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THREE RECURSIVE APPROACHES FOR DECISION PROCESSES WITH A CONVERGING BRANCH SYSTEM

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Abstract

In this paper, we consider a decision process model with a converging branch system that is a nonserial transition system. The model is treated by three approaches. Thus we introduce three types of recursive equations by using a dynamic programming technique.

1. Introduction

Nonserial dynamic programming was proposed by Nemhauser [5] and has been widely discussed [1, 2, 3]. Nonserial dynamic systems are classified into the four structures: diverging branch systems, converging branch systems, feedback loop systems, and feedforward loop systems. Herein, we focus on a converging branch system and propose a finite-stage decision process model with a converging branch system. In the model, more than two initial states are given, the states are converged on the process, and finally the process is terminated at a final state.

We formulate the model in Section 2, and in Section 3 we discuss three recursive approaches to the model and introduce three types of recursive equations by using a dynamic programming technique. We give a numerical example in Section 4.

2. Notation and formulation

We introduce a finite-stage decision process model with a converging branch system.

1. X, a nonempty finite set, is the state space. The initial states $x_1, x_2, \ldots, x_L$ $(\in X)$ are specified at the beginning of the process. The process progresses through states $x_{L+1}, x_{L+2}, \ldots, x_{N-1} \in X$ with a converging branch system and is terminated at state $x_N \in X$.
2. U, a nonempty finite set, is the decision space. u_n $(\in U)$ represents the selected action for state x_n, $n = 1, 2, \ldots, N-1$. We denote the power set of U by 2^U:

   $$2^U = \{A : \text{a set} \mid A \subset U\}.$$

   Furthermore, we denote by U a point-to-set valued mapping from X to $2^U \backslash \{\phi\}$. $U(x)$, called the feasible decision space, represents the set of all feasible decisions in state x.

3. The transition matrix $E = (e_{ij}) \in \{0,1\}^{N \times N}$ is defined by

$$e_{ij} = \begin{cases} 1 & (\text{if } x_j \text{ is the next state to } x_i) \\ 0 & (\text{otherwise}), \end{cases}$$

and let $I_j = \{i \mid e_{ij} = 1\}$ $(j = L+1, L+2, \dots, N)$.

Let $G_r(U)$ be the graph of $U(\cdot)$:

$$G_r(U) = \{(x,u) \mid u \in U(x),\ x \in X\},$$

and, for a set A, $\#A$ means the number of elements in A. When an index set $I = \{m_1, m_2, \dots, m_M\}$ $(m_1 < m_2 < \cdots < m_M)$ is given, the corresponding sequence

$$(x_{m_1}, u_{m_1}, x_{m_2}, u_{m_2}, \dots, x_{m_M}, u_{m_M})$$

is denoted by $(x_m, u_m \mid m \in I)$. Similarly

$$(x_{m_1}, x_{m_2}, \dots, x_{m_M}) \qquad \text{and} \qquad (u_{m_1}, u_{m_2}, \dots, u_{m_M})$$

are denoted by $(x_m \mid m \in I)$ and $(u_m \mid m \in I)$, respectively.

4. $r_n : G_r(U) \to \mathbf{R}$ $(n = 1, 2, \dots, N-1)$ are the reward functions, where $\mathbf{R} = (-\infty, \infty)$. A decision u_n selected in state x_n confers a reward $r_n(x_n, u_n)$. The function $k : X \to \mathbf{R}$ is the terminal reward function.
5. $f_n : G_r(U)^{\#I_n} \to X$ $(n = L+1, L+2, \dots, N)$ are the converging transition laws. If a process in states $(x_m \mid m \in I_n)$ selects actions $(u_m \mid m \in I_n)$, it proceeds deterministically to the next state $f_n(x_m, u_m \mid m \in I_n)$.

Then our model is formulated as follows:

$$\begin{aligned} \text{(P)} \quad \text{Max } & r_1(x_1, u_1) + r_2(x_2, u_2) + \cdots + r_{N-1}(x_{N-1}, u_{N-1}) + k(x_N) \\ \text{s.t. } & x_n = f_n(x_m, u_m \mid m \in I_n) \qquad n = L+1, L+2, \dots, N \\ & u_n \in U(x_n) \qquad n = 1, 2, \dots, N-1. \end{aligned}$$

EXAMPLE 2.1. *Let* $N = 8$, $L = 3$, *and* $e_{14} = e_{25} = e_{37} = e_{46} = e_{57} = e_{68} = e_{78} = 1$ *(*$e_{ij} = 0$ *for the other pairs* (i, j)*). Then, for the given initial states* x_1, x_2, x_3, *the other states* x_4, x_5, x_6, x_7, x_8 *are determined by*

$$\begin{aligned} x_4 &= f_4(x_1, u_1), && x_1 \in X,\ u_1 \in U(x_1) \\ x_5 &= f_5(x_2, u_2), && x_2 \in X,\ u_2 \in U(x_2) \\ x_6 &= f_6(x_4, u_4), && x_4 \in X,\ u_4 \in U(x_4) \\ x_7 &= f_7(x_3, u_3, x_5, u_5), && x_3, x_5 \in X,\ u_3 \in U(x_3),\ u_5 \in U(x_5) \\ x_8 &= f_8(x_6, u_6, x_7, u_7), && x_6, x_7 \in X,\ u_6 \in U(x_6),\ u_7 \in U(x_7) \end{aligned}$$

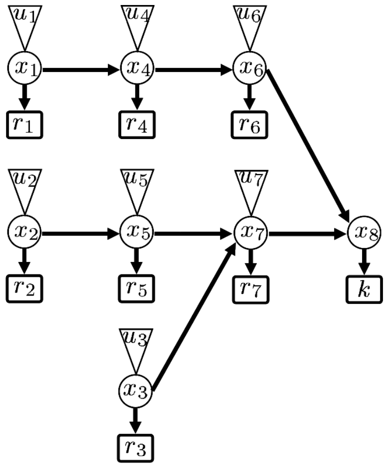

Figure 1. State transition tree for Example 2.1

(see Figure 1). In this case,

$$I_4 = \{1\}, \qquad I_5 = \{2\}, \qquad I_6 = \{4\}, \qquad I_7 = \{3,5\}, \qquad I_8 = \{6,7\}$$

and the problem is described as follows:

$$\begin{aligned} &Max\ \ r_1(x_1,u_1) + r_2(x_2,u_2) + \cdots + r_7(x_7,u_7) + k(x_8) \\ &s.t.\ \ x_n = f_n(x_m, u_m \,|\, m \in I_n) \qquad n = 4,5,\ldots,8 \\ &\qquad u_n \in U(x_n) \qquad n = 1,2,\ldots,7. \end{aligned}$$

□

3. Three recursive methods

3.1. Backward recursive equation I

We now give the first recursive method for the problem (P). First, P_n $(n = 1,2,\ldots)$ denotes a set of indexes of x_m satisfying the condition that the distance between x_m and the final state x_N in the state transition tree for (P) equals n. Especially, $P_0 = \{N\}$. We consider the following subproblems with the initial states $(x_m \,|\, m \in P_n)$ and the optimal value functions are denoted by $V^n(\cdot)$:

$$V^0(x_N) = k(x_N), \qquad x_N \in X$$

$$V^n(x_m \,|\, m \in P_n) = \max_{\substack{u_m \in U(x_m) \\ (m \in \bigcup_{l=1}^n P_l)}} \left[\sum_{m \in \bigcup_{l=1}^n P_l} r_m(x_m,u_m) + k(x_N) \right],$$

$$(x_m \,|\, m \in P_n) \in X^{\# P_n}, \qquad n = 1,2,\ldots.$$

We note that $x_h \in P_l$ $(l < n)$ that appears in above objective function is determined consecutively by the initial states $(x_m \mid m \in P_n)$ and decisions $(u_m \mid m \in \bigcup_{l=1}^{n} P_l)$ through the transition law f_h as follows

$$x_h = f_h(x_m, u_m \mid m \in I_h),$$

because, for $l = 0, 1, \ldots, n-1$, $h \in P_l$ implies $I_h \subset P_{l+1}$.

