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Weak second order explicit exponential Runge-Kutta methods for stochastic differential equations

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Abstract

We propose new explicit exponential Runge-Kutta methods for the weak approximation of solutions of stiff Itô stochastic differential equations (SDEs). These methods have weak order two for multi-dimensional, non-commutative SDEs with a semilinear drift term, whereas they are of order two or three for semilinear ordinary differential equations. These methods are A-stable in the mean square sense for a scalar linear test equation whose drift and diffusion terms have complex coefficients. We perform numerical experiments to compare the performance of these methods with an existing explicit stabilized method of weak order two.

1 Introduction

For stiff ordinary differential equations (ODEs), there are some classes of explicit methods that are well suited. One such class is the class of Runge-Kutta Chebyshev (RKC) methods. They are useful for stiff problems whose eigenvalues lie near the negative real axis. Van der Houwen and Sommeijer [30] have constructed a family of first order RKC methods. Abdulle and Medovikov [3] have modified this class and have proposed a family of second order RKC methods. Another suitable class of methods is the class of explicit exponential Runge-Kutta (RK) methods for semilinear problems [10, 14, 15, 16, 21, 26]. Although these methods were proposed many years ago, they have not been regarded as practical until recently because of the cost of calculations for matrix exponentials, especially for large problems. In order to overcome this problem, new methods have been proposed [12, 14, 15, 16].

Similarly, for stochastic differential equations (SDEs) explicit RK methods that have excellent stability properties have been developed. Abdulle and Cirilli [1] have proposed a family of explicit stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. Their methods have strong order one half and weak order one for non-commutative Stratonovich SDEs, whereas they reduce to the first order RKC methods when applied to ODEs. Abdulle and Li [2] have proposed SROCK methods of the same order for non-commutative Itô SDEs. Komori and Burrage [19] have developed these ideas and have proposed weak second order SROCK methods for non-commutative Stratonovich SDEs. If the methods are applied to ODEs, they reduce to the second order RKC methods of Abdulle and Medovikov [3]. Komori and Burrage [20] have also proposed strong first order SROCK methods for non-commutative Itô and Stratonovich SDEs, which reduce to the first or second order RKC methods for ODEs. The weak second order SROCK methods given by Komori and Burrage [19] have the advantage that the stability region is large along the negative real axis, but they still have a drawback, that is, their stability region is not so wide. In order to overcome this drawback, Abdulle, Vilmart and Zygalakis [5] have proposed a new family of weak second order SROCK methods for non-commutative Itô SDEs, in which another family of second order RKC methods is embedded.

On the other hand, Shi, Xiao and Zhang [28] have proposed an exponential Euler scheme for the strong approximation of solutions of SDEs with multiplicative noise driven by a scalar Wiener process. Cohen [7] and Tocino [29] have proposed exponential integrators for second order SDEs with a semilinear drift term and multiplicative noise. Adamu [6], Geiger, Lord and Tambue [11], and Lord and Tambue [22] have proposed exponential integrators for stochastic partial differential equations with a semilinear drift term and multiplicative noise. Komori and Burrage [18] have proposed another explicit exponential Euler scheme for non-commutative Itô SDEs with a semilinear drift term, which is of strong order one half and A-stable in the MS.

In the present paper, we derive stochastic exponential Runge-Kutta (SERK) methods for the weak approximation of solutions of non-commutative Itô SDEs with a semilinear drift term. We will achieve this on the basis of the derivative free Milstein-Talay (DFMT) method proposed by Abdulle et al. [4, 5] and explicit exponential RK methods for ODEs proposed by Hochbruck and Ostermann [15]. In Section 2 we will briefly introduce explicit exponential RK methods for ODEs. In Section 3 we will derive our SERK methods, and in Section 4 we will give their stability analysis. Section 5 will present numerical results and Section 6 our conclusions.

2 Explicit exponential RK methods for ODEs

We consider autonomous semilinear ODEs given by

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{2. 1}$$

where $\boldsymbol{y}$ is an $\mathbb{R}^d$-valued function on $[0, \infty)$, A is a $d \times d$ matrix and $\boldsymbol{f}$ is an $\mathbb{R}^d$-valued nonlinear function on $\mathbb{R}^d$. In order to introduce some exponential RK methods for (2. 1), we make the following assumption [15]:

Assumption 2.1 *For a given time $T > 0$, (2.1) satisfies the conditions below.*

(1) *There exists a constant C such that*

$$\left\| \mathrm{e}^{tA} \right\| \le C$$

for all $t \in [0, T]$.

(2) *The nonlinear function $\boldsymbol{f}$ is (locally) Lipschitz continuous in a local region U which contains the exact solution $\boldsymbol{y}$ on $[0, T]$, that is,*

$$\{\boldsymbol{y}(t) \mid t \in [0, T]\} \subset U.$$

(3) *The solution $\boldsymbol{y}$ is a sufficiently smooth function on $[0, T]$ and $\boldsymbol{f}$ is sufficiently often differentiable in U. All occurring derivatives of $\boldsymbol{y}$ and $\boldsymbol{f}$ are uniformly bounded in $[0, T]$ and U, respectively.*

Remark that the global error estimation of all exponential RK methods introduced in this section can be influenced by the constant C [15].

By the variation-of-constants formula, the solution of (2. 1) is

$$\boldsymbol{y}(t_{n+1}) = \mathrm{e}^{Ah}\boldsymbol{y}(t_n) + \int_{t_n}^{t_{n+1}} \mathrm{e}^{A(t_{n+1}-s)}\boldsymbol{f}(\boldsymbol{y}(s))\mathrm{d}s. \tag{2. 2}$$

Let $\boldsymbol{y}_n$ denote a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (2. 1) for an equidistant grid point $t_n \stackrel{\text{def}}{=} nh$ $(n = 1, 2, \ldots, M)$ with step size $h = T/M < 1$ (M is a natural number). By interpolating $\boldsymbol{f}(\boldsymbol{y}(s))$ at $\boldsymbol{f}(\boldsymbol{y}_n)$ only, we obtain the simplest exponential scheme for (2. 1) [16]:

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{Ah}\boldsymbol{y}_n + h\varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n), \tag{2. 3}$$

where $\varphi_1(Z) \stackrel{\text{def}}{=} Z^{-1}(\mathrm{e}^Z - I)$ and I stands for the $d \times d$ identity matrix. This is called the explicit exponential Euler method.

Higher order exponential RK methods have been proposed in [15, 16]. For example, the following is a one-parameter family of second order exponential RK methods:

$$\begin{aligned} \boldsymbol{Y}_1 &= \mathrm{e}^{c_2hA}\boldsymbol{y}_n + c_2h\varphi_1(c_2hA)\boldsymbol{f}(\boldsymbol{y}_n), \\ \boldsymbol{y}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{1}{c_2}\varphi_2(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) + \frac{1}{c_2}h\varphi_2(hA)\boldsymbol{f}(\boldsymbol{Y}_1), \end{aligned} \tag{2. 4}$$

where c_2 is a parameter and $\varphi_2(Z) \overset{\text{def}}{=} Z^{-2}(\mathrm{e}^Z - I - Z)$. In addition to Assumption 2.1, let us assume that there exists a constant C such that

$$\left\| hA \sum_{k=1}^{n-1} \mathrm{e}^{khA} \right\| \leq C$$

for $n = 2, 3, \ldots, M$. (Note that the global error estimation of the following family of exponential RK methods can be also influenced by the above constant C [15].) Then, a two-parameter family of third order exponential RK methods is given by

$$\begin{aligned}
\boldsymbol{Y}_1 &= \mathrm{e}^{c_2 hA}\boldsymbol{y}_n + c_2 h\varphi_1(c_2 hA)\boldsymbol{f}(\boldsymbol{y}_n), \\
\boldsymbol{Y}_2 &= \mathrm{e}^{c_3 hA}\boldsymbol{y}_n + h\left\{c_3\varphi_1(c_3 hA) - \psi(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) + h\psi(hA)\boldsymbol{f}(\boldsymbol{Y}_1), \\
\boldsymbol{y}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{\gamma+1}{\gamma c_2 + c_3}\varphi_2(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) \\
&\quad + \frac{h}{\gamma c_2 + c_3}\varphi_2(hA)\left\{\gamma\boldsymbol{f}(\boldsymbol{Y}_1) + \boldsymbol{f}(\boldsymbol{Y}_2)\right\},
\end{aligned} \tag{2. 5}$$

where c_2, c_3 and γ are parameters satisfying

$$2(\gamma c_2 + c_3) = 3\left(\gamma c_2^2 + c_3^2\right) \tag{2. 6}$$

and $\psi(Z) \overset{\text{def}}{=} \gamma c_2\varphi_2(c_2 Z) + \frac{c_3^2}{c_2}\varphi_2(c_3 Z)$.

