Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 64, 2017, pp. 13–19

A DIRECT PROOF OF SOME RECENT GENERALIZATION OF BOTH ĆIRIĆ'S AND BOGIN'S FIXED POINT THEOREMS

Misako Kikkawa and Tomonari Suzuki

Abstract

We give a direct proof of some recent generalization of both Ćirić's and Bogin's fixed point theorems.

1. Introduction

In 1974 and 1976, Ćirić and Bogin, respectively, proved the following very interesting fixed point theorems, which are generalizations of the Banach contraction principle [1, 3].

Theorem 1 (Ćirić [4]). *Let T be a quasi-contraction on a complete metric space (X, d), that is, there exists $\rho \in [0, 1)$ such that*

$$d(Tx, Ty) \le \rho \max\{d(x, y), d(x, Ty), d(Tx, y), d(x, Tx), d(y, Ty)\} \tag{1}$$

for any $x, y \in X$. Then T has a unique fixed point.

Theorem 2 (Bogin [2]). *Let T be a mapping on a complete metric space (X, d). Assume that there exist $r \in [0, 1)$ and $s, t \in (0, 1/2)$ satisfying $r + 2s + 2t = 1$ and*

$$d(Tx, Ty) \le rd(x, y) + sd(x, Ty) + sd(Tx, y) + td(x, Tx) + td(y, Ty) \tag{2}$$

for any $x, y \in X$. Then T has a unique fixed point.

Though Inequalities (1) and (2) are similar, Theorems 1 and 2 are independent. Indeed, in Section 4, we give examples which tell that. Motivated by this fact, very recently, we prove the following generalization of Theorems 1 and 2. See also [5, 6].

Theorem 3 ([7]). *Let T be a mapping on a complete metric space (X, d). Assume that there exist $q \in (0, \infty)$, $r \in [0, 1)$ and $s, t \in (0, 1/2)$ satisfying $r + 2s + 2t = 1$ and*

$$d(Tx, Ty)^q \le rd(x, y)^q + sd(x, Ty)^q + sd(Tx, y)^q$$
$$+ td(x, Tx)^q + td(y, Ty)^q$$

2010 *Mathematics Subject Classification.* 54H25.

Key words and phrases. Ćirić's fixed point theorem, Bogin's fixed point theorem.

The second author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

for any $x, y \in X$. *Then* T *has a unique fixed point* z. *Moreover* $\{T^n x\}$ *converges to* z *for any* $x \in X$.

In [7], we give a proof of Theorem 3 in a very general setting, that is, we have not given a direct proof of Theorem 3. In this paper, for readers' convenience, we give a direct proof of Theorem 3.

2. Lemmas

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers. In order to prove Theorem 3, we need the following lemmas. All the lemmas are proved in [7].

Lemma 4 ([7]). *Let* $r \in [0, 1)$ *and* $s, t \in (0, 1/2)$ *satisfy* $r + 2s + 2t = 1$. *Define subsets* I_0 *and* I *of* $\mathbf{N} \times \mathbf{N}$ *by*

$$I_0 = \{(m, n) : m, n \in \mathbf{N} \cup \{0\}, m \le n\}$$
$$I = \{(m, n) : m, n \in \mathbf{N}, m < n\}.$$

Let a function B *from* I_0 *into* $[0, \infty)$ *satisfying the following:*

$$\begin{aligned} B(m, n) &\le rB(m-1, n-1) + sB(m-1, n) + sB(m, n-1) \\ &\quad + tB(m-1, m) + tB(n-1, n) \qquad \text{for } (m, n) \in I \\ B(n, n) &= 0 \qquad \text{for } n \in \mathbf{N} \cup \{0\} \\ B(0, n) &\le 1 \qquad \text{for } n \in \mathbf{N}. \end{aligned}$$

Then $\lim_n B(n, n+1) = 0$ *holds.*

Lemma 5 ([7]). *Let* $p \in (0, 1)$ *and* $\{a_n\}$ *be a real sequence in* $(0, \infty)$ *converging to* 0. *Then*

$$\lim_{n\to\infty} \sum_{j=1}^{n} p^{n-j} a_j = 0$$

holds.

Lemma 6 ([7]). *Let* X, d, T *be as in Theorem* 3. *Then* $\{T^n x\}$ *is bounded for any* $x \in X$.