Theorem 3.1. *The following recursive equations hold:*

$$V^0(x_N) = k(x_N), \qquad x_N \in X$$

$$V^n(x_m \mid m \in P_n) = \max_{\substack{u_m \in U(x_m) \\ (m \in P_n)}} \left[\sum_{m \in P_n} r_m(x_m, u_m) + V^{n-1}(x_m \mid m \in P_{n-1}) \right]$$

$$(x_m \mid m \in P_n) \in X^{\# P_n}, \qquad n = 1, 2, \ldots,$$

where, in the second term of the objective function, if $1 \le m \le L$, x_m *is given* (x_m *is the initial state*). *Otherwise,* x_m *is given by* $x_m = f_m(x_l, u_l \mid l \in I_m)$.

Proof. By the definition of the subproblems,

$$V^n(x_m \mid m \in P_n) = \max_{\substack{u_m \in U(x_m) \\ (m \in \bigcup_{l=1}^{n} P_l)}} \left[\sum_{m \in \bigcup_{l=1}^{n} P_l} r_m(x_m, u_m) + k(x_N) \right], \qquad n = 1, 2, \ldots.$$

Since

$$P_n \cap P_m = \phi \qquad (n \ne m),$$

we have

$$P_n \cap \bigcup_{l=1}^{n-1} P_l = \phi.$$

Therefore

$$\begin{aligned}
&V^n(x_m \mid m \in P_n) \\
&= \max_{\substack{u_m \in U(x_m) \\ (m \in P_n)}} \left[\max_{\substack{u_m \in U(x_m) \\ (m \in \bigcup_{l=1}^{n-1} P_l)}} \left[\sum_{m \in P_n} r_m(x_m, u_m) + \sum_{m \in \bigcup_{l=1}^{n-1} P_l} r_m(x_m, u_m) + k(x_N) \right] \right] \\
&= \max_{\substack{u_m \in U(x_m) \\ (m \in P_n)}} \left[\sum_{m \in P_n} r_m(x_m, u_m) + \max_{\substack{u_m \in U(x_m) \\ (m \in \bigcup_{l=1}^{n-1} P_l)}} \left[\sum_{m \in \bigcup_{l=1}^{n-1} P_l} r_m(x_m, u_m) + k(x_N) \right] \right] \\
&= \max_{\substack{u_m \in U(x_m) \\ (m \in P_n)}} \left[\sum_{m \in P_n} r_m(x_m, u_m) + V^{n-1}(x_m \mid m \in P_{n-1}) \right]. \qquad \square
\end{aligned}$$

Example 3.1. *We consider the problem given by Example* 2.1. *Then*

$$P_0 = \{8\}, \qquad P_1 = \{6,7\}, \qquad P_2 = \{3,4,5\}, \qquad P_3 = \{1,2\},$$

and the subproblems become

$$\begin{aligned}
V^0(x_8) &= k(x_8)\\
V^1(x_6,x_7) &= \max_{u_m \in U(x_m)\ (m=6,7)} [r_6(x_6,u_6) + r_7(x_7,u_7) + k(x_8)]\\
V^2(x_3,x_4,x_5) &= \max_{u_m \in U(x_m)\ (3 \le m \le 7)} [r_3(x_3,u_3) + r_4(x_4,u_4) + r_5(x_5,u_5)\\
&\qquad + r_6(x_6,u_6) + r_7(x_7,u_7) + k(x_8)]\\
V^3(x_1,x_2) &= \max_{u_m \in U(x_m)\ (1 \le m \le 7)} [r_1(x_1,u_1) + r_2(x_2,u_2) + r_3(x_3,u_3) + r_4(x_4,u_4)\\
&\qquad + r_5(x_5,u_5) + r_6(x_6,u_6) + r_7(x_7,u_7) + k(x_8)].
\end{aligned}$$

By Theorem 3.1, *we get the following backward recursive equations:*

$$\begin{aligned}
V^0(x_8) &= k(x_8)\\
V^1(x_6,x_7) &= \max_{u_m \in U(x_m)\ (m=6,7)} [r_6(x_6,u_6) + r_7(x_7,u_7) + V^0(f_8(x_6,u_6,x_7,u_7))]\\
V^2(x_3,x_4,x_5) &= \max_{u_m \in U(x_m)\ (m=3,4,5)} [r_3(x_3,u_3) + r_4(x_4,u_4) + r_5(x_5,u_5)\\
&\qquad + V^1(f_6(x_4,u_4), f_7(x_3,u_3,x_5,u_5))]\\
V^3(x_1,x_2) &= \max_{u_m \in U(x_m)\ (m=1,2)} [r_1(x_1,u_1) + r_2(x_2,u_2) + V^2(x_3, f_4(x_1,u_1), f_5(x_2,u_2))]. \qquad \square
\end{aligned}$$

3.2. Forward recursive equation

Next, we introduce a forward recursive method for the problem (P). Starting with each initial state $x_1, x_2, \ldots, x_L$, subproblems are generated consecutively in a forward direction and the optimal values functions are denoted by $W^n(\cdot)$. For more details, see the following procedure.

Step 1. Let

$$I = \{1,2,\ldots,L\}$$

and

$$W^n(x_n) = 0, \qquad J_n = \phi, \qquad n = 1,2,\ldots,L.$$

Step 2. For each $n \notin I$ satisfying $I_n \subset I$, let

$$J_n = I_n \cup \left(\bigcup_{m \in I_n} J_m \right)$$

and

$$\bar{J}_n = (J_n \backslash \{1, 2, \ldots, L\}) \cup \{n\}.$$

If $n < N$, define the subproblem terminated with x_n as follows:

$$W^n(x_n) = \max_{\substack{(x_m, u_m \mid m \in J_n); \\ f_m(x_l, u_l \mid l \in I_m) = x_m \ (m \in \bar{J}_n)}} \left[\sum_{m \in J_n} r_m(x_m, u_m) \right], \qquad x_n \in X.$$

Otherwise (i.e. if $n = N$),

$$W^N(x_N) = \max_{\substack{(x_m, u_m \mid m \in J_N); \\ f_m(x_l, u_l \mid l \in I_m) = x_m \ (m \in \bar{J}_N)}} \left[\sum_{m \in J_N} r_m(x_m, u_m) + k(x_N) \right], \qquad x_N \in X.$$

According to the final state x_n, the corresponding subproblem may have no feasible solution. In that case, the value of $W^n(x_n)$ is regarded as "$-\infty$".

Step 3. if I equals $\{1, 2, \ldots, N-1\}$, stop. Otherwise, update I as

$$I \leftarrow I \cup \{n \mid I_n \subset I\}.$$

Then, go to Step 2. □

We note that, in Step 2,

$$J_N = \{1, 2, \ldots, N-1\}, \qquad \bar{J}_N = \{L+1, L+2, \ldots, N\}$$

hold. Therefore $\max_{x_N \in X} W^N(x_N)$ gives the optimal value of the original problem (P).