3 Weak second order SERK methods

We shall now derive SERK methods of weak order two by utilizing some results for a well-designed existing stochastic Runge-Kutta (SRK) method. For this, we give a brief introduction to the SRK method in the first subsection. After this, we will present SERK methods in the second subsection.

3.1 The derivative free Milstein-Talay method

Similarly to the case of ODEs, we are concerned with autonomous SDEs with a semilinear drift term given by

$$\mathrm{d}\boldsymbol{y}(t) = (A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t)))\mathrm{d}t + \sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{y}(t))\mathrm{d}W_j(t), \quad t > 0, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{3. 1}$$

where $\boldsymbol{g}_j$, $j = 1, 2, \ldots, m$ are $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$, the $W_j(t)$, $j = 1, 2, \ldots, m$ are independent Wiener processes and $\boldsymbol{y}_0$ is independent of $W_j(t) - W_j(0)$ for $t > 0$.

In order to deal with weak approximations for (3. 1), let $\boldsymbol{g}_0(\boldsymbol{y})$ denote $A\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y})$ and let us consider the following DFMT method [4, 5]:

$$
\begin{aligned}
&\boldsymbol{K}_1=\boldsymbol{y}_n+h\boldsymbol{g}_0(\boldsymbol{y}_n), \qquad \boldsymbol{K}_2=\boldsymbol{K}_1+\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{y}_n)\xi_j,\\
&\boldsymbol{y}_{n+1}=\boldsymbol{y}_n+\frac{h}{2}\left\{\boldsymbol{g}_0\left(\boldsymbol{y}_n\right)+\boldsymbol{g}_0\left(\boldsymbol{K}_2\right)\right\}\\
&\qquad+\frac{1}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{y}_n+h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\zeta_{kj}\right)-\boldsymbol{g}_j\left(\boldsymbol{y}_n-h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\zeta_{kj}\right)\right\}\\
&\qquad+\frac{\sqrt{h}}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\frac{\boldsymbol{y}_n+\boldsymbol{K}_1}{2}+\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\chi_k\right)\right.\\
&\qquad\qquad\left.+\boldsymbol{g}_j\left(\frac{\boldsymbol{y}_n+\boldsymbol{K}_1}{2}-\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\chi_k\right)\right\}\xi_j,
\end{aligned}
\tag{3. 2}
$$

where the χ_j and ξ_j, $j=1,2,\ldots,m$ are discrete random variables satisfying

$$
P(\chi_j=\pm 1)=\frac{1}{2}, \quad P(\xi_j=\pm\sqrt{3})=\frac{1}{6}, \quad P(\xi_j=0)=\frac{2}{3}
$$

and the ζ_{kj}, $j,k=1,2,\ldots,m$ are given by

$$
\zeta_{kj}\stackrel{\text{def}}{=}\begin{cases}(\xi_j\xi_j-1)/2 & (j=k),\\ (\xi_k\xi_j-\chi_k)/2 & (j<k),\\ (\xi_k\xi_j+\chi_j)/2 & (j>k).\end{cases}
$$

Let $C_P^L(\mathbb{R}^d,\mathbb{R})$ denote the family of L times continuously differentiable real-valued functions on $\mathbb{R}^d$, whose partial derivatives of order less than or equal to L have polynomial growth. Whenever we deal with weak convergence of order q, we will make the following assumption [17, p. 474]:

Assumption 3.1 *All moments of the initial value* $\boldsymbol{y}_0$ *exist and* $\boldsymbol{g}_j$ $(j=0,1,\ldots,m)$ *are Lipschitz continuous with all their components belonging to* $C_P^{2(q+1)}(\mathbb{R}^d,\mathbb{R})$.

Then, we can give the definition of weak convergence of order q [17, p. 327]:

Definition 3.1 *When discrete approximations* $\boldsymbol{y}_n$ *are given by a numerical method, we say that the method is of weak (global) order* q *if for all* $G\in C_P^{2(q+1)}(\mathbb{R}^d,\mathbb{R})$, *constants* $C>0$ *(independent of* h*) and* $\delta_0>0$ *exist, such that*

$$
|E[G(\boldsymbol{y}(T)]-E[G(\boldsymbol{y}_M)]|\le Ch^q, \quad h\in(0,\delta_0).
$$

In order to consider numerical methods of weak order q, the following theorem is very useful, which has been originally proposed by Milstein [24] (see [25, p. 100]) and which is very often utilized by other researchers [4, 5, 27].

Theorem 3.1 *In addition to Assumption 3.1, suppose that the following conditions hold:*

(1) *for sufficiently large* r, *the moments* $E[\|\boldsymbol{y}_n\|^{2r}]$ *exist and are uniformly bounded with respect to* M *and* $n=0,1,\ldots,M$;

(2) *for all $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$, the local error estimation*

$$\left| E[G(\boldsymbol{y}(t_{n+1}))] - E[G(\boldsymbol{y}_{n+1})] \right| \leq |K(\boldsymbol{y}_n)| h^{q+1}$$

holds if $\boldsymbol{y}(t_n) = \boldsymbol{y}_n$, where $K \in C_P^0(\mathbb{R}^d, \mathbb{R})$.

Then, the scheme that gives $\boldsymbol{y}_n$ $(n = 0, 1, \ldots, M)$ is of weak (global) order q.

The second condition concerning the local error in the theorem provides us order conditions for an SRK method to be of weak order q [27]. In addition, the DFMT method is of weak order two [4]. These facts give us a way of deriving new SRK methods of weak order two [4, 5]. For this, we propose a useful lemma to give a sufficient condition for SRK methods based on the DFMT method to satisfy the second condition in Theorem 3.1.

Lemma 3.1 *For an approximate solution $\boldsymbol{y}_n$, let $\boldsymbol{y}_{n+1}$ be given by (3. 2). For the $\boldsymbol{y}_n$, let $\hat{\boldsymbol{y}}_{n+1}$ be given by*

$$\begin{aligned}
\hat{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} &+ \frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{Y}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_0\left(\boldsymbol{Y}_2\right)\xi_j\right) \\
&+ \frac{1}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_3 + h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_3\right)\zeta_{kj}\right) - \boldsymbol{g}_j\left(\boldsymbol{Y}_3 - h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_3\right)\zeta_{kj}\right)\right\} \\
&+ \frac{\sqrt{h}}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_4 + \sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_5\right)\chi_k\right)\right. \\
&\qquad\qquad \left. + \boldsymbol{g}_j\left(\boldsymbol{Y}_4 - \sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_5\right)\chi_k\right)\right\}\xi_j
\end{aligned}$$

and assume that $\tilde{\boldsymbol{y}}_n$ and $\boldsymbol{Y}_i, i = 1, 2, \ldots, 5$ have no random variable and satisfy

$$\begin{aligned}
&\tilde{\boldsymbol{y}}_{n+1} + \frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{Y}_1\right) = \boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{y}_n\right) + \frac{h^2}{2}\boldsymbol{g}_0'\left(\boldsymbol{y}_n\right)\boldsymbol{g}_0'\left(\boldsymbol{y}_n\right) + O\left(h^3\right), \qquad (3.\ 3) \\
&\boldsymbol{Y}_i = \boldsymbol{y}_n + h\boldsymbol{a}_i + O\left(h^2\right) \quad (i = 1, 2, 3, 5), \\
&\boldsymbol{Y}_4 = \boldsymbol{y}_n + \frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{y}_n\right) + h^2\boldsymbol{a}_4 + O\left(h^3\right),
\end{aligned}$$

where $\boldsymbol{a}_i, i = 1, 2, \ldots, 5$ are vectors independent of h. (Note that the symbol $O(h^p)$ represents terms $\boldsymbol{x}$ such that $\|\boldsymbol{x}\| \leq |K(\boldsymbol{y}_n)|h^p$ for an $K \in C_P^0(\mathbb{R}^d, \mathbb{R})$ and a small $h > 0$.) Then, for all $G \in C_P^r(\mathbb{R}^d, \mathbb{R})$ $(r \geq 3)$

$$E\left[G\left(\hat{\boldsymbol{y}}_{n+1}\right)\right] - E\left[G\left(\boldsymbol{y}_{n+1}\right)\right] = O\left(h^3\right).$$