3. A direct proof

In this section, we give a direct proof of Theorem 3.

Proof of Theorem 3. Fix $x \in X$. Then by Lemma 6, $\{T^n x\}$ is bounded. So, there exists a positive real number M such that $M > d(T^m x, T^n x)^q$ for any $m, n \in \mathbf{N} \cup \{0\}$. Define a function B by

$$B(m, n) = \frac{1}{M} d(T^m x, T^n x)^q$$

for $m, n \in \mathbf{N} \cup \{0\}$ with $m \leq n$. Then all the assumptions of Lemma 4 are satisfied. So by Lemma 4 we obtain

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x)^q = M \lim_{n \to \infty} B(n, n+1) = 0.$$

For any $m, n \in \mathbf{N}$, we have

$$d(T^{n+1} x, T^{m+1} x)^q \leq r d(T^n x, T^m x)^q + s d(T^n x, T^{m+1} x)^q + s d(T^{n+1} x, T^m x)^q$$
$$+ t d(T^n x, T^{n+1} x)^q + t d(T^m x, T^{m+1} x)^q$$

and hence

$$f(m+1) \leq r f(m) + s f(m+1) + s f(m) + t g(m),$$

where $f(m) = \limsup_n d(T^n x, T^m x)^q$ and $g(m) = d(T^m x, T^{m+1} x)^q$. Putting

$$\alpha := \frac{r+s}{r+s+2t} < 1 \qquad \text{and} \qquad \beta = \frac{t}{r+s+2t},$$

we obtain

$$f(m+1) \leq \alpha f(m) + \beta g(m).$$

Using this inequality, we have

$$\begin{aligned} f(m+1) &\leq \alpha f(m) + \beta g(m) \\ &\leq \alpha^2 f(m-1) + \alpha\beta g(m-1) + \beta g(m) \\ &\leq \alpha^3 f(m-2) + \alpha^2 \beta g(m-2) + \alpha\beta g(m-1) + \beta g(m) \\ &\leq \cdots \leq \alpha^m f(1) + \sum_{j=1}^{m} \alpha^{m-j} \beta g(j) \end{aligned}$$

for $m \in \mathbf{N}$. Noting $\lim_m g(m) = 0$, we have $\lim_m f(m) = 0$ by Lemma 5. So we have

$$\lim_{m \to \infty} \limsup_{n \to \infty} d(T^n x, T^m x) = 0.$$

Using this, we have

$$\begin{aligned}\limsup_{k,\ell\to\infty} d(T^k x, T^\ell x) &\le \limsup_{k,\ell\to\infty} \limsup_{n\to\infty}(d(T^k x, T^n x) + d(T^n x, T^\ell x)) \\ &\le \limsup_{k,\ell\to\infty}\left(\limsup_{n\to\infty} d(T^k x, T^n x) + \limsup_{n\to\infty} d(T^n x, T^\ell x)\right) \\ &= \lim_{k\to\infty} \limsup_{n\to\infty} d(T^k x, T^n x) + \lim_{\ell\to\infty} \limsup_{n\to\infty} d(T^n x, T^\ell x) = 0.\end{aligned}$$

Therefore we obtain that $\{T^n x\}$ is a Cauchy sequence. Since X is complete, $\{T^n x\}$ converges to some point $z \in X$. Since

$$\begin{aligned} d(Tz, T^{n+1}x)^q \le\ & rd(z, T^n x)^q + sd(z, T^{n+1}x)^q + sd(Tz, T^n x)^q \\ & + td(z, Tz)^q + td(T^n x, T^{n+1}x)^q, \end{aligned}$$

we have $(1 - s - t)d(Tz, z)^q \le 0$. Since $1 - s - t = r + s + t > 0$, we obtain $Tz = z$. In order to prove that the fixed point z is unique, we let $w \in X$ be a fixed point of T. Then we have

$$\begin{aligned} d(z, w)^q &= d(Tz, Tw)^q \\ &\le rd(z, w)^q + sd(z, Tw)^q + sd(Tz, w)^q + td(z, Tz)^q + td(w, Tw)^q \\ &= (r + 2s)d(z, w)^q. \end{aligned}$$

Since $r + 2s < 1$, we have $d(z, w)^q = 0$, which implies $z = w$. We have shown that the fixed point is unique. □

4. Examples

In this section, we give two examples, which tell that Theorems 1 and 2 are independent.