THEOREM 3.2. *The value functions $W^n(\cdot)$ satisfy the following forward recursive equations:*

$$W^n(x_n) = 0, \qquad x_n \in X,\ n = 1, 2, \ldots, L$$

$$W^n(x_n) = \max_{\substack{(x_m, u_m \mid m \in I_n); \\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \sum_{m \in I_n} (W^m(x_m) + r_m(x_m, u_m))$$

$$x_n \in X,\ n = L+1, L+2, \ldots, N-1$$

$$W^N(x_N) = \max_{\substack{(x_m, u_m \mid m \in I_N); \\ f_n(x_m, u_m \mid m \in I_N) = x_N}} \left[\sum_{m \in I_N} (W^m(x_m) + r_m(x_m, u_m)) + k(x_N) \right], \qquad x_N \in X.$$

PROOF. If $L < n < N$, by the definition of the subproblems,

$$W^n(x_n) = \max_{\substack{(x_m, u_m \mid m \in J_n); \\ f_m(x_l, u_l \mid l \in I_m) = x_m \ (m \in \bar{J}_n)}} \left[\sum_{m \in J_n} r_m(x_m, u_m) \right],$$

and, by the definition of J_n, we have

$$J_n = I_n \cup \left(\bigcup_{m \in I_n} J_m \right) \qquad \text{and} \qquad I_n \cap \left(\bigcup_{m \in I_n} J_m \right) = \phi.$$

Then,

$$\begin{aligned}
\bar{J}_n &= (J_n \backslash \{1, 2, \dots, L\}) \cup \{n\} \\
&= \left(\left(I_n \cup \left(\bigcup_{m \in I_n} J_m \right) \right) \backslash \{1, 2, \dots, L\} \right) \cup \{n\} \\
&= I_n \cup \left(\left(\bigcup_{m \in I_n} J_m \right) \backslash \{1, 2, \dots, L\} \right) \cup \{n\} \\
&= I_n \cup \left(\bigcup_{m \in I_n} (J_m \backslash \{1, 2, \dots, L\}) \right) \cup \{n\} \\
&= \left(\bigcup_{m \in I_n} ((J_m \backslash \{1, 2, \dots, L\}) \cup \{m\}) \right) \cup \{n\} \\
&= \left(\bigcup_{m \in I_n} \bar{J}_m \right) \cup \{n\}.
\end{aligned}$$

Hence, because

$$n \notin \left(\bigcup_{m \in I_n} \bar{J}_m \right),$$

we have

$$\text{“}m \in J_n \quad \Leftrightarrow \quad m \in I_n \ \text{or} \ m \in \bigcup_{m \in I_n} J_m\text{”}, \qquad I_n \cap \left(\bigcup_{m \in I_n} J_m \right) = \phi.$$

These facts imply that

$$\text{“}m \in \bar{J}_n \quad \Leftrightarrow \quad m = n \ \text{or} \ m \in \bigcup_{m \in I_n} \bar{J}_m\text{”}, \qquad n \notin \left(\bigcup_{m \in I_n} \bar{J}_m \right).$$

hold. Therefore

$$\begin{aligned}
W^n(x_n) &= \max_{\substack{(x_m, u_m \mid m \in I_n);\\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \left[\max_{\substack{(x_m, u_m \mid m \in \bigcup_{l \in I_n} J_l);\\ f_m(x_l, u_l \mid l \in I_m) = x_m \;\; (m \in \bigcup_{l \in I_n} \bar{J}_l)}} \left[\sum_{m \in I_n} r_m(x_m, u_m) + \sum_{m \in \bigcup_{l \in I_n} J_l} r_m(x_m, u_m) \right] \right] \\
&= \max_{\substack{(x_m, u_m \mid m \in I_n);\\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \left[\sum_{m \in I_n} r_m(x_m, u_m) + \max_{\substack{(x_m, u_m \mid m \in \bigcup_{l \in I_n} J_l);\\ f_m(x_l, u_l \mid l \in I_m) = x_m \;\; (m \in \bigcup_{l \in I_n} \bar{J}_l)}} \left[\sum_{m \in I_n} \sum_{i \in J_m} r_i(x_i, u_i) \right] \right] \\
&= \max_{\substack{(x_m, u_m \mid m \in I_n);\\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \left[\sum_{m \in I_n} r_m(x_m, u_m) + \sum_{m \in I_n} \max_{\substack{(x_l, u_l \mid l \in J_m);\\ f_l(x_i, u_i \mid i \in I_l) = x_l \;\; (l \in \bar{J}_m)}} \left[\sum_{i \in J_m} r_i(x_i, u_i) \right] \right] \\
&= \max_{\substack{(x_m, u_m \mid m \in I_n);\\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \left[\sum_{m \in I_n} r_m(x_m, u_m) + \sum_{m \in I_n} W^m(x_m) \right] \\
&= \max_{\substack{(x_m, u_m \mid m \in I_n);\\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \left[\sum_{m \in I_n} (r_m(x_m, u_m) + W^m(x_m)) \right].
\end{aligned}$$

When $n = N$, by the definition of the subproblems,

$$\begin{aligned}
W^N(x_N) = \max_{\substack{(x_m, u_m \mid m \in I_N);\\ f_N(x_m, u_m \mid m \in I_N) = x_N}} \Bigg[\max_{\substack{(x_m, u_m \mid m \in \bigcup_{l \in I_n} J_l);\\ f_m(x_l, u_l \mid l \in I_m) = x_m \;\; (m \in \bigcup_{l \in I_N} \bar{J}_l)}} \Bigg[& \sum_{m \in I_N} r_m(x_m, u_m) \\
& + \sum_{m \in \bigcup_{l \in I_N} J_l} r_m(x_m, u_m) + k(x_N) \Bigg] \Bigg].
\end{aligned}$$

Similarly, we can show the following equality:

$$W^n(x_n) = \max_{\substack{(x_m, u_m \mid m \in I_n);\\ f_n(x_m, u_m \mid m \in I_n) = x_n}} \left[\sum_{m \in I_n} (r_m(x_m, u_m) + W^m(x_m)) + k(x_N) \right]. \qquad \square$$

EXAMPLE 3.2. *We consider again the problem given by Example* 2.1. *First, for the initial states* x_1, x_2, x_3, *let*

$$W^1(x_1) = W^2(x_2) = W^3(x_3) = 0,$$

and the subproblems become as follows:

$$W^4(x_4) = \max_{(x_1,u_1);f_4(x_1,u_1)=x_4} [r_1(x_1,u_1)]$$

$$W^5(x_5) = \max_{(x_2,u_2);f_5(x_2,u_2)=x_5} [r_2(x_2,u_2)]$$

$$W^6(x_6) = \max_{(x_1,u_1,x_4,u_4);f_4(x_1,u_1)=x_4,f_6(x_4,u_4)=x_6} [r_1(x_1,u_1) + r_4(x_4,u_4)]$$

$$W^7(x_7) = \max_{\substack{(x_m,u_m\,|\,m\in\{2,3,5\};\\ f_5(x_2,u_2)=x_5,f_7(x_3,u_3,x_5,u_5)=x_7}} [r_2(x_2,u_2) + r_3(x_3,u_3) + r_5(x_5,u_5)]$$

$$W^8(x_8) = \max_{\substack{(x_m,u_m\,|\,m\in\{1,2,\dots,7\};f_4(x_1,u_1)=x_4,f_5(x_2,u_2)=x_5,\\ f_6(x_4,u_4)=x_6,f_7(x_3,u_3,x_5,u_5)=x_7,f_8(x_6,u_6,x_7,u_7)=x_8}} [r_1(x_1,u_1) + r_2(x_2,u_2)$$
$$+ r_3(x_3,u_3) + r_4(x_4,u_4) + r_5(x_5,u_5) + r_6(x_6,u_6) + r_7(x_7,u_7) + k(x_8)].$$

Then, by using Theorem 3.2, *the following forward recursive equations hold:*

$$W^1(x_1) = W^2(x_2) = W^3(x_3) = 0$$

$$W^4(x_4) = \max_{(x_1,u_1);f_4(x_1,u_1)=x_4} [W^1(x_1) + r_1(x_1,u_1)]$$

$$W^5(x_5) = \max_{(x_2,u_2);f_5(x_2,u_2)=x_5} [W^2(x_2) + r_2(x_2,u_2)]$$

$$W^6(x_6) = \max_{(x_4,u_4);f_6(x_4,u_4)=x_6} [W^4(x_4) + r_4(x_4,u_4)]$$

$$W^7(x_7) = \max_{\substack{(x_3,u_3,x_5,u_5);\\ f_7(x_3,u_3,x_5,u_5)=x_7}} [W^3(x_3) + W^5(x_5) + r_3(x_3,u_3) + r_5(x_5,u_5)]$$

$$W^8(x_8) = \max_{\substack{(x_6,u_6,x_7,u_7);\\ f_8(x_6,u_6,x_7,u_7)=x_8}} [W^6(x_6) + W^7(x_7) + r_6(x_6,u_6) + r_7(x_7,u_7) + k(x_8)]. \qquad \square$$

3.3. Backward recursive equation II

The results in this subsection are discussed in [4]. When we construct the subproblems, starting with the initial target state sequence $Q = (x_N)$, we add x_n to Q in the order that coincides one of the node order for a depth-first search for a state transition tree with the root x_N. Without loss of generality, we regard that index order as

$$N \to N-1 \to N-2 \to \cdots \to 2 \to 1,$$

by renumbering the state index.