Proof. As

$$
\begin{aligned}
&\boldsymbol{g}_0\left(\boldsymbol{Y}_1+\sqrt{h}\sum_{j=1}^m \boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right)\\
&=\boldsymbol{g}_0\left(\boldsymbol{Y}_1\right)+\sqrt{h}\sum_{j=1}^m \boldsymbol{g}_0'\left(\boldsymbol{y}_n\right)\boldsymbol{g}_j\left(\boldsymbol{y}_n\right)\xi_j+\frac{h}{2}\sum_{j,k=1}^m \boldsymbol{g}_0''\left(\boldsymbol{y}_n\right)\left[\boldsymbol{g}_j(\boldsymbol{y}_n),\boldsymbol{g}_k(\boldsymbol{y}_n)\right]\xi_j\xi_k\\
&\quad+\frac{h^{3/2}}{6}\sum_{j,k,l=1}^m \boldsymbol{g}_0'''\left(\boldsymbol{y}_n\right)\left[\boldsymbol{g}_j(\boldsymbol{y}_n),\boldsymbol{g}_k(\boldsymbol{y}_n),\boldsymbol{g}_l(\boldsymbol{y}_n)\right]\xi_j\xi_k\xi_l\\
&\quad+h^{3/2}\sum_{j=1}^m \boldsymbol{g}_0'\left(\boldsymbol{y}_n\right)\boldsymbol{g}_j'\left(\boldsymbol{y}_n\right)\boldsymbol{a}_2\xi_j+h^{3/2}\sum_{j=1}^m \boldsymbol{g}_0''\left(\boldsymbol{y}_n\right)\left[\boldsymbol{a}_1,\boldsymbol{g}_j(\boldsymbol{y}_n)\right]\xi_j+O\left(h^2\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\tilde{\boldsymbol{y}}_{n+1}+\frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{Y}_1+\sqrt{h}\sum_{j=1}^m \boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right)-\left\{\boldsymbol{y}_n+\frac{h}{2}\left(\boldsymbol{g}_0\left(\boldsymbol{y}_n\right)+\boldsymbol{g}_0\left(\boldsymbol{K}_2\right)\right)\right\}\\
&=h^{5/2}\boldsymbol{r}_1+O\left(h^3\right)
\end{aligned}
$$

from (3. 3), where

$$
\boldsymbol{r}_1=\frac{1}{2}\sum_{j=1}^m\left\{\boldsymbol{g}_0'\left(\boldsymbol{y}_n\right)\boldsymbol{g}_j'\left(\boldsymbol{y}_n\right)\boldsymbol{a}_2+\boldsymbol{g}_0''\left(\boldsymbol{y}_n\right)\left[\boldsymbol{a}_1-\boldsymbol{g}_0(\boldsymbol{y}_n),\boldsymbol{g}_j(\boldsymbol{y}_n)\right]\right\}\xi_j.
$$

As

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^m\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_3+h\sum_{k=1}^m \boldsymbol{g}_k\left(\boldsymbol{Y}_3\right)\zeta_{kj}\right)-\boldsymbol{g}_j\left(\boldsymbol{Y}_3-h\sum_{k=1}^m \boldsymbol{g}_k\left(\boldsymbol{Y}_3\right)\zeta_{kj}\right)\right\}\\
&=h\sum_{j,k=1}^m \boldsymbol{g}_j'\left(\boldsymbol{y}_n\right)\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\zeta_{kj}+h^2\boldsymbol{r}_2+O\left(h^3\right)
\end{aligned}
$$

where

$$
\boldsymbol{r}_2=\sum_{j,k=1}^m\left\{\boldsymbol{g}_j''\left(\boldsymbol{y}_n\right)\left[\boldsymbol{a}_3,\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\right]+\boldsymbol{g}_j'\left(\boldsymbol{y}_n\right)\boldsymbol{g}_k'\left(\boldsymbol{y}_n\right)\boldsymbol{a}_3\right\}\zeta_{kj},
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^m\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_3+h\sum_{k=1}^m \boldsymbol{g}_k\left(\boldsymbol{Y}_3\right)\zeta_{kj}\right)-\boldsymbol{g}_j\left(\boldsymbol{Y}_3-h\sum_{k=1}^m \boldsymbol{g}_k\left(\boldsymbol{Y}_3\right)\zeta_{kj}\right)\right\}\\
&\quad-\frac{1}{2}\sum_{j=1}^m\left\{\boldsymbol{g}_j\left(\boldsymbol{y}_n+h\sum_{k=1}^m \boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\zeta_{kj}\right)-\boldsymbol{g}_j\left(\boldsymbol{y}_n-h\sum_{k=1}^m \boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\zeta_{kj}\right)\right\}\\
&=h^2\boldsymbol{r}_2+O\left(h^3\right).
\end{aligned}
$$

As

$$\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_4+\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_5\right)\chi_k\right)+\boldsymbol{g}_j\left(\boldsymbol{Y}_4-\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_5\right)\chi_k\right)\right\}\xi_j$$
$$=\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{y}_n+\frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{y}_n\right)+\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\chi_k\right)\right.$$
$$\left.+\boldsymbol{g}_j\left(\boldsymbol{y}_n+\frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{y}_n\right)-\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\chi_k\right)\right\}\xi_j+2h^2\boldsymbol{r}_3+O\left(h^{5/2}\right)$$

where

$$\boldsymbol{r}_3=\sum_{j=1}^{m}\left\{\boldsymbol{g}_j'\left(\boldsymbol{y}_n\right)\boldsymbol{a}_4+\frac{1}{2}\sum_{k,l=1}^{m}\boldsymbol{g}_j''\left(\boldsymbol{y}_n\right)\left[\boldsymbol{g}_k\left(\boldsymbol{y}_n\right),\boldsymbol{g}_l'\left(\boldsymbol{y}_n\right)\boldsymbol{a}_5\right]\chi_k\chi_l\right\}\xi_j,$$

we have

$$\frac{\sqrt{h}}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_4+\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_5\right)\chi_k\right)+\boldsymbol{g}_j\left(\boldsymbol{Y}_4-\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_5\right)\chi_k\right)\right\}\xi_j$$
$$-\frac{\sqrt{h}}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\frac{1}{2}\left(\boldsymbol{y}_n+\boldsymbol{K}_1\right)+\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\chi_k\right)\right.$$
$$\left.+\boldsymbol{g}_j\left(\frac{1}{2}\left(\boldsymbol{y}_n+\boldsymbol{K}_1\right)-\sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\chi_k\right)\right\}\xi_j$$
$$=h^{5/2}\boldsymbol{r}_3+O\left(h^3\right).$$

From these results,

$$\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}=h^2\boldsymbol{r}_2+h^{5/2}\boldsymbol{r}_1+h^{5/2}\boldsymbol{r}_3+O\left(h^3\right).$$

From this and (3. 2), thus,

$$\begin{aligned}&G\left(\hat{\boldsymbol{y}}_{n+1}\right)-G\left(\boldsymbol{y}_{n+1}\right)\\&=G'\left(\boldsymbol{y}_{n+1}\right)\left(\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right)+O\left(h^4\right)\\&=G'\left(\boldsymbol{y}_n+\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{y}_n\right)\xi_j+O(h)\right)\left(\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right)+O\left(h^4\right)\\&=G'\left(\boldsymbol{y}_n\right)\left(\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right)+h^{5/2}G''\left(\boldsymbol{y}_n\right)\left[\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{y}_n\right)\xi_j,\boldsymbol{r}_2\right]+O\left(h^3\right).\end{aligned}$$

Consequently, we obtain

$$E\left[G\left(\hat{\boldsymbol{y}}_{n+1}\right)\right]-E\left[G\left(\boldsymbol{y}_{n+1}\right)\right]=O\left(h^3\right)$$

because

$$E[\boldsymbol{r}_1]=E[\boldsymbol{r}_2]=E[\boldsymbol{r}_3]=E[\xi_j\boldsymbol{r}_2]=0\quad(j=1,2,\ldots,m).$$

$\square$

3.2 SERK methods

We shall propose weak second order SERK methods for (3. 1). As a simple case, let us begin with

$$
\begin{aligned}
\boldsymbol{y}_{n+1} = \boldsymbol{Y}_1 + h\varphi_2(hA)\left\{\boldsymbol{f}\left(\boldsymbol{Y}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}(\boldsymbol{y}_n)\right\} \\
+\sqrt{h}\left(\mathrm{e}^{\frac{h}{2}A} - I\right)\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j + \boldsymbol{H}.
\end{aligned}
\tag{3. 4}
$$

Here and in what follows, we set $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ and $\boldsymbol{H}$ by

$$
\begin{aligned}
\boldsymbol{Y}_1 &\stackrel{\text{def}}{=} \mathrm{e}^{hA}\boldsymbol{y}_n + h\varphi_1(hA)\boldsymbol{f}(\boldsymbol{y}_n), \quad \boldsymbol{Y}_2 \stackrel{\text{def}}{=} \mathrm{e}^{\frac{h}{2}A}\boldsymbol{y}_n + \frac{h}{2}\varphi_1\left(\frac{h}{2}A\right)\boldsymbol{f}(\boldsymbol{y}_n) \\
\boldsymbol{H} &\stackrel{\text{def}}{=} \frac{1}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_1 + h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_1\right)\zeta_{kj}\right)\right. \\
&\qquad\left. -\boldsymbol{g}_j\left(\boldsymbol{Y}_1 - h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_1\right)\zeta_{kj}\right)\right\} \\
&\quad + \frac{\sqrt{h}}{2}\sum_{j=1}^{m}\left\{\boldsymbol{g}_j\left(\boldsymbol{Y}_2 + \sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_2\right)\chi_k\right)\right. \\
&\qquad\left. +\boldsymbol{g}_j\left(\boldsymbol{Y}_2 - \sqrt{\frac{h}{2}}\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_2\right)\chi_k\right)\right\}\xi_j.
\end{aligned}
\tag{3. 5}
$$

If the diffusion terms vanish, (3. 4) is equivalent to (2. 4) with $c_2 = 1$.