EXAMPLE 7. Define a subset X of a Banach space $(\mathbf{R}, |\cdot|)$ by

$$X = \{0, 2, 3\}.$$

Define a mapping T on X by

$$\begin{aligned} &T3 = 2 \\ &Ta = 0 \qquad \text{for } a \in \{0, 2\}. \end{aligned}$$

Then T satisfies the assumption of Theorem 1, but does not satisfy that of Theorem 2.

Proof. We first show that T satisfies the assumption of Theorem 1. Fix $x, y \in X$ with $x < y$. In the case where $(x, y) = (0, 3), (2, 3)$, we have

$$\begin{aligned} d(Tx, Ty) = 2 &= \frac{2}{3} d(Tx, y) \\ &\le \frac{2}{3} \max\{d(x, y), d(x, Ty), d(Tx, y), d(x, Tx), d(y, Ty)\}. \end{aligned}$$

In the other case, we have

$$d(Tx, Ty) = 0 \le \frac{2}{3} \max\{d(x, y), d(x, Ty), d(Tx, y), d(x, Tx), d(y, Ty)\}.$$

Hence T satisfies (1). We next show that T does not satisfy the assumption of Theorem 2. We put $x = 2$ and $y = 3$. Then we have

$$\begin{aligned} &rd(x, y) + sd(x, Ty) + sd(Tx, y) + td(x, Tx) + td(y, Ty) \\ &= r + 3s + 3t \le \frac{3}{2} r + 3s + 3t \\ &= \frac{3}{2} < 2 = d(Tx, Ty) \end{aligned}$$

for any $r \in [0, 1)$ and $s, t \in (0, 1/2)$ with $r + 2s + 2t = 1$. Hence T does not satisfy (2). □

Example 8. Define a subset X of a Banach space $(\mathbf{R}^2, \|\cdot\|_\infty)$ by

$$X = \{(0, 0), (\pm 1, \pm 1)\}.$$

Define a mapping T on X by

$$\begin{aligned} &T(0, 0) = (0, 0) \\ &T(a, -1) = (a, 1) \qquad \text{for } a \in \{\pm 1\} \\ &T(a, 1) = (0, 0) \qquad \text{for } a \in \{\pm 1\}. \end{aligned}$$

Then T satisfies the assumption of Theorem 2, but does not satisfy that of Theorem 1.

Proof. Put $r = 0$, $s = t = 1/4$ and

$$A = rd(x, y) + sd(x, Ty) + sd(Tx, y) + td(x, Tx) + td(y, Ty).$$

In order to show that T satisfies the assumption of Theorem 2, we only have to verify $d(Tx,Ty) \le A$ in the following cases.

$x,\ y$	$d(Tx,Ty)$	$d(x,Ty)$	$d(Tx,y)$	$d(x,Tx)$	$d(y,Ty)$	A
$(1,1),\ (0,0)$	0	1	0	1	0	1/2
$(1,1),\ (-1,1)$	0	1	1	1	1	1
$(1,1),\ (1,-1)$	1	0	1	1	2	1
$(1,1),\ (-1,-1)$	1	2	1	1	2	3/2
$(1,-1),\ (0,0)$	1	1	1	2	0	1
$(1,-1),\ (-1,1)$	1	1	2	2	1	3/2
$(1,-1),\ (-1,-1)$	2	2	2	2	2	2

Therefore $d(Tx,Ty) \le A$ holds. We next show that T does not satisfy the assumption of Theorem 1. We put $x = (1,-1)$ and $y = (-1,-1)$. Then we have

$$\rho \max\{d(x,y), d(x,Ty), d(Tx,y), d(x,Tx), d(y,Ty)\} \le 2\rho < 2 = d(Tx,Ty)$$

for any $\rho \in [0,1)$. Therefore T does not satisfy (1). □

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(M. Kikkawa)
Department of Mathematics
Faculty of Science, Saitama University
Sakura, Saitama 338-8570, *Japan*
E-mail: mi-sa-ko-kikkawa@jupiter.sannet.ne.jp

(T. Suzuki)
Department of Basic Sciences
Faculty of Engineering
Kyushu Institute of Technology
Tobata, Kitakyushu 804-8550*, Japan*
E-mail: suzuki-t@mns.kyutech.ac.jp