Specifically, when we consider the decision process in Example 2.1, the indexes are renumbered as shown in Figure 2. Then, the sequence of the target state sequence becomes

$$(x_8) \to (x_7, x_8) \to (x_6, x_7, x_8) \to \cdots \to (x_1, x_2, \ldots, x_7, x_8)$$

and we consider the corresponding subproblems and the optimal value functions $v^n(\cdot)$ as follows:

$$\begin{aligned}
v^8(x_8) &= k(x_8) \\
v^7(x_7; x_3, u_3) &= \max_{u_7 \in U(x_7)} [r_7(x_7, u_7) + k(x_8)] \\
v^6(x_6; x_3, u_3, x_5, u_5) &= \max_{u_m \in U(x_m)\ (m=6,7)} [r_6(x_6, u_6) + r_7(x_7, u_7) + k(x_8)] \\
v^5(x_5; x_3, u_3) &= \max_{u_m \in U(x_m)\ (5 \le m \le 7)} [r_5(x_5, u_5) + r_6(x_6, u_6) + r_7(x_7, u_7) + k(x_8)] \\
v^4(x_4; x_3, u_3) &= \max_{u_m \in U(x_m)\ (4 \le m \le 7)} [r_4(x_4, u_4) + r_5(x_5, u_5) + r_6(x_6, u_6) \\
&\qquad + r_7(x_7, u_7) + k(x_8)] \\
v^3(x_3) &= \max_{u_m \in U(x_m)\ (3 \le m \le 7)} [r_3(x_3, u_3) + r_4(x_4, u_4) + r_5(x_5, u_5) \\
&\qquad + r_6(x_6, u_6) + r_7(x_7, u_7) + k(x_8)] \\
v^2(x_2) &= \max_{u_m \in U(x_m)\ (2 \le m \le 7)} [r_2(x_2, u_2) + r_3(x_3, u_3) + r_4(x_4, u_4) + r_5(x_5, u_5) \\
&\qquad + r_6(x_6, u_6) + r_7(x_7, u_7) + k(x_8)] \\
v^1(x_1) &= \max_{u_m \in U(x_m)\ (1 \le m \le 7)} [r_1(x_1, u_1) + r_2(x_2, u_2) + r_3(x_3, u_3) + r_4(x_4, u_4) \\
&\qquad + r_5(x_5, u_5) + r_6(x_6, u_6) + r_7(x_7, u_7) + k(x_8)].
\end{aligned}$$

We give the general form of subproblems.

$\underline{n = N}$

For state sequence (x_N), the subproblem is given by

$$v^N(x_N) = k(x_N), \qquad x_N \in X.$$

$\underline{n < N}$

For state sequence $(x_n, x_{n+1}, \ldots, x_N)$, the subproblem is given by

$$\begin{aligned}
&v^n(x_n; (x_m, u_m \mid m \in J_n)) \\
&\quad = \max_{u_m \in U(x_m)\ (m=n,n+1,\ldots,N-1)} [r_n(x_n, u_n) + r_{n+1}(x_{n+1}, u_{n+1}) + \cdots + k(x_N)], \\
&\quad\quad x_n \in X,\ x_m \in X,\ u_m \in U(x_m)\ \ (m \in J_n),
\end{aligned}$$

where

$$J_n = \bigcup_{l=n+1}^{N} \{ j \in I_l \mid j < n \}.$$

The set of indexes that indicate the initial states is denoted by I_{Init}. For example, we have $I_{\mathrm{Init}} = \{1, 4, 6\}$ for the decision process shown in Figure 2.

Then the following recursive relations are shown in [4].

Proposition 3.1. *Put $J_N = \phi$, then,*

(i) *if $n + 1 \notin I_{\mathrm{Init}}$,*

$$J_n = J_{n+1} \cup \{ j \in I_{n+1} \mid j < n \}.$$

(ii) *if $n + 1 \in I_{\mathrm{Init}}$,*

$$J_n = J_{n+1} \backslash \{n\}.$$

Theorem 3.3. *We have the following backward recursive equations:*

$$v^N(x_N) = k(x_N), \qquad x_N \in X$$

$$v^n(x_n; (x_m, u_m \mid m \in J_n))$$
$$= \max_{u_n \in U(x_n)} [r_n(x_n, u_n) + v^{n+1}(f_{n+1}(x_m, u_m \mid m \in I_{n+1}); (x_m, u_m \mid m \in J_{n+1}))],$$
$$x_n \in X,\ n + 1 \notin I_{\mathrm{Init}}$$

$$v^n(x_n; (x_m, u_m \mid m \in J_n))$$
$$= \max_{u_n \in U(x_n)} [r_n(x_n, u_n) + v^{n+1}(x_{n+1}; (x_m, u_m \mid m \in J_{n+1}))], \qquad x_n \in X,\ n + 1 \in I_{\mathrm{Init}}.$$

Example 3.3. We consider the decision process shown in Figure 2. First, by Proposition 3.1, we can get J_j $(j = 8, 7, \ldots, 1)$ as follows:

$$J_8 = \phi.$$

Since $j = 7 \in I_8 = \{3, 7\}$ and $8 \notin I_{\mathrm{Init}}$,

$$J_7 = J_8 \cup \{ j \in I_8 \mid j < 7\} = \phi \cup \{3\} = \{3\}.$$

Similarly,

$$j = 6 \in I_7 = \{5, 6\},\ 7 \notin I_{\mathrm{Init}}$$
$$\Rightarrow \quad J_6 = J_7 \cup \{ j \in I_7 \mid j < 6\} = \{3\} \cup \{5\} = \{3, 5\},$$
$$j = 5 \in I_7 = \{5, 6\},\ 6 \in I_{\mathrm{Init}}$$
$$\Rightarrow \quad J_5 = J_6 \backslash \{5\} = \{3, 5\} \backslash \{5\} = \{3\},$$