Theorem 3.2 *Let $\boldsymbol{g}_0(\boldsymbol{y})$ denote $A\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y})$ and suppose that (3. 1) satisfies Assumption 3.1 for $q = 2$. Suppose also that $\boldsymbol{g}_j'(\boldsymbol{y})\boldsymbol{g}_k(\boldsymbol{y})$ $(j, k = 1, 2, \ldots, m)$ satisfy the linear growth condition:*

$$
\left\|\boldsymbol{g}_j'(\boldsymbol{y})\boldsymbol{g}_k(\boldsymbol{y})\right\| \leq C\left(1 + \|\boldsymbol{y}\|\right) \tag{3. 6}
$$

for a constant $C > 0$. Then, (3. 4) is of weak order two.

Proof. First, let us consider

$$
\hat{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} + \frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) + \boldsymbol{H},
$$

where

$$
\tilde{\boldsymbol{y}}_{n+1} = \boldsymbol{y}_n + \frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{y}_n\right), \qquad \boldsymbol{K}_1 = \boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{y}_n\right).
$$

By Lemma 3.1, the local error of this method is of weak order three because

$$
\boldsymbol{Y}_1 = \boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{y}_n\right) + O\left(h^2\right), \qquad \boldsymbol{Y}_2 = \boldsymbol{y}_n + \frac{h}{2}\boldsymbol{g}_0\left(\boldsymbol{y}_n\right) + \frac{h^2}{8}A\boldsymbol{g}_0\left(\boldsymbol{y}_n\right) + O\left(h^3\right).
$$

Using $\boldsymbol{g}_0(\boldsymbol{y}) = A\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y})$, we can rewrite this as follows:

$$\begin{aligned}\hat{\boldsymbol{y}}_{n+1} &= \boldsymbol{y}_n + \frac{h}{2}\left(A\boldsymbol{y}_n + \boldsymbol{f}\left(\boldsymbol{y}_n\right)\right) + \frac{h}{2}A\boldsymbol{K}_1 \\ &+\frac{h}{2}\boldsymbol{f}\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) + \frac{h^{3/2}}{2}A\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j + \boldsymbol{H}.\end{aligned} \tag{3. 7}$$

The last term is the same in (3. 4) and (3. 7). If $\boldsymbol{g}_j \equiv \boldsymbol{0}$ for $j = 1, 2, \ldots, m$, then

$$\boldsymbol{y}_{n+1} - \hat{\boldsymbol{y}}_{n+1} = O\left(h^3\right)$$

because (3. 4) and (3. 7) are of order two for semilinear ODEs. Hence, all that remains concerning the local error is to check the difference between

$$h\varphi_2(hA)\left\{\boldsymbol{f}\left(\boldsymbol{Y}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}\left(\boldsymbol{Y}_1\right)\right\} + \sqrt{h}\left(\mathrm{e}^{\frac{h}{2}A} - I\right)\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j$$

and

$$\frac{h}{2}\left\{\boldsymbol{f}\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}\left(\boldsymbol{K}_1\right)\right\} + \frac{h^{3/2}}{2}A\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j.$$

As $\boldsymbol{Y}_1 = \boldsymbol{K}_1 + O\left(h^2\right)$, we have

$$\begin{aligned}&h\varphi_2(hA)\left\{\boldsymbol{f}\left(\boldsymbol{Y}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}\left(\boldsymbol{Y}_1\right)\right\} + \sqrt{h}\left(\mathrm{e}^{\frac{h}{2}A} - I\right)\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j \\ &= \frac{h}{2}\left\{\boldsymbol{f}\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}\left(\boldsymbol{K}_1\right)\right\} + \frac{h^{3/2}}{2}A\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j + h^{5/2}\boldsymbol{r} + O\left(h^3\right),\end{aligned}$$

where

$$\boldsymbol{r} = \frac{1}{2}\sum_{j=1}^{m}\left\{\frac{1}{4}A^2\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right) + \frac{1}{3}A\boldsymbol{f}'\left(\boldsymbol{K}_1\right)\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\right\}\xi_j.$$

Since $E[\boldsymbol{r}] = \boldsymbol{0}$, the local error of (3. 4) is also of weak order three.

As a sufficient condition for (1) in Theorem 3.1, it is known that the following two inequalities hold for sufficiently small $h > 0$:

$$\left\|E\left[\boldsymbol{y}_{n+1} - \boldsymbol{y}_n \mid \boldsymbol{y}_n\right]\right\| \le C\left(1 + \|\boldsymbol{y}_n\|\right)h, \qquad \left\|\boldsymbol{y}_{n+1} - \boldsymbol{y}_n\right\| \le X_n\left(1 + \|\boldsymbol{y}_n\|\right)\sqrt{h},$$

where C is a positive constant and X_n is a random variable which has moments of all orders [25, p. 102]. From the definition of $\boldsymbol{Y}_1$ in (3. 4) and the linear growth of $\boldsymbol{g}_j'\boldsymbol{g}_k$, we have

$$\begin{aligned}&\frac{1}{2}\left\|\boldsymbol{g}_j\left(\boldsymbol{Y}_1 + h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_1\right)\zeta_{kj}\right) - \boldsymbol{g}_j\left(\boldsymbol{Y}_1 - h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{Y}_1\right)\zeta_{kj}\right)\right\| \\ &\le C_1\left\|\boldsymbol{g}_j'\left(\boldsymbol{y}_n\right)h\sum_{k=1}^{m}\boldsymbol{g}_k\left(\boldsymbol{y}_n\right)\zeta_{kj}\right\| \\ &\le C_2\left(1 + \|\boldsymbol{y}_n\|\right)h\end{aligned}$$

for constants $C_1, C_2 > 0$. As $\boldsymbol{g}_j(\boldsymbol{y})$ $(j = 0, 1, \ldots, m)$ are Lipschitz continuous, they also satisfy the linear growth conditions of them. From these facts, we can see that the two inequalities requested above hold for (3. 4). Consequently, (3. 4) is of weak order two by Theorem 3.1. □

As another case, let us consider the family of SERK methods given by

$$\begin{aligned}\boldsymbol{y}_{n+1} = \boldsymbol{Y}_1 + \frac{h}{\gamma c_2 + c_3}\varphi_2(hA)\Bigg\{\gamma \boldsymbol{f}\left(\boldsymbol{Y}_3 + b_1\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) \\ +\boldsymbol{f}\left(\boldsymbol{Y}_4 + b_2\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - (\gamma+1)\boldsymbol{f}(\boldsymbol{y}_n)\Bigg\} \\ + \sqrt{h}\left(\mathrm{e}^{\frac{h}{2}A} - I\right)\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j + \boldsymbol{H},\end{aligned} \tag{3. 8}$$

where

$$\begin{aligned}\boldsymbol{Y}_3 &= \mathrm{e}^{c_2 hA}\boldsymbol{y}_n + c_2 h\varphi_1(c_2 hA)\boldsymbol{f}(\boldsymbol{y}_n), \\ \boldsymbol{Y}_4 &= \mathrm{e}^{c_3 hA}\boldsymbol{y}_n + h\left\{c_3\varphi_1(c_3 hA) - \psi(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) + h\psi(hA)\boldsymbol{f}(\boldsymbol{Y}_3)\end{aligned}$$

and b_1 and b_2 are parameters as well as c_2, c_3 and γ satisfying (2. 6). If the diffusion terms vanish, (3. 8) is equivalent to (2. 5).

