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Complex energy method in four-body Faddeev-Yakubovsky equations

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The Complex Energy Method [Prog. Theor. Phys. **109**, 869L (2003)] is applied to the four-body Faddeev-Yakubovsky equations in the four-nucleon system. We obtain a well converged solution in all energy regions below and above the four-nucleon breakup threshold.

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Calculations for scattering systems in configuration space require boundary conditions which increase in complexity with growing particle numbers. These boundary conditions appear in the form of Green's functions in momentum space which carry singularities of increasing complexity. Green's functions are expressed as $G_0 = 1/(E + i\varepsilon - H_0)$ where E and H_0 are the total and kinetic energy, respectively, and the limit $\varepsilon \to 0$ has to be taken. In the two-body system there is one (relative) momentum variable p and G_0 has a pole in the complex p plane. It is easy to handle it using the principal value prescription and (half) the residue theorem (PVR). In the three-body case there arises already a difficulty in the form of so-called moving singularities [1,2], however, PVR is still applicable [3], or one can use the contour deformation (CD) [4–7] technique. Summarizing these techniques, first one takes the limiting value $\varepsilon \to 0$ and next the equation is solved avoiding the integration path on the complex plane. This is illustrated in Fig. 1.

The situation is more complicated in the four-body system. Employing a separable potential and a separable expansion technique for the three-body and [2+2] subamplitudes, the four-body Faddeev-Yakubovsky (FY) equations [8] for four identical particles can be expressed as

$$\begin{pmatrix} \mathcal{M}_{\alpha\alpha} & \mathcal{M}_{\alpha\beta} \\ \mathcal{M}_{\beta\alpha} & \mathcal{M}_{\beta\beta} \end{pmatrix} = \begin{pmatrix} \pm\mathcal{E} & \mathcal{F}_1 \\ 2(\mathcal{F}_1^T + \mathcal{F}_2^T) & 0 \end{pmatrix} + \begin{pmatrix} \pm\mathcal{E} & \mathcal{F}_1 \\ 2(\mathcal{F}_1^T + \mathcal{F}_2^T) & 0 \end{pmatrix} \times \begin{pmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{G} \end{pmatrix} \begin{pmatrix} \mathcal{M}_{\alpha\alpha} & \mathcal{M}_{\alpha\beta} \\ \mathcal{M}_{\beta\alpha} & \mathcal{M}_{\beta\beta} \end{pmatrix}, \quad (1)$$

where the $\mathcal{M}$'s are the four-body amplitudes, α and β indicate [3+1] and [2+2] configurations (see Fig. 2), $\mathcal{E}$ is an exchange term from [3+1] to [3+1] configurations of which the plus sign corresponds to the four bosons and minus sign to the fermions. $\mathcal{F}$'s are exchange terms from [2+2] to [3+1] where the subscripts are related to the two diagrams in Fig. 3. More details may be found in Ref. [9]. Further $\mathcal{H}$ and $\mathcal{G}$ are the three-body and [2+2] propagators and they have a similar nature as Green's function in the two-body Lippmann-Schwinger (LS) equation.

If one stays below the three-body breakup threshold the FY equations can be solved with PVR, since only two-body singularities occur [10–16]. Above the three-body threshold but still below the four-body threshold the FY equations have also been solved applying the CD techniques [13–18]. There in the $\mathcal{E}$ and $\mathcal{F}$'s terms occur two-body propagators whose nature is similar to the three-body Green's function in the Born term of the three-body Alt-Grassberger-Sandhas [19] or Amado-Mitra-Faddeev-Lovelace (e.g., Ref. [20]) equations. However, above the four-body threshold the four-body Green's function depends on all (relative) momenta and the

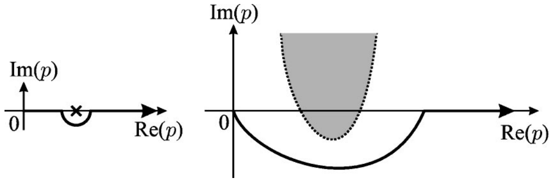

FIG. 1. Illustration of integration paths for PVR (left) and CD (right). The cross in the left figure indicates a fixed pole. Moving singularities occur in the shaded area of the right figure.