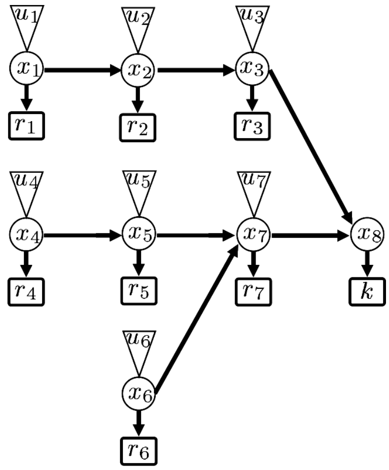

Figure 2. Transition tree of Example 3.3

$$
\begin{aligned}
& j = 4 \in I_5 = \{4\},\ 5 \notin I_{\mathrm{Init}} \\
& \qquad \Rightarrow \quad J_4 = J_5 \cup \{j \in I_5 \mid j < 4\} = \{3\} \cup \phi = \{3\}, \\
& j = 3 \in I_8 = \{3, 7\},\ 4 \in I_{\mathrm{Init}} \\
& \qquad \Rightarrow \quad J_3 = J_4 \backslash \{3\} = \{3\} \backslash \{3\} = \phi, \\
& j = 2 \in I_3 = \{2\},\ 3 \notin I_{\mathrm{Init}} \\
& \qquad \Rightarrow \quad J_2 = J_3 \cup \{j \in I_3 \mid j < 2\} = \phi \cup \phi = \phi, \\
& j = 1 \in I_2 = \{1\},\ 2 \notin I_{\mathrm{Init}} \\
& \qquad \Rightarrow \quad J_1 = J_2 \cup \{j \in I_2 \mid j < 1\} = \phi \cup \phi = \phi.
\end{aligned}
$$

Then, we have the following recursive equations by Theorem 3.1:

$$
\begin{aligned}
v^8(x_8) &= k(x_8) \\
v^7(x_7; (x_m, u_m \mid m \in J_7)) &= \max_{u_7 \in U(x_7)} [r_7(x_7, u_7) + v^8(f_8(x_m, u_m \mid m \in I_8); (x_m, u_m \mid m \in J_8))] \\
v^7(x_7; (x_m, u_m \mid m \in \{3\})) &= \max_{u_7 \in U(x_7)} [r_7(x_7, u_7) + v^8(f_8(x_m, u_m \mid m \in \{3, 7\}); (x_m, u_m \mid m \in \phi))] \\
v^7(x_7; x_3, u_3) &= \max_{u_7 \in U(x_7)} [r_7(x_7, u_7) + v^8(f_8(x_3, u_3, x_7, u_7))]
\end{aligned}
$$

and

$$v^6(x_6; (x_m, u_m \mid m \in J_6)) = \max_{u_6 \in U(x_6)} [r_6(x_6, u_6) + v^7(f_7(x_m, u_m \mid m \in I_7); (x_m, u_m \mid m \in J_7))]$$

$$v^6(x_6; (x_m, u_m \mid m \in \{3, 5\})) = \max_{u_6 \in U(x_6)} [r_6(x_6, u_6) + v^7(f_7(x_m, u_m \mid m \in \{5, 6\});$$

$$(x_m, u_m \mid m \in \{3\}))]$$

$$v^6(x_6; x_3, u_3, x_5, u_5) = \max_{u_6 \in U(x_6)} [r_6(x_6, u_6) + v^7(f_7(x_5, u_5, x_6, u_6); x_3, u_3)].$$

Similarly, we have

$$v^5(x_5; x_3, u_3) = \max_{u_5 \in U(x_5)} [r_5(x_5, u_5) + v^6(x_6; x_3, u_3, x_5, u_5)]$$

$$v^4(x_4; x_3, u_3) = \max_{u_4 \in U(x_4)} [r_4(x_4, u_4) + v^5(f_5(x_4, u_4); x_3, u_3)]$$

$$v^3(x_3) = \max_{u_3 \in U(x_3)} [r_3(x_3, u_3) + v^4(x_4; x_3, u_3)]$$

$$v^2(x_2) = \max_{u_2 \in U(x_2)} [r_2(x_2, u_2) + v^3(f_3(x_2, u_2))]$$

$$v^1(x_1) = \max_{u_1 \in U(x_1)} [r_1(x_1, u_1) + v^2(f_2(x_1, u_1))]. \qquad \square$$

4. Numerical example

We consider the decision process problem given by Example 2.1 with the following data:

$$X = \{s_1, s_2\}, \qquad U(x) = U = \{a_1, a_2\} \ ({}^{\forall} x \in X)$$

$$x_1 = s_1, \qquad x_2 = s_2, \qquad x_3 = s_1$$

$f_4(x, u)$

$x \backslash u$	a_1	a_2
s_1	s_2	s_1
s_2	—	—

$f_5(x, u)$

$x \backslash u$	a_1	a_2
s_1	—	—
s_2	s_1	s_2

$f_6(x, u)$

$x \backslash u$	a_1	a_2
s_1	s_2	s_1
s_2	s_1	s_2

$f_7(x, u, y, v)$

$(x, u) \backslash (y, v)$	(s_1, a_1)	(s_1, a_2)	(s_2, a_1)	(s_2, a_2)
(s_1, a_1)	s_2	s_2	s_1	s_2
(s_1, a_2)	s_1	s_2	s_2	s_1

$f_8(x, u, y, v)$

$(x, u) \backslash (y, v)$	(s_1, a_1)	(s_1, a_2)	(s_2, a_1)	(s_2, a_2)
(s_1, a_1)	s_1	s_1	s_2	s_2
(s_1, a_2)	s_2	s_2	s_1	s_2
(s_2, a_1)	s_1	s_2	s_2	s_1
(s_2, a_2)	s_1	s_2	s_1	s_1

and

(x,u)	(s_1,a_1)	(s_1,a_2)	(s_2,a_1)	(s_2,a_2)
$r_1(x,u)$	4	3	–	–
$r_2(x,u)$	–	–	3	2
$r_3(x,u)$	3	4	–	–
$r_4(x,u)$	2	3	3	1
$r_5(x,u)$	1	2	4	2
$r_6(x,u)$	2	3	3	4
$r_7(x,u)$	4	5	2	3

x	s_1	s_2
$k(x)$	5	2

.

4.1. Compute with backward recursive equation I

First, for the terminal state x_8, we get

$$V^0(s_1) = k(s_1) = 5, \qquad V^0(s_2) = k(s_2) = 2.$$

By using the backward recursive equation given by Theorem 3.1, we compute $V^1(s_1, s_1)$ and the corresponding optimal decision function $\sigma_1^*(s_1, s_1)$:

$$\begin{aligned}
V^1(s_1,s_1) = &\max_{u_6,u_7\in U}[r_6(s_1,u_6)+r_7(s_1,u_7)+V^0(f_8(s_1,u_6,s_1,u_7))]\\
&= [r_6(s_1,a_1)+r_7(s_1,a_1)+V^0(f_8(s_1,a_1,s_1,a_1))]\\
&\quad \vee [r_6(s_1,a_1)+r_7(s_1,a_2)+V^0(f_8(s_1,a_1,s_1,a_2))]\\
&\quad \vee [r_6(s_1,a_2)+r_7(s_1,a_1)+V^0(f_8(s_1,a_2,s_1,a_1))]\\
&\quad \vee [r_6(s_1,a_2)+r_7(s_1,a_2)+V^0(f_8(s_1,a_2,s_1,a_2))]\\
&= [2+4+V^0(s_1)] \vee [2+5+V^0(s_1)] \vee [3+4+V^0(s_2)] \vee [3+5+V^0(s_2)]\\
&= 11 \vee 12 \vee 9 \vee 10 = 12, \qquad \sigma_1^*(s_1,s_1) = (u_6^*, u_7^*) = (a_1, a_2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
V^1(s_1,s_2) &= 6 \vee 7 \vee 10 \vee 8 = 10, \qquad \sigma_1^*(s_1,s_2) = (u_6^*, u_7^*) = (a_2, a_1)\\
V^1(s_2,s_1) &= 12 \vee 10 \vee 13 \vee 11 = 13, \qquad \sigma_1^*(s_2,s_1) = (u_6^*, u_7^*) = (a_2, a_1)\\
V^1(s_2,s_2) &= 7 \vee 11 \vee 11 \vee 12 = 12, \qquad \sigma_1^*(s_2,s_2) = (u_6^*, u_7^*) = (a_2, a_2).
\end{aligned}$$