Theorem 3.3 *Let $\boldsymbol{g}_0(\boldsymbol{y})$ denote $A\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y})$ and suppose that (3. 1) satisfies Assumption 3.1 for $q = 2$. Suppose also that $\boldsymbol{g}_j'(\boldsymbol{y})\boldsymbol{g}_k(\boldsymbol{y})$ $(j, k = 1, 2, \ldots, m)$ satisfy (3. 6). Then, (3. 8) is of weak order two if the parameters satisfy*

$$\frac{\gamma b_1 + b_2}{\gamma c_1 + c_2} = 1, \qquad \frac{\gamma b_1^2 + b_2^2}{\gamma c_1 + c_2} = 1 \tag{3. 9}$$

as well as (2. 6).

Proof. The last term is the same in (3. 7) and (3. 8). If $\boldsymbol{g}_j \equiv \boldsymbol{0}$ for $j = 1, 2, \ldots, m$, then

$$\boldsymbol{y}_{n+1} - \hat{\boldsymbol{y}}_{n+1} = O\left(h^3\right)$$

because (3. 7) and (3. 8) are of order two and three for semilinear ODEs, respectively. Hence, all that remains concerning the local error is to check the difference between

$$\begin{aligned}\frac{h}{\gamma c_2 + c_3}\varphi_2(hA)\Bigg\{\gamma \boldsymbol{f}\left(\boldsymbol{Y}_3 + b_1\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \gamma\boldsymbol{f}\left(\boldsymbol{Y}_3\right) \\ +\boldsymbol{f}\left(\boldsymbol{Y}_4 + b_2\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}\left(\boldsymbol{Y}_4\right)\Bigg\} \\ + \sqrt{h}\left(\mathrm{e}^{\frac{h}{2}A} - I\right)\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\end{aligned}$$

and

$$\frac{h}{2}\left\{\boldsymbol{f}\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j\right) - \boldsymbol{f}\left(\boldsymbol{K}_1\right)\right\} + \frac{h^{3/2}}{2}A\sum_{j=1}^{m}\boldsymbol{g}_j\left(\boldsymbol{Y}_2\right)\xi_j.$$

As $\boldsymbol{Y}_3 = \boldsymbol{K}_1 + (c_2-1)h\boldsymbol{g}_0(\boldsymbol{y}_n) + O(h^2)$ and $\boldsymbol{Y}_4 = \boldsymbol{K}_1 + (c_3-1)h\boldsymbol{g}_0(\boldsymbol{y}_n) + O(h^2)$, using (3. 9) we have

$$\begin{aligned}
&\frac{1}{\gamma c_2 + c_3}\left\{\gamma \boldsymbol{f}\left(\boldsymbol{Y}_3 + b_1\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\right) - \gamma\boldsymbol{f}(\boldsymbol{Y}_3)\right.\\
&\qquad\qquad \left. + \boldsymbol{f}\left(\boldsymbol{Y}_4 + b_2\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\right) - \boldsymbol{f}(\boldsymbol{Y}_4)\right\}\\
&= \sqrt{h}\sum_{j=1}^{m}\boldsymbol{f}'(\boldsymbol{K}_1)\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j + \frac{h}{2}\sum_{j,k=1}^{m}\boldsymbol{f}''(\boldsymbol{K}_1)\left[\boldsymbol{g}_j(\boldsymbol{Y}_2), \boldsymbol{g}_k(\boldsymbol{Y}_2)\right]\xi_j\xi_k + h^{3/2}\boldsymbol{r}_1\\
&\quad + \frac{\gamma b_2^3 + b_3^3}{6(\gamma c_2 + c_3)}h^{3/2}\sum_{j,k,l=1}^{m}\boldsymbol{f}'''(\boldsymbol{K}_1)\left[\boldsymbol{g}_j(\boldsymbol{Y}_2), \boldsymbol{g}_k(\boldsymbol{Y}_2), \boldsymbol{g}_l(\boldsymbol{Y}_2)\right]\xi_j\xi_k\xi_l + O\left(h^2\right),
\end{aligned}$$

where

$$\boldsymbol{r}_1 = \left(\frac{\gamma b_2 c_2 + b_3 c_3}{\gamma c_2 + c_3} - 1\right)\sum_{j=1}^{m}\boldsymbol{f}''(\boldsymbol{K}_1)\left[\boldsymbol{g}_0(\boldsymbol{y}_n), \boldsymbol{g}_j(\boldsymbol{Y}_2)\right]\xi_j.$$

On the other hand,

$$\begin{aligned}
&\boldsymbol{f}\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\right) - \boldsymbol{f}(\boldsymbol{K}_1)\\
&= \sqrt{h}\sum_{j=1}^{m}\boldsymbol{f}'(\boldsymbol{K}_1)\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j + \frac{h}{2}\sum_{j,k=1}^{m}\boldsymbol{f}''(\boldsymbol{K}_1)\left[\boldsymbol{g}_j(\boldsymbol{Y}_2), \boldsymbol{g}_k(\boldsymbol{Y}_2)\right]\xi_j\xi_k\\
&\quad + \frac{h^{3/2}}{6}\sum_{j,k,l=1}^{m}\boldsymbol{f}'''(\boldsymbol{K}_1)\left[\boldsymbol{g}_j(\boldsymbol{Y}_2), \boldsymbol{g}_k(\boldsymbol{Y}_2), \boldsymbol{g}_l(\boldsymbol{Y}_2)\right]\xi_j\xi_k\xi_l + O\left(h^2\right).
\end{aligned}$$

By utilizing these results and $h\phi_2(hA) = (h/2)I + (h^2/6)A + O(h^3)$, thus, we obtain

$$\begin{aligned}
&\frac{h}{\gamma c_2 + c_3}\varphi_2(hA)\left\{\gamma \boldsymbol{f}\left(\boldsymbol{Y}_3 + b_1\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\right) - \gamma\boldsymbol{f}(\boldsymbol{Y}_3)\right.\\
&\qquad\qquad \left. + \boldsymbol{f}\left(\boldsymbol{Y}_4 + b_2\sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\right) - \boldsymbol{f}(\boldsymbol{Y}_4)\right\}\\
&\quad + \sqrt{h}\left(\mathrm{e}^{\frac{h}{2}A} - I\right)\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\\
&= \frac{h}{2}\left\{\boldsymbol{f}\left(\boldsymbol{K}_1 + \sqrt{h}\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\right) - \boldsymbol{f}(\boldsymbol{K}_1)\right\} + \frac{h^{3/2}}{2}A\sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{Y}_2)\xi_j\\
&\quad + \frac{h^{5/2}}{2}\boldsymbol{r}_1 + h^{5/2}\boldsymbol{r}_2 + O\left(h^3\right),
\end{aligned}$$

where

$$\boldsymbol{r}_2 = \frac{1}{12}\left(\frac{\gamma b_2^3 + b_3^3}{\gamma c_2 + c_3} - 1\right)\sum_{j,k,l=1}^{m} \boldsymbol{f}'''(\boldsymbol{K}_1)\left[\boldsymbol{g}_j(\boldsymbol{Y}_2), \boldsymbol{g}_k(\boldsymbol{Y}_2), \boldsymbol{g}_l(\boldsymbol{Y}_2)\right]\xi_j\xi_k\xi_l$$
$$+\frac{1}{2}\sum_{j=1}^{m}\left\{\frac{1}{4}A^2\boldsymbol{g}_j(\boldsymbol{Y}_2) + \frac{1}{3}A\boldsymbol{f}'(\boldsymbol{K}_1)\boldsymbol{g}_j(\boldsymbol{Y}_2)\right\}\xi_j.$$

Since $E[\boldsymbol{r}_1] = E[\boldsymbol{r}_2] = \boldsymbol{0}$, the local error of (3. 8) is of weak order three. In a similar way to the proof of Theorem 3.2, we can see that (1) in Theorem 3.1 holds. Consequently, (3. 8) is of weak order two if the parameters satisfy (2. 6) and (3. 9). □

Remark 3.1 *As a simple solution of (2. 6) and (3. 9), we can find*

$$c_2 = \frac{1}{2}, \quad c_3 = 1, \quad \gamma = 4, \quad b_1 = \frac{6 \pm \sqrt{6}}{10}, \quad b_2 = \frac{3 \mp 2\sqrt{6}}{5}$$

(double sign in order). For this solution, the intermediate values $\boldsymbol{Y}_3$ and $\boldsymbol{Y}_4$ satisfy

$$\boldsymbol{Y}_3 = \boldsymbol{Y}_2, \qquad \boldsymbol{Y}_4 = \boldsymbol{Y}_1 + h\psi(hA)\left\{\boldsymbol{f}(\boldsymbol{Y}_2) - \boldsymbol{f}(\boldsymbol{y}_n)\right\}.$$

4 MS stability analysis for SERK methods

Let us investigate the stability properties of our SERK methods. We consider the following scalar test SDE [13]:

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sum_{j=1}^{m}\sigma_j y(t)\mathrm{d}W_j(t), \qquad t > 0, \qquad y(0) = y_0, \tag{4. 1}$$

where $y_0 \neq 0$ with probability one (w. p. 1) and where λ and σ_j $(1 \le j \le m)$ are complex numbers satisfying

$$2\Re(\lambda) + \sum_{j=1}^{m}\left|\sigma_j\right|^2 < 0. \tag{4. 2}$$

Because of (4. 2), the solution of (4. 1) is MS stable ($\lim_{t\to\infty} E[|y(t)|^2] = 0$).

