^2)v(k,\mu)]E(k)}{k(1-k^2)(k^2+v(k,\mu))}.$$

PROOF. (i) We first see from (3.1) that

$$v_\mu(k,\mu) = -\frac{k^2}{(\mu-k^2)^2} = -\frac{v(k,\mu)^2}{k^2}.$$

By a standard calculation with (ii) of Proposition 2.2,

$$\begin{aligned}
\mathscr{A}_\mu &= \mathscr{M}_v(v(k,\mu),k)v_\mu(k,\mu) \\
&= \sqrt{\frac{(1+v(k,\mu))(k^2+v(k,\mu))}{v(k,\mu)}} \\
&\quad \cdot \left(-\frac{(k^2+v(k,\mu))K(k) - v(k,\mu)E(k)}{2v(k,\mu)(1+v(k,\mu))(k^2+v(k,\mu))}\right) \cdot \left(-\frac{v(k,\mu)^2}{k^2}\right) \\
&= \frac{1}{2}\left(\sqrt{\frac{(1+v(k,\mu))(k^2+v(k,\mu))}{v(k,\mu)}}\right)^{-1} \cdot \frac{(k^2+v(k,\mu))K(k) - v(k,\mu)E(k)}{k^2}.
\end{aligned}$$

(ii) In the similarly as the proof of (i) of the lemma, we obtain

$$v_k(k,\mu) = \frac{2k\mu}{(\mu-k^2)^2} = \frac{2}{k} v(k,\mu)(1+v(k,\mu))$$

and by Proposition 2.2,

$$\begin{aligned}
\mathscr{A}_k &= \mathscr{M}_k(v(k,\mu),k) + \mathscr{M}_v(v(k,\mu),k)v_k(k,\mu) \\
&= m(k,v(k,\mu)) \frac{kE(k)}{(k^2+v(k,\mu))(1-k^2)} \\
&\quad - m(k,v(k,\mu)) \left[\frac{(k^2+v(k,\mu))K(k) - v(k,\mu)E(k)}{2v(k,\mu)(1+v(k,\mu))(k^2+v(k,\mu))}\right] \cdot \frac{2}{k} v(k,\mu)(1+v(k,\mu))
\end{aligned}$$

$$
\begin{aligned}
&= m(k, v(k,\mu)) \frac{kE(k)}{(k^2 + v(k,\mu))(1-k^2)} \\
&\quad - m(k, v(k,\mu)) \left[\frac{(k^2 + v(k,\mu))K(k) - v(k,\mu)E(k)}{k(k^2 + v(k,\mu))} \right] \\
&= -m(k, v(k,\mu)) \frac{(1-k^2)(k^2 + v(k,\mu))K(k) - [k^2 + (1-k^2)v(k,\mu)]E(k)}{k(1-k^2)(k^2 + v(k,\mu))}. \qquad \square
\end{aligned}
$$

Lemma 3.2. *Let Σ and $\mathscr{A}$ as in Theorem B. Then, for $(k,\mu) \in \Sigma$,*

$$\mathscr{A}_\mu(k,\mu) > 0.$$

Proof. Fix $k \in (0,1)$ arbitrarily, and consider

$$\mathscr{B}_1(v;k) := (K(k) - E(k))v + K(k)$$

for $v \in [-1, +\infty)$. It is easy to see from $K(k) - E(k) > 0$ that if $v < -1$, then

$$\mathscr{B}_1(v;k) > \mathscr{B}_1(-1;k) = E(k) - (1-k^2)K(k).$$

By (i) and (ii) of Proposition 2.1, the function $g_1(k) := E(k) - (1-k^2)K(k)$ satisfies

$$
\begin{aligned}
\frac{d}{dk} g_1(k) &= \frac{E(k) - K(k)}{k} + 2kK(k) - (1-k^2)\frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \\
&= kK(k).
\end{aligned}
$$

So $g_1(k) > 0$ for all $k \in (0,1)$ and hence, $\mathscr{B}_1(v;k) > 0$ for every $\mu \in (-1, +\infty)$.

By combining this fact with (i) of Lemma 3.1, we obtain $\mathscr{A}_\mu > 0$ for $(k,\mu) \in \Sigma$. Thus it completes a proof. $\square$

Lemma 3.3. *Let Σ_0 and $\mathscr{A}$ as in Theorem B. Then, for $(k,\mu) \in \Sigma_0$,*

$$\mathscr{A}_k(k,\mu) < 0.$$

Proof. Fix $k \in (0,1)$ arbitrarily, and consider

$$\mathscr{B}_2(v;k) := (1-k^2)(K(k) - E(k))v + k^2[(1-k^2)K(k) - E(k)]$$

for $v \in [-1, -k^2]$. It is easy to see from $K(k) - E(k) > 0$ that for $v \in (-1, -k^2)$,

$$\mathscr{B}_2(v;k) < \mathscr{B}_2(-k^2;k) = -k^4 E(k) < 0.$$

Hence, $\mathscr{B}_2(v;k) < 0$ for every $\mu \in (-1, -k^2)$.