Next, we compute V^2 and the corresponding optimal decision function σ_2^*:

$$\begin{aligned}
&V^2(s_1,s_1,s_1)\\
&\quad = \max_{u_3,u_4,u_5\in U}[r_3(s_1,u_3)+r_4(s_1,u_4)+r_5(s_1,u_5)+V^1(f_6(s_1,u_4), f_7(s_1,u_3,s_1,u_5))]
\end{aligned}$$

$$
\begin{aligned}
&= [r_3(s_1,a_1) + r_4(s_1,a_1) + r_5(s_1,a_1) + V^1(f_6(s_1,a_1), f_7(s_1,a_1,s_1,a_1))] \\
&\quad \vee [r_3(s_1,a_2) + r_4(s_1,a_1) + r_5(s_1,a_1) + V^1(f_6(s_1,a_1), f_7(s_1,a_2,s_1,a_1))] \\
&\quad \vee [r_3(s_1,a_1) + r_4(s_1,a_1) + r_5(s_1,a_2) + V^1(f_6(s_1,a_1), f_7(s_1,a_1,s_1,a_2))] \\
&\quad \vee [r_3(s_1,a_2) + r_4(s_1,a_1) + r_5(s_1,a_2) + V^1(f_6(s_1,a_1), f_7(s_1,a_2,s_1,a_2))] \\
&\quad \vee [r_3(s_1,a_1) + r_4(s_1,a_2) + r_5(s_1,a_1) + V^1(f_6(s_1,a_2), f_7(s_1,a_1,s_1,a_1))] \\
&\quad \vee [r_3(s_1,a_2) + r_4(s_1,a_2) + r_5(s_1,a_1) + V^1(f_6(s_1,a_2), f_7(s_1,a_2,s_1,a_1))] \\
&\quad \vee [r_3(s_1,a_1) + r_4(s_1,a_2) + r_5(s_1,a_2) + V^1(f_6(s_1,a_2), f_7(s_1,a_1,s_1,a_2))] \\
&\quad \vee [r_3(s_1,a_2) + r_4(s_1,a_2) + r_5(s_1,a_2) + V^1(f_6(s_1,a_2), f_7(s_1,a_2,s_1,a_2))] \\
&= [3+2+1+V^1(s_2,s_2)] \vee [4+2+1+V^1(s_2,s_1)] \\
&\quad \vee [3+2+2+V^1(s_2,s_2)] \vee [4+2+2+V^1(s_2,s_2)] \\
&\quad \vee [3+3+1+V^1(s_1,s_2)] \vee [4+3+1+V^1(s_1,s_1)] \\
&\quad \vee [3+3+2+V^1(s_1,s_2)] \vee [4+3+2+V^1(s_1,s_2)] \\
&= 18 \vee 20 \vee 19 \vee 20 \vee 17 \vee 20 \vee 18 \vee 19 = 20
\end{aligned}
$$

$$\sigma_2^*(s_1,s_1,s_1) = (u_3^*, u_4^*, u_5^*) = (a_2,a_1,a_1), (a_2,a_1,a_2), (a_2,a_2,a_1)$$

$$V^2(s_1,s_1,s_2) = 22, \qquad \sigma_2^*(s_1,s_1,s_2) = (a_1,a_1,a_1), (a_2,a_1,a_1), (a_1,a_2,a_1)$$

$$V^2(s_1,s_2,s_1) = 20, \qquad \sigma_2^*(s_1,s_2,s_1) = (a_2,a_1,a_1), (a_2,a_2,a_2)$$

$$V^2(s_1,s_2,s_2) = 22, \qquad \sigma_2^*(s_1,s_2,s_2) = (a_2,a_1,a_1).$$

Finally, we compute V^3 and the corresponding optimal decision function σ_3^*:

$$
\begin{aligned}
V^3(s_1,s_2) &= \max_{u_1,u_2 \in U} [r_1(s_1,u_1) + r_2(s_2,u_2) + V^2(s_1, f_4(s_1,u_1), f_5(s_2,u_2))] \\
&= [4+3+V^2(s_1,s_2,s_1)] \vee [4+2+V^2(s_1,s_2,s_2)] \\
&\quad \vee [3+3+V^2(s_1,s_1,s_1)] \vee [3+2+V^2(s_1,s_1,s_2)] \\
&= [4+3+20] \vee [4+2+22] \vee [3+3+20] \vee [3+2+22] \\
&= 27 \vee 28 \vee 26 \vee 27 = 28, \qquad \sigma_3^*(s_1,s_2) = (u_1^*, u_2^*) = (a_1,a_2).
\end{aligned}
$$

Thus, the optimal value is $V^3(s_1,s_2) = 28$ and an optimal state-decision sequence is given as follows:

$$
\begin{aligned}
&(x_1, x_2) = (s_1, s_2)\\
&\quad \to (u_1^*, u_2^*) = \sigma_3^*(s_1, s_2) = (a_1, a_2)\\
&\quad \to (x_4, x_5, x_3) = (f_4(s_1, a_1), f_5(s_2, a_2), s_1) = (s_2, s_2, s_1)\\
&\quad \to (u_4^*, u_5^*, u_3^*) = \sigma_2^*(s_2, s_2, s_1) = (a_1, a_1, a_1)\\
&\quad \to (x_6, x_7) = (f_6(s_2, a_1), f_7(s_1, a_1, s_2, a_1)) = (s_1, s_1)\\
&\quad \to (u_6^*, u_7^*) = \sigma_1^*(s_1, s_1) = (a_1, a_2)\\
&\quad \to x_8 = f_8(s_1, a_1, s_1, a_2) = s_1.
\end{aligned}
$$

4.2. Compute with forward recursive equation

First, for the initial states $x_1 = s_1$, $x_2 = s_2$, $x_3 = s_1$, we get

$$W^1(s_1) = W^2(s_2) = W^3(s_1) = 0$$

By using the forward recursive equation in subsection 3.2, we compute $W^4(s_1)$ and the corresponding optimal decision function $\tau_4^*(s_1)$:

$$
\begin{aligned}
W^4(s_1) &= \max_{\substack{(x_1, u_1);\\ f_4(x_1, u_1) = s_1}} [W^1(x_1) + r_1(x_1, u_1)] = \max_{(x_1, u_1) \in \{(s_1, a_2)\}} [W^1(x_1) + r_1(x_1, u_1)]\\
&= W^1(s_1) + r_1(s_1, a_2) = 0 + 3 = 3, \qquad \tau_4^*(s_1) = (x_1, u_1^*) = (s_1, a_2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
W^4(s_2) &= 4, \qquad \tau_4^*(s_2) = (x_1, u_1^*) = (s_1, a_1)\\
W^5(s_1) &= 3, \qquad \tau_5^*(s_1) = (x_2, u_2^*) = (s_2, a_1)\\
W^5(s_2) &= 2, \qquad \tau_5^*(s_2) = (x_2, u_2^*) = (s_2, a_2)
\end{aligned}
$$

and

$$
\begin{aligned}
W^6(s_1) &= \max_{(x_4, u_4); f_6(x_4, u_4) = s_1} [W^4(x_4) + r_4(x_4, u_4)]\\
&= \max_{(x_4, u_4) \in \{(s_1, a_2), (s_2, a_1)\}} [W^4(x_4) + r_4(x_4, u_4)]\\
&= \max[W^4(s_1) + r_4(s_1, a_2), W^4(s_2) + r_4(s_2, a_1)]\\
&= \max[3 + 3, 4 + 3] = 7, \qquad \tau_6^*(s_1) = (x_4, u_4^*) = (s_2, a_1)\\
W^6(s_2) &= 5, \qquad \tau_6^*(s_2) = (x_4, u_4^*) = (s_1, a_1), (s_2, a_2).
\end{aligned}
$$