When an SRK method is applied to (4. 1), it is generally expressed by

$$y_{n+1} = R\left(h, \lambda, \{\sigma_j\}_{j=1}^m, \boldsymbol{\eta}\right) y_n,$$

where $\boldsymbol{\eta}$ is a random vector whose elements are random variables appeared in the method. The method is said to be MS-stable for particular h, λ, σ_j $(j = 1, 2, \dots, m)$ if

$$E\left[\left|R\left(h, \lambda, \{\sigma_j\}_{j=1}^m, \boldsymbol{\eta}\right)\right|^2\right] < 1,$$

which means that $E[|y_n|^2] \to 0$ as $n \to \infty$ for the given h, λ, σ_j $(j = 1, 2, \dots, m)$. Further, the method is said to be A-stable in the MS if it is MS-stable for any $h > 0$ whenever (4. 2) holds [13].

Theorem 4.1 *The SERK method (3. 4) is A-stable in the MS for the test equation (4. 1).*

Proof. If we apply (3. 4) to (4. 2), then, we have

$$y_{n+1} = R\left(h, \lambda, \{\sigma_j\}_{j=1}^m, \{\xi_j\}_{j=1}^m, \{\zeta_{jk}\}_{j,k=1}^m\right) y_n,$$

where

$$R\left(h, \lambda, \{\sigma_j\}_{j=1}^m, \{\xi_j\}_{j=1}^m, \{\zeta_{jk}\}_{j,k=1}^m\right) = \mathrm{e}^{h\lambda}\left\{1 + \sum_{j=1}^m \sqrt{h}\sigma_j\xi_j + \sum_{j=1,k}^m h\sigma_j\sigma_k\zeta_{kj}\right\}.$$

From this, the MS stability function $\hat{R}$ of (3. 4) is given by

$$\hat{R}(p_r, q) \stackrel{\text{def}}{=} E\left[|R|^2\right] = \mathrm{e}^{2p_r}\left(1 + q + \frac{q^2}{2}\right),$$

where $p_r \stackrel{\text{def}}{=} \Re(\lambda)h$ and $q \stackrel{\text{def}}{=} \sum_{j=1}^m |\sigma_j|^2 h$. As we can rewrite (4. 2) by $2p_r + q < 0$, we have

$$\hat{R}(p_r, q) < \mathrm{e}^{2p_r}(1 - 2p_r + 2p_r^2).$$

The function in the right-hand side is less than 1 for any $p_r < 0$. Thus, $\hat{R}(p_r, q) < 1$ whenever $2p_r + q < 0$. Consequently, (3. 4) is A-stable in the MS. □

Theorem 4.2 *The SERK method (3. 8) is A-stable in the MS for (4. 1).*

Proof. The method (3. 8) is equivalent to (3. 4) except the second term, and both second terms disappear when they are applied to (4. 2). By Theorem 4.1, thus, (3. 8) is also A-stable in the MS □

As a comparison, let us look at stability properties of the SROCK2 method. When $m = 1$, its MS stability function is given by

$$\hat{R}(p, q) = |A(p)|^2 + |B(p)|^2 q + |C(p)|^2 \frac{q^2}{2},$$

where $p \stackrel{\text{def}}{=} \lambda h$ and $A(p), B(p), C(p)$ are polynomial functions of p. For details, see [5]. Now, we can plot the MS stability domain, that is, $\{(p, q) \mid \hat{R}(p, q) < 1\}$. For the SROCK2 method with six stages, the MS stability domain and its profile are given in Figure 1. The MS stability domain is indicated by the colored part in the left of the figure, and p_i denotes $\Im(\lambda)h$. The other part enclosed by the mesh indicates the domain in which the solution of the test SDE is MS stable. In the right part of the figure, the colored area indicates the profile of the MS stability domain when $p_i = 0$. We can see that the MS stability domain is large along the negative axis of p_r, but it is thin in the axis of p_i. On the other hand, we plot the MS stability domain of our SERK methods in Figure 2.

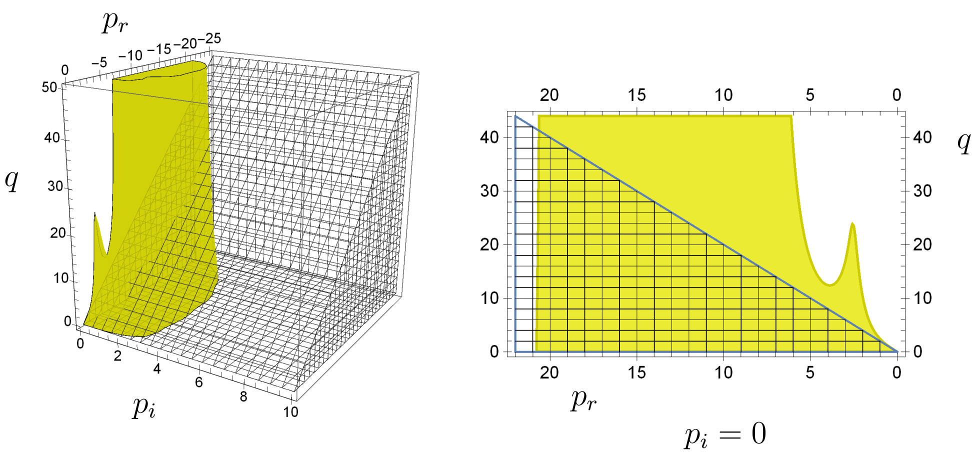

Figure 1: MS stability domain (left) and its profile (right) for the SROCK2 method with six stages

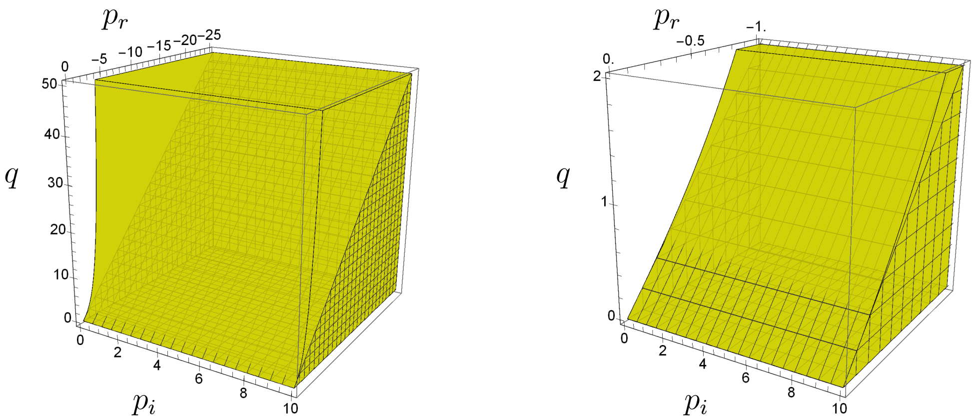

Figure 2: MS stability domain for our SERK methods

5 Numerical Experiments

In Section 3, we have derived our SERK methods. For example, (3. 4) is of weak order two and deterministic order two. In what follows, let us call this the SERKW2D2 method. As we have seen in Remark 3.1, (3. 8) with $c_2 = 1/2, c_3 = 1, \gamma = 4, b_1 = (6+\sqrt{6})/10, b_2 = (3-2\sqrt{6})/5$ is of weak order two and deterministic order three. Let us call this the SERKW2D3 method. As an implementation of the SROCK2 method, we do not directly use the Fortran codes from http://anmc.epfl.ch/Pdf/srock2.zip, but have implemented C codes by including rectp.f from the Fortran codes. Thus, the SROCK2 method in our C codes has the same parameter values as that in the Fortran codes.