By combining this fact with (ii) of Lemma 3.1, we obtain $\mathscr{A}_k > 0$ for $(k,\mu) \in \Sigma_0$. Thus it completes a proof. $\square$

PROOF OF THEOREM 1. Suppose $k \in (0,1)$ and $0 < j < n$ (we omit a proof of the case $j = 0$). We apply the implicit function theorem to Theorem B:

$$\frac{d}{dk}\mu_j^n(k) = -\frac{A_k(k, \mu_j^n(k))}{A_\mu(k, \mu_j^n(k))}.$$

By Lemmas 3.2 and 3.3 with $(k, \mu_j^n(k)) \in \Sigma_0$, we conclude that $(d/dk)\mu_j^n(k) > 0$ for any $k \in (0,1)$.

Thus it complete the proof. □

PROOF OF THEOREM 2. Suppose $k \in (0,1)$. In the case $j = 0$ we observe from (i) of Theorem A that

$$\tilde{\mu}_0^n(k) = -4n^2(1-k^2)K(k)^2.$$

By using (ii) of Proposition 2.1,

$$\begin{aligned}\frac{d}{dk}\tilde{\mu}_0^n(k) &= 4n^2\left[2kK(k)^2 - (1-k^2)\cdot 2K(k)\cdot\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\right]\\ &= \frac{8n^2K(k)}{k}(K(k)-E(k)) > 0.\end{aligned}$$

Now we suppose that $0 < j < n$. Since

$$\tilde{\mu}_j^n(k) = \mu_j^n(k)4n^2K(k)^2,$$

we obtain from the implicite function theorem and Lemmas 3.2 that

$$\begin{aligned}\frac{d}{dk}\tilde{\mu}_j^n(k) &= 4n^2\left[-\frac{\mathscr{A}_k(k,\mu_j^n(k))}{\mathscr{A}_\mu(k,\mu_j^n(k))}K(k)^2 + \mu_j^n(k)\cdot 2K(k)K'(k)\right]\\ &= \frac{4n^2K(k)}{\mathscr{A}_\mu(k,\mu_j^n(k))}\cdot(-\mathscr{A}_k(k,\mu_j^n(k))K(k) + 2\mu_j^n(k)\mathscr{A}_\mu(k,\mu_j^n(k))K'(k))\\ &= \frac{4n^2K(k)}{\mathscr{A}_\mu(k,\mu_j^n(k))}\cdot\frac{1}{m(k,\nu_j^n(k))}\\ &\quad\cdot\left[\frac{(1-k^2)(k^2+\nu_j^n(k))K(k) - [k^2+(1-k^2)\nu_j^n(k)]E(k)}{k(1-k^2)(k^2+\nu_j^n(k))}m(k,\nu_j^n(k))^2K(k)\right.\\ &\quad\left.+\frac{k^2(1+\nu_j^n(k))}{\nu_j^n(k)}\,\frac{(k^2+\nu_j^n(k))K(k)-\nu_j^n(k)E(k)}{k^2}\cdot\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\right]\end{aligned}$$

$$= \frac{4n^2K(k)}{m(k, v_j^n(k))\mathscr{A}_\mu(k, \mu_j^n(k))} \cdot \frac{1 + v_j^n(k)}{k(1-k^2)v_j^n(k)}$$

$$\cdot [(1-k^2)(k^2 + v_j^n(k))K(k)^2 - [k^2 + (1-k^2)v_j^n(k)]K(k)E(k)$$

$$- ((k^2 + v_j^n(k))K(k) - v_j^n(k)E(k))((1-k^2)K(k) - E(k))]$$

$$= \frac{4n^2(1 + v_j^n(k))K(k)E(k)(K(k) - E(k))}{k(1-k^2)m(k, v_j^n(k))\mathscr{A}_\mu(k, \mu_j^n(k))},$$

where

$$v_j^n(k) = \frac{k^2}{\mu_j^n(k) - k^2}.$$

Since $\mu_j^n(k) \in (k^2 - 1, 0)$, we conclude that $(d/dk)\tilde{\mu}_j^n(k) > 0$. Thus it complete the proof. □

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