Next, because

$$\begin{aligned}
&\{(x_3, u_3, x_5, u_5) \mid f_7(x_3, u_3, x_5, u_5) = s_1\} \\
&\qquad = \{(s_1, a_1, s_2, a_1), (s_1, a_2, s_1, a_1), (s_1, a_2, s_2, a_2)\} \\
&\{(x_3, u_3, x_5, u_5) \mid f_7(x_3, u_3, x_5, u_5) = s_2\} \\
&\qquad = \{(s_1, a_1, s_1, a_1), (s_1, a_1, s_1, a_2), (s_1, a_1, s_2, a_2), (s_1, a_2, s_1, a_2), (s_1, a_2, s_2, a_1)\},
\end{aligned}$$

we have

$$\begin{aligned}
W^7(s_1) &= \max_{(x_3, u_3, x_5, u_5); f_7(x_3, u_3, x_5, u_5) = s_1} [W^3(x_3) + W^5(x_5) + r_3(x_3, u_3) + r_5(x_5, u_5)] \\
&= \max[W^3(s_1) + W^5(s_2) + r_3(s_1, a_1) + r_5(s_2, a_1), \\
&\qquad W^3(s_1) + W^5(s_1) + r_3(s_1, a_2) + r_5(s_1, a_1), \\
&\qquad W^3(s_1) + W^5(s_2) + r_3(s_1, a_2) + r_5(s_2, a_2)] \\
&= \max[0 + 2 + 3 + 4, 0 + 3 + 4 + 1, 0 + 2 + 4 + 2] = \max[9, 8, 8] \\
&= 9, \qquad \tau_7^*(s_1) = (x_3, u_3^*, x_5, u_5^*) = (s_1, a_1, s_2, a_1) \\
W^7(s_2) &= 10, \qquad \tau_7^*(s_2) = (x_3, u_3^*, x_5, u_5^*) = (s_1, a_2, s_2, a_1).
\end{aligned}$$

Finally, because

$$\begin{aligned}
&\{(x_6, u_6, x_7, u_7) \mid f_8(x_6, u_6, x_7, u_7) = s_1\} \\
&\qquad = \{(s_1, a_1, s_1, a_1), (s_1, a_1, s_1, a_2), (s_1, a_2, s_2, a_1), (s_2, a_1, s_1, a_1), \\
&\qquad\qquad (s_2, a_1, s_2, a_2), (s_2, a_2, s_1, a_1), (s_2, a_2, s_2, a_1), (s_2, a_2, s_2, a_2)\} \\
&\{(x_6, u_6, x_7, u_7) \mid f_8(x_6, u_6, x_7, u_7) = s_2\} \\
&\qquad = \{(s_1, a_1, s_2, a_1), (s_1, a_1, s_2, a_2), (s_1, a_2, s_1, a_1), (s_1, a_2, s_1, a_2), \\
&\qquad\qquad (s_1, a_2, s_2, a_2), (s_2, a_1, s_1, a_2), (s_2, a_1, s_2, a_1), (s_2, a_2, s_1, a_2)\},
\end{aligned}$$

we have

$$\begin{aligned}
W^8(s_1) &= \max_{\substack{(x_6, u_6, x_7, u_7); \\ f_8(x_6, u_6, x_7, u_7) = s_1}} [W^6(x_6) + W^7(x_7) + r_6(x_6, u_6) + r_7(x_7, u_7) + k(x_8)] \\
&= \max[W^6(s_1) + W^7(s_1) + r_6(s_1, a_1) + r_7(s_1, a_1) + k(s_1), \\
&\qquad W^6(s_1) + W^7(s_1) + r_6(s_1, a_1) + r_7(s_1, a_2) + k(s_1), \\
&\qquad W^6(s_1) + W^7(s_2) + r_6(s_1, a_2) + r_7(s_2, a_1) + k(s_1),
\end{aligned}$$

$$
\begin{aligned}
&W^6(s_2) + W^7(s_1) + r_6(s_2, a_1) + r_7(s_1, a_1) + k(s_1),\\
&W^6(s_2) + W^7(s_2) + r_6(s_2, a_1) + r_7(s_2, a_2) + k(s_1),\\
&W^6(s_2) + W^7(s_1) + r_6(s_2, a_2) + r_7(s_1, a_1) + k(s_1),\\
&W^6(s_2) + W^7(s_2) + r_6(s_2, a_2) + r_7(s_2, a_1) + k(s_1),\\
&W^6(s_2) + W^7(s_2) + r_6(s_2, a_2) + r_7(s_2, a_2) + k(s_1)]
\end{aligned}
$$

$$
\begin{aligned}
= \max[&7+9+2+4+5, 7+9+2+5+5, 7+10+3+2+5,\\
&5+9+3+4+5, 5+10+3+3+5, 5+9+4+4+5,\\
&5+10+4+2+5, 5+10+4+3+5]\\
= \max[&27, 28, 27, 26, 26, 27, 26, 27] = 28
\end{aligned}
$$

$$\tau_8^*(s_1) = (x_6, u_6^*, x_7, u_7^*) = (s_1, a_1, s_1, a_2)$$

$$W^8(s_2) = 26, \qquad \tau_8^*(s_2) = (x_6, u_6^*, x_7, u_7^*) = (s_1, a_2, s_1, a_2).$$

Thus, the optimal value is

$$\max[W^8(s_1), W^8(s_2)] = \max[28, 26] = 28, \qquad x_8 = s_1$$

and we get an optimal state-decision sequence as follows:

$$
\begin{aligned}
x_8 = s_1 \quad &\Rightarrow \quad (x_6, u_6^*, x_7, u_7^*) = \tau_8^*(s_1) = (s_1, a_1, s_1, a_2)\\
&\rightarrow \quad x_6 = s_1, \quad u_6^* = a_1\\
&\qquad \Rightarrow \quad (x_4, u_4^*) = \tau_6^*(s_1) = (s_2, a_1)\\
&\qquad\quad \rightarrow \quad x_4 = s_2, \quad u_4^* = a_1\\
&\qquad\qquad \Rightarrow \quad (x_1, u_1^*) = \tau_4^*(s_2) = (s_1, a_1)\\
&\qquad\qquad\quad \rightarrow \quad x_1 = s_1, \quad u_1^* = a_1\\
&\rightarrow \quad x_7 = s_1, \quad u_7^* = a_2\\
&\qquad \Rightarrow \quad (x_3, u_3^*, x_5, u_5^*) = \tau_7^*(s_1) = (s_1, a_1, s_2, a_1)\\
&\qquad\quad \rightarrow \quad x_3 = s_1, \quad u_3^* = a_1\\
&\qquad\quad \rightarrow \quad x_5 = s_2, \quad u_5^* = a_1\\
&\qquad\qquad \Rightarrow \quad (x_2, u_2^*) = \tau_5^*(s_2) = (s_2, a_2)\\
&\qquad\qquad\quad \rightarrow \quad x_2 = s_2, \quad u_2^* = a_2.
\end{aligned}
$$

4.3. Compute with backward recursive equation II

In Example 3.3, we modified the indexes. But, to solve our numerical example, we need to return that modification back to the original. Then, the backward recursive equations in Example 3.3 become

$$\begin{aligned}
v^8(x_8) &= k(x_8) \\
v^7(x_7; x_6, u_6) &= \max_{u_7 \in U}[r_7(x_7, u_7) + v^8(f_8(x_6, u_6, x_7, u_7))] \\
v^3(x_3; x_5, u_5, x_6, u_6) &= \max_{u_3 \in U}[r_3(x_3, u_3) + v^7(f_7(x_3, u_3, x_5, u_5); x_6, u_6)] \\
v^5(x_5; x_6, u_6) &= \max_{u_5 \in U}[r_5(x_5, u_5) + v^3(x_3; x_5, u_5, x_6, u_6)] \\
v^2(x_2; x_6, u_6) &= \max_{u_2 \in U}[r_2(x_2, u_2) + v^5(f_5(x_2, u_2); x_6, u_6)] \\
v^6(x_6) &= \max_{u_6 \in U}[r_6(x_6, u_6) + v^2(x_2; x_6, u_6)] \\
v^4(x_4) &= \max_{u_4 \in U}[r_4(x_4, u_4) + v^6(f_6(x_4, u_4))] \\
v^1(x_1) &= \max_{u_1 \in U}[r_1(x_1, u_1) + v^4(f_4(x_1, u_1))].
\end{aligned}$$