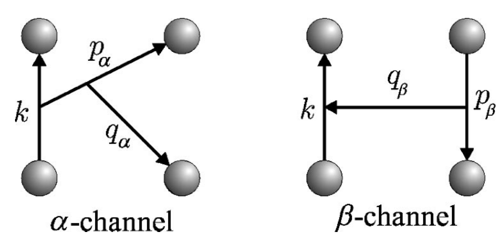

FIG. 2. The two partitions in four-particle system. k, p's, and q's are standard Jacobi momenta.

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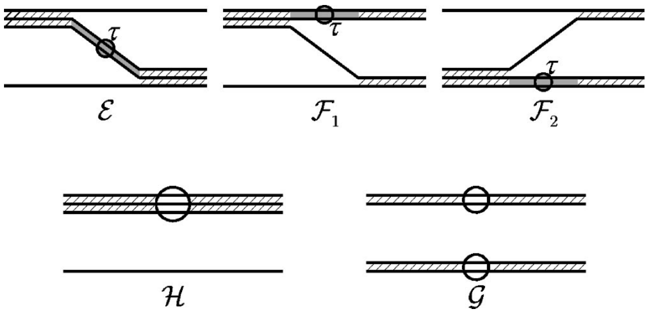

FIG. 3. Diagrams for the $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{H}$, and $\mathcal{G}$ ingredients in Eq. (1). The shaded parts with τ indicate the two-body propagators.

behavior of those singularities is quite complicated. Thus neither PVR nor CD techniques have been successfully extended at energies above the four-body breakup threshold and we are not aware of a solution in this energy region.

Recently the complex energy method (CEM) [21] has been revived and applied to the two- and three-nucleon system. The first step of CEM is to solve the equation with some finite ε's. These calculations are easily carried out since there are no singularities on the real momentum axis. After obtaining solutions with various ε's, the limiting value $\varepsilon \to 0$ is taken numerically with an analytical continuation method.

Our aim is to generate solutions applying this method to the FY equations in all energy regions including energies above the four-body breakup threshold. We performed calculations in the $J^\pi=0^+$ and $T=0$ state for the four-nucleon system. For this feasibility study the $J^\pi=1/2^+$ state is included in the three-body subsystem and the $^1S_0$ and $^3S_1$-$^3D_1$ states in the two-body subsystem. All allowed spins and angular momenta within this restriction are included; in short, there are 14 channels. The Coulomb force is neglected. The Yamaguchi potential [22,23] is employed as the nucleon-nucleon interaction. The potential V has a separable form as

$$V_{\ell\ell'}(k,k') = g_\ell(k)\lambda g_{\ell'}(k'), \quad (2)$$

where g_ℓ's are the two-body form factors for the partial waves ℓ as

$$g_0(k) = \frac{1}{k^2+\beta^2}, \quad g_2(k) = \frac{Ck^2}{(k^2+\beta^2)^2}, \quad (3)$$

and k's are the initial and final momenta between the two nucleons, respectively. We adopt the parameters λ, β, and

TABLE I. Parameters of the Yamaguchi potential.

State	λ (MeV fm^{-1})	State	β (fm^{-1})	C (fm^0)
$^1S_0$	−68.942 626	$^1S_0$	1.130 0	
$^3S_1$-$^3D_1$	−74.506 955	$^3S_1$	1.241 2	
		$^3D_1$	1.947 6	−4.4950154

C given in Table I. We represent the three-body and [2+2] subamplitudes by rank-4 separable forms employing the energy dependent pole expansion (EDPE) [24] method. We take the nucleon mass as 938.918 97 MeV which is the average of those for proton and neutron, and $\hbar c = 197.327\,054$ MeV fm. The integrations are cut off at 200 fm^{-1} for k, at 40 fm^{-1} for p's, and at 16 fm^{-1} for q's (see Fig. 3).

The FY equations are solved at four energies: (i) 1.5 MeV above the $3N+N$ threshold, (ii) 1 MeV above the $2N+2N$ threshold, (iii) 1 MeV below the four-body breakup threshold, and (iv) 12 MeV above it (see Fig. 4). We define the $i\varepsilon$ term of the four-body Green's function as $i\varepsilon+\zeta$, where ε and ζ are real. Thus G_0 turns into $G_0=1/(E+i\varepsilon+\zeta-H_0)$. Solutions of the FY equation satisfy uniqueness even at the limit for $\varepsilon \to 0$, which is not the case for simple four-body LS equation [25]. Therefore the results by the analytical continuation do not depend on the choice of ε's within the radius of convergence. Thus we empirically choose 0.5 MeV as the minimum ε value for the cases (i)–(iii) and 0.75 MeV for the case (iv) (see crosses in Fig. 4), with attention only to a better numerics. They are increased in steps of 0.125 MeV. ζ is chosen as 0 and ±0.125 MeV.