In order to confirm the performance of the methods, we investigate some statistics in numerical experiments. As first two examples, let us consider the following scalar, nonstiff, nonlinear SDEs [5, 9] for which some functions of the exact solution are analytically

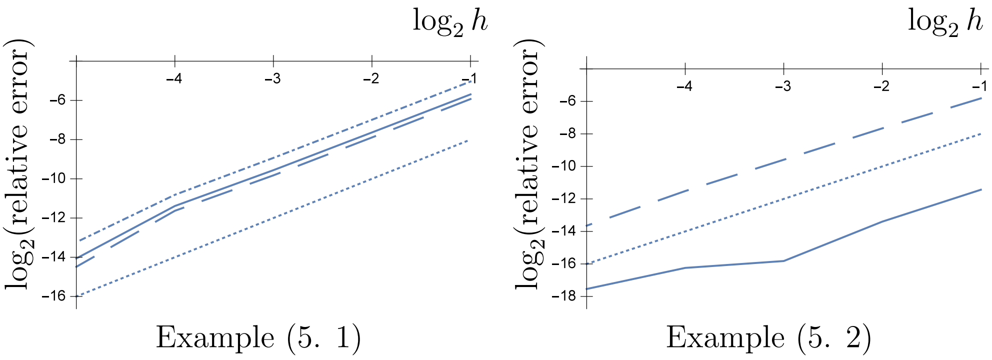

Figure 3: Log-log plots of the relative error versus h in the examples (5. 1) and (5. 2) (Solid: SERKW2D2, dash-dotted: SERKW2D3, dash: SROCK2, dotted: reference line with slope 2)

obtained. One of the examples is

$$\begin{aligned} \mathrm{d}y(t) &= \left(\frac{1}{4}y(t) + \frac{1}{2}\sqrt{(y(t))^2+1}\right)\mathrm{d}t + \sqrt{\frac{(y(t))^2+1}{2}}\mathrm{d}W(t), \quad t>0, \\ y(0) &= 0 \text{ (w.p.1)}. \end{aligned} \tag{5. 1}$$

For the solution $y(t)$, $E[(\text{arcsinh}\ y(t))^2] = t^2/4 + t/2$. The other is

$$\begin{aligned} \mathrm{d}y(t) &= y(t)\mathrm{d}t + \sum_{j=1}^{10} \frac{1}{a_j}\sqrt{y(t) + \frac{1}{b_j}}\mathrm{d}W_j(t), \quad t>0, \\ y(0) &= 1 \text{ (w.p.1)}, \end{aligned} \tag{5. 2}$$

where $a_1 = 10$, $a_2 = a_8 = 15$, $a_3 = a_7 = a_9 = 20$, $a_4 = a_6 = a_{10} = 25$, $a_5 = 40$, $b_1 = b_6 = 2$, $b_2 = b_7 = 4$, $b_3 = b_8 = 5$, $b_4 = b_9 = 10$, $b_5 = b_{10} = 20$. For the solution $y(t)$,

$$E\left[(y(t))^2\right] = \left(-68013 - 458120\mathrm{e}^t + 14926133\mathrm{e}^{2t}\right)/14400000.$$

In these examples, using the Mersenne twister algorithm [23] we simulate 1024×10^6 independent trajectories for a given h, and seek numerical approximations to $E[(\text{arcsinh}\ y(1))^2]$ and $E[(y(1))^2]$ for (5. 1) and (5. 2), respectively. The results are indicated in Figures 3. The solid, dash-dotted and dash lines denote the SERKW2D2 method, the SERKW2D3 method, and the SROCK2 method with 13 stages [5], respectively. The dotted one is a reference line with slope 2. Note that the results of the SERKW2D2 and SERKW2D3 methods in (5. 2) are the same because the drift term is linear. As a whole, we can observe that all methods achieve theoretical convergence order (weak order two), although the error of the SERKW2D2 method seems to be influenced by statistical errors when $h = 2^{-4}, 2^{-5}$, in the right plot.

In order to deal with stiff cases, let us consider the following SDE

$$\begin{aligned} \mathrm{d}\boldsymbol{y}(t) &= \begin{bmatrix} \alpha & 1 \\ -\omega^2 & \alpha \end{bmatrix}\boldsymbol{y}(t)\mathrm{d}t + \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}\boldsymbol{y}(t)\mathrm{d}W(t), \quad t>0, \\ \boldsymbol{y}(0) &= [1\ 1]^\top \text{ (w.p.1)} \end{aligned} \tag{5. 3}$$

Table 1: Step size for Numerical stability in (5. 3)

	Method	Step size	Absolute errors
Case 1)	SROCK2 (10 stages)	$h = 1/2$	9.1×10^{-5} (stable)
	SERKW2D2	$h = 1/2$	9.1×10^{-5} (stable)
Case 2)	SROCK2 (3 stages)	$h = 1/2^9$	1.9 (stable)
	SROCK2 (all stages)	$h = 1/2^8$	∞ (unstable)
	SERKW2D2	$h = 1/2$	4.1×10^{-5} (stable)
Case 3)	SROCK2 (3 stages)	$h = 1/2^7$	9.1×10^{-5} (stable)
	SROCK2 (5 stages)	$h = 1/2^6$	9.1×10^{-5} (stable)
	SROCK2 (all stages)	$h = 1/2^5$	∞ (unstable)
	SERKW2D2	$h = 1/2$	9.1×10^{-5} (stable)

for $\alpha, \omega, \sigma \in \mathbb{R}$. Since the eigenvalues of the matrix in the drift term are $\alpha \pm i\omega$, $\lim_{t\to\infty} E[\|\boldsymbol{y}(t)\|^2] = 0$ holds if $2\alpha + \sigma^2 < 0$. We investigate three cases:

$$\text{Case 1)} \quad \alpha = -100,\ \omega = 1,\ \sigma = \sqrt{199}, \qquad \text{Case 2)} \quad \alpha = -\tfrac{1}{4},\ \omega = 30\pi,\ \sigma = \tfrac{1}{4},$$
$$\text{Case 3)} \quad \alpha = -100,\ \omega = 30\pi,\ \sigma = \sqrt{199}.$$

In this example, we simulate 1×10^6 independent trajectories for a given h until $t = 10$ and seek numerical solutions to $E[\|\boldsymbol{y}(10)\|^2]$ by the SROCK2 and SERKW2D2 methods. Note that the SERKW2D2 and SERKW2D3 methods are equivalent for (5. 3) because the drift term is linear. For the solution $\boldsymbol{y}(t)$ in each case, we have

$$\begin{aligned}
&\text{Case 1)} \quad E\left[(y_1(10))^2\right] = \{1 + \sin(20)\}\mathrm{e}^{-20}, \quad E\left[(y_2(10))^2\right] = \{1 - \sin(20)\}\mathrm{e}^{-20},\\
&\text{Case 2)} \quad E\left[(y_1(10))^2\right] = E\left[(y_2(10))^2\right] = \mathrm{e}^{-35/8},\\
&\text{Case 3)} \quad E\left[(y_1(10))^2\right] = E\left[(y_2(10))^2\right] = \mathrm{e}^{-10}.
\end{aligned}$$

Table 1 gives numerical results, which indicate how small step size is necessary for each method to solve (5. 3) numerically stablely. In Case 1) the SROCK2 method with 10 stages can solve it for $h = 1/2$, but those with less than 10 stages cannot. In Case 2) the SROCK2 method cannot solve the SDE for $h = 1/2^8$ even if we make the stage number large. This is understandable because increasing stage number does not lead to making the MS stability domain large enough in the axis of p_i. Remember Fig. 1. In Case 3) the SROCK2 method with three stages cannot solve the SDE for $h = 1/2^6$, but those with five stages can. However, for $h = 1/2^5$ the SROCK2 method cannot solve by making the stage number large. On the other hand, the SERKW2D2 method can solve for $h = 1/2$ in all cases.