We employ the point method [26] as an analytical continuation technique in CEM. Its convergence behavior is shown in Table II where the phase shift δ and the inelasticity parameter η are defined by $S=\eta\exp(2i\delta)$. Here S is the S matrix of elastic $3N+N$ scattering and is related to the on-shell amplitude of $\mathcal{M}_{\alpha\alpha}$ in Eq. (1) ($\mathcal{M}^{\rm on}_{\alpha\alpha}$) as $S=1-2i\kappa\mathcal{M}^{\rm on}_{\alpha\alpha}$ where κ is the on-shell momentum.

In case (i) η must be 1 due to unitarity and our result satisfies it within six digits. Also in cases (ii) and (iii) we reach a very high accuracy. In case (iv), we still obtain converged solutions within four digits. In the cases (i) and (ii) our results agree very well with the solutions based on PVR.

We showed that well converged solutions of the FY

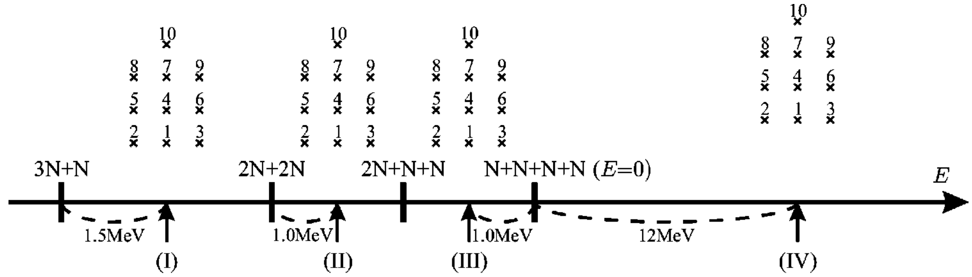

FIG. 4. Illustration of threshold energies for the $4N$ system. We choose $E=0$ at the four-body threshold. The various energies for the calculations are measured relative to the thresholds. The crosses indicate the complex energies where we solve the FY equations in the CEM. They are numbered by n for each choice of the energy region (i–iv).

TABLE II. Phase shifts (in degrees) and inelasticity parameters for $3N+N \rightarrow 3N+N$ elastic scattering. $n_{\max}$ denotes the number of sample energies which are included in the point method. For instance, $n_{\max}=5$ means that the solutions from $n=1$ to 5 (see Fig. 4) are included. The row PVR shows results from a direct solution of the FY equations using PVR. The agreement is perfect.

	(i)		(ii)		(iii)		(iv)	
$n_{\max}$	δ	η	δ	η	δ	η	δ	η
1	53.31351	1.018810	14.62234	0.675088	−8.0617	0.692391	−62.093	0.83875
2	46.21307	0.973139	10.47942	0.853658	−5.9428	0.815698	−61.965	0.75946
3	44.27898	0.989232	12.38204	0.948787	−5.5022	0.899150	−61.620	0.74499
4	44.27129	1.000623	12.38254	0.948044	−5.5101	0.898666	−61.676	0.74570
5	44.34441	0.999787	12.38211	0.948046	−5.5095	0.898656	−61.682	0.74589
6	44.34157	0.999994	12.38284	0.948053	−5.5094	0.898655	−61.669	0.74580
7	44.34022	1.000005	12.38198	0.948070	−5.5096	0.898654	−61.666	0.74570
8	44.34012	0.999997	12.38198	0.948069	−5.5095	0.898657	−61.669	0.74581
9	44.34013	0.999999	12.38198	0.948069	−5.5096	0.898656	−61.670	0.74580
10	44.34016	1.000000	12.38198	0.948069	−5.5095	0.898657	−61.669	0.74582
PVR	44.34016	1.000000	12.38198	0.948069				

equations are obtained in all energy regions. In relation to the application of EDPE we confirmed that converged solutions are obtained in the cases (i)–(iii). In the case (iv), however, there is a report that EDPE is not applicable [27]. We also found that EDPE did not converge. Therefore, in this study we just kept the rank fixed by (4). We plan to investigate this problem in a forthcoming study.

Further we shall include higher partial waves and employ realistic nucleon-nucleon forces to discuss physics. One expects that evidence for three-nucleon forces is more pronounced in the high energy region and the presented method is applicable there, now in the four-nucleon system.

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