The fourth example comes from a stochastic Burgers equation with white noise in time only. Da Prato and Gatarek [8] have proved the existence and uniqueness of the global solution of a scalar Burgers equation with multiplicative noise driven by a scalar Wiener

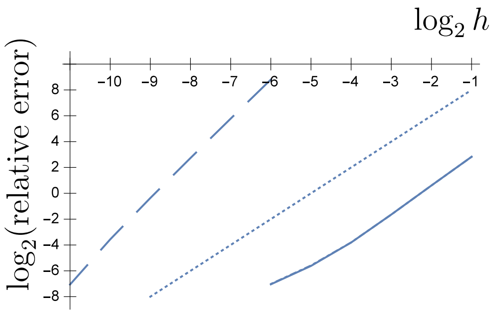

Figure 4: Log-log plot of the relative error of the variance versus h (Solid: SERKW2D2, dash-dotted: SERKW2D3, dash: SROCK2, dotted: reference line with slope 2)

process. Now, we consider an extended version of their equation:

$$\begin{aligned}
\mathrm{d}u(t,x) &= \left(\frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial u}{\partial x}(t,x)\right)\mathrm{d}t + k_1 u(t,x)\mathrm{d}W_1(t) \\
&\qquad + k_2\sqrt{1+\big(u(t,x)\big)^2}\,\mathrm{d}W_2(t), \qquad t>0, \quad x\in[0,1], \qquad (5.\ 4)\\
u(t,0) &= u(t,1) = 0 \ \text{(w.p.1)}, \qquad t>0,\\
u(0,x) &= 2\sin(\pi x)\ \text{(w.p.1)}, \qquad x\in[0,1],
\end{aligned}$$

where $k_1, k_2 \in \mathbb{R}$. If we discretize the space interval by $N+2$ equidistant points x_i $(0 \le i \le N+1)$ and define a vector-valued function by $\boldsymbol{y}(t) \overset{\text{def}}{=} [u(t,x_1)\ u(t,x_2)\ \cdots\ u(t,x_N)]^\top$, then we obtain the following non-commutative SDE

$$\begin{aligned}
&\mathrm{d}\boldsymbol{y}(t) = \big(A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t))\big)\mathrm{d}t + k_1\boldsymbol{y}(t)\mathrm{d}W_1(t) + \boldsymbol{b}\big(\boldsymbol{y}(t)\big)\mathrm{d}W_2(t), \quad t>0,\\
&\boldsymbol{y}(0) = [2\sin(\pi x_1)\ 2\sin(\pi x_2)\ \cdots\ 2\sin(\pi x_N)]^\top \ \text{(w. p. 1)}
\end{aligned} \qquad (5.\ 5)$$

by applying the central difference scheme to (5. 4), where

$$A \overset{\text{def}}{=} (N+1)^2 \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \mathbf{0} & & & 1 & -2 \end{bmatrix},$$

$$\boldsymbol{f}(\boldsymbol{y}) \overset{\text{def}}{=} \frac{N+1}{2}\begin{bmatrix} y_1y_2 \\ y_2(y_3-y_1) \\ \vdots \\ y_{N-1}(y_N - y_{N-2}) \\ y_N(-y_{N-1}) \end{bmatrix}, \qquad \boldsymbol{b}(\boldsymbol{y}) \overset{\text{def}}{=} k_2 \begin{bmatrix} \sqrt{1+y_1^2} \\ \sqrt{1+y_2^2} \\ \vdots \\ \sqrt{1+y_N^2} \end{bmatrix}.$$

For $N = 127$, $k_1 = 2$ and $k_2 = 3/2$, we seek an approximation to the variance of each element of $\boldsymbol{y}(t)$ at $t = 1$. As we do not know the exact solution of the SDE, we seek numerical approximations by the SROCK2 method with six stages for $h = 2^{-12}$ and use them instead of the exact variance.

In this example, we simulate 1024×10^4 independent trajectories for a given h. In order to solve the SDE numerically stably with reasonable cost by the SROCK2 method, we

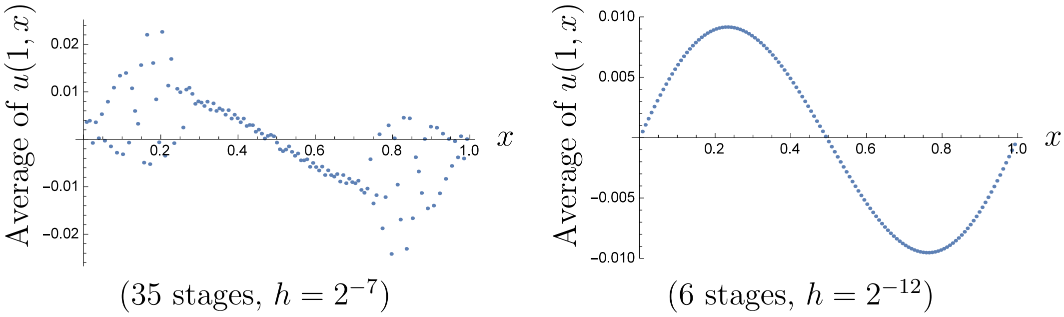

Figure 5: Approximations to $E[u(t,x)]$ at $t = 1$ given by the SROCK2 method

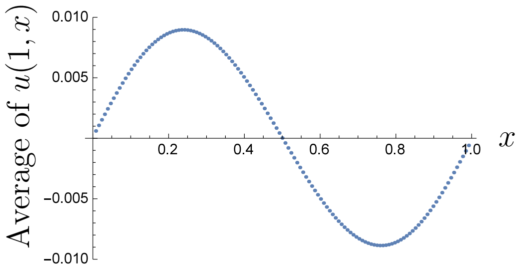

Figure 6: Approximation to $E[u(t,x)]$ at $t = 1$ given by the SERKW2D2 method for $h = 2^{-6}$

set the stage number of the method at 49, 35, 24, 17, 12 and 8 corresponding to the step size $2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}$ and 2^{-11}, respectively. The results are indicated in Figure 4. The solid, dash-dotted and dash lines denote the SERKW2D2 method, the SERKW2D3 method and the SROCK2 method, respectively. The dotted one is a reference line with slope 2. The figure indicates that the SERKW2D2 and SERKW2D3 methods have almost the same error, whereas the SROCK2 method seems to be inferior to them.

In Figures 5 and 6 plots for $E[u(t,x)]$ at $t = 1$ are shown. These are obtained by the SROCK method with 35 stages for $h = 2^{-7}$, the SROCK method with 6 stages for $h = 2^{-12}$ and the SERKW2D2 method for $h = 2^{-6}$.

Finally, Table 2 indicates comparisons of computational cost for each method in one step and one trajectory. In the table, n_e, n_r and n_m stand for the number of evaluations on the drift or diffusion coefficients, the number of generated pseudo random numbers and the number of the products of a matrix exponential function with a vector, respectively.

6 Concluding remarks

We have derived explicit SERK methods which achieve weak order two for non-commutative Itô SDEs with a semilinear drift term, and simultaneously achieve order two or three for ODEs. Using a scalar test SDE with complex coefficients, we have investigated the stability properties of the methods. As a result, we have proved that they are A-stable in the MS for the test SDE. To our best knowledge, there seems to be no weak second order method for which the A-stability in the MS is proven using the test SDE with complex coefficients, except a drift-implicit method of weak order two and deterministic order two

Table 2: comparisons of computational cost in one step and one trajectory

Method	n_e	n_r	n_m
SROCK2 with s stages	$s + 5m + 2$	$2m$	0
SERKW2D2	$6m + 2$	$2m$	6
SERKW2D3	$6m + 4$	$2m$	7

in [4]. In addition, as one of explicit stabilized methods we have picked up the SROCK2 method, and have plotted its MS stability domain.

In order to check numerical performance of the methods as well as their stability properties, we have performed four numerical experiments. In the first two experiments, scalar, nonstiff, nonlinear SDEs have been considered. The experiments have confirmed the theoretical convergence order, weak order two for our SERK methods and the SROCK2 method.

In the third experiment, we have dealt with three stiff cases. The experiment indicates that if the imaginary part of eigenvalues in the drift term is large, the SROCK2 method needs a very small step size for stability, whereas the SERK methods do not need.

In the last experiment, we have considered a stochastic Burgers equation with white noise, and compared our SERK methods with the SROCK2 method with several stages. This experiment has shown the superiority of the SERK methods to the SROCK2 method in terms of computational accuracy for relatively large step size.

Finally, we should make the following remarks. As we have seen, we can apply our methods to SDEs with a semilinear drift term and they have very good performance if the stiffness of the problem is in the matrix A, not in the nonlinear function $\boldsymbol{f}$. The SROCK2 method is applicable to more general SDEs without such restriction and they can also cope with stiff problems by increasing the stage number. When the dimension of a system of SDEs is not large and the stiffness is very strong, our methods will have a significant advantage over the SROCK2 method. This is because the method has to increase the stage number significantly, which leads to high computational cost. On the other hand, when the dimension of SDEs is very large, the SROCK2 method can still cope with high dimensional stiff SDEs by just increasing the stage number, but our methods need techniques in order to calculate matrix exponentials efficiently, such as known methods. Although we have not used such approaches for matrix exponentials in this paper, the application of these techniques will give considerably important impact on our methods to challenge very high dimensional SDEs with a semilinear drift term.

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