First, for the terminal state x_8, we get

$$v^8(s_1) = k(s_1) = 5, \qquad v^8(s_2) = k(s_2) = 2.$$

We compute $v_7(s_1; s_1, a_1)$ and the corresponding optimal decision function $\pi_7^*(s_1; s_1, a_1)$:

$$\begin{aligned}
v^7(s_1; s_1, a_1) &= \max_{u_7 \in U}[r_7(s_1, u_7) + v^8(f_8(s_1, a_1, s_1, u_7))] \\
&= [r_7(s_1, a_1) + v^8(f_8(s_1, a_1, s_1, a_1))] \vee [r_7(s_1, a_2) + v^8(f_8(s_1, a_1, s_1, a_2))] \\
&= [4 + v^8(s_1)] \vee [5 + v^8(s_1)] = [4 + 5] \vee [5 + 5] = 10
\end{aligned}$$

$$\pi_7^*(s_1; s_1, a_1) = a_2.$$

Similarly,

$$v^7(s_1; s_1, a_2) = [4 + 2] \vee [5 + 2] = 7, \qquad \pi_7^*(s_1; s_1, a_2) = a_2$$

$$v^7(s_1; s_2, a_1) = 9, \qquad \pi_7^*(s_1; s_2, a_1) = a_1, \qquad v^7(s_1; s_2, a_2) = 9, \qquad \pi_7^*(s_1; s_2, a_2) = a_1$$

$$v^7(s_2; s_1, a_1) = 5, \qquad \pi_7^*(s_2; s_1, a_1) = a_2, \qquad v^7(s_2; s_1, a_2) = 7, \qquad \pi_7^*(s_2; s_1, a_2) = a_1$$

$$v^7(s_2; s_2, a_1) = 8, \qquad \pi_7^*(s_2; s_2, a_1) = a_2, \qquad v^7(s_2; s_2, a_2) = 8, \qquad \pi_7^*(s_2; s_2, a_2) = a_2.$$

Moreover, we compute $v^3, v^5, v^2, \dots, v^1$ and the corresponding optimal decision functions $\pi_3^*, \pi_4^*, \pi_2^*, \dots, \pi_1^*$ as follows:

$$
\begin{aligned}
&v^3(s_1; s_1, a_1, s_1, a_1) \\
&\qquad = \max_{u_3 \in U} [r_3(s_1, u_3) + v^7(f_7(s_1, u_3, s_1, a_1); s_1, a_1)] \\
&\qquad = [r_3(s_1, a_1) + v^7(f_7(s_1, a_1, s_1, a_1); s_1, a_1)] \vee [r_3(s_1, a_2) + v^7(f_7(s_1, a_2, s_1, a_1); s_1, a_1)] \\
&\qquad = [3 + v^7(s_2; s_1, a_1)] \vee [4 + v^7(s_1; s_1, a_1)] = [3 + 5] \vee [4 + 10] = 14 \\
&\pi_3^*(s_1; s_1, a_1, s_1, a_1) = a_2
\end{aligned}
$$

$$
\begin{aligned}
v^3(s_1; s_1, a_1, s_1, a_2) &= 11, & \pi_3^*(s_1; s_1, a_1, s_1, a_2) &= a_2 \\
v^3(s_1; s_1, a_1, s_2, a_1) &= 13, & \pi_3^*(s_1; s_1, a_1, s_2, a_1) &= a_2 \\
v^3(s_1; s_1, a_1, s_2, a_2) &= 14, & \pi_3^*(s_1; s_1, a_1, s_2, a_2) &= a_2 \\
v^3(s_1; s_1, a_2, s_1, a_1) &= 9, & \pi_3^*(s_1; s_1, a_2, s_1, a_1) &= a_2 \\
v^3(s_1; s_1, a_2, s_1, a_2) &= 11, & \pi_3^*(s_1; s_1, a_2, s_1, a_2) &= a_2 \\
v^3(s_1; s_1, a_2, s_2, a_1) &= 13, & \pi_3^*(s_1; s_1, a_2, s_2, a_1) &= a_2 \\
v^3(s_1; s_1, a_2, s_2, a_2) &= 13, & \pi_3^*(s_1; s_1, a_2, s_2, a_2) &= a_2 \\
v^3(s_1; s_2, a_1, s_1, a_1) &= 13, & \pi_3^*(s_1; s_2, a_1, s_1, a_1) &= a_1 \\
v^3(s_1; s_2, a_1, s_1, a_2) &= 11, & \pi_3^*(s_1; s_2, a_1, s_1, a_2) &= a_2 \\
v^3(s_1; s_2, a_1, s_2, a_1) &= 12, & \pi_3^*(s_1; s_2, a_1, s_2, a_1) &= a_1, a_2 \\
v^3(s_1; s_2, a_1, s_2, a_2) &= 12, & \pi_3^*(s_1; s_2, a_1, s_2, a_2) &= a_1, a_2 \\
v^3(s_1; s_2, a_2, s_1, a_1) &= 14, & \pi_3^*(s_1; s_2, a_2, s_1, a_1) &= a_2 \\
v^3(s_1; s_2, a_2, s_1, a_2) &= 11, & \pi_3^*(s_1; s_2, a_2, s_1, a_2) &= a_2 \\
v^3(s_1; s_2, a_2, s_2, a_1) &= 13, & \pi_3^*(s_1; s_2, a_2, s_2, a_1) &= a_2 \\
v^3(s_1; s_2, a_2, s_2, a_2) &= 13, & \pi_3^*(s_1; s_2, a_2, s_2, a_2) &= a_2
\end{aligned}
$$

$$
\begin{aligned}
v^5(s_1; s_1, a_1) &= 15, & \pi_5^*(s_1; s_1, a_1) &= a_1, & v^5(s_1; s_1, a_2) &= 13, & \pi_5^*(s_1; s_1, a_2) &= a_2 \\
v^5(s_1; s_2, a_1) &= 15, & \pi_5^*(s_1; s_2, a_1) &= a_2, & v^5(s_1; s_2, a_2) &= 15, & \pi_5^*(s_1; s_2, a_2) &= a_1, a_2 \\
v^5(s_2; s_1, a_1) &= 17, & \pi_5^*(s_2; s_1, a_1) &= a_1, & v^5(s_2; s_1, a_2) &= 15, & \pi_5^*(s_2; s_1, a_2) &= a_1 \\
v^5(s_2; s_2, a_1) &= 16, & \pi_5^*(s_2; s_2, a_1) &= a_1, & v^5(s_2; s_2, a_2) &= 16, & \pi_5^*(s_2; s_2, a_2) &= a_1
\end{aligned}
$$

$$v^2(s_2; s_1, a_1) = 19, \quad \pi_2^*(s_2; s_1, a_1) = a_2, \quad v^2(s_2; s_1, a_2) = 17, \quad \pi_2^*(s_2; s_1, a_2) = a_2$$
$$v^2(s_2; s_2, a_1) = 18, \quad \pi_2^*(s_2; s_2, a_1) = a_1, a_2, \quad v^2(s_2; s_2, a_2) = 18, \quad \pi_2^*(s_2; s_2, a_2) = a_1, a_2$$
$$v^6(s_1) = 21, \quad \pi_6^*(s_1) = a_1, \quad v^6(s_2) = 22, \quad \pi_6^*(s_2) = a_2$$
$$v^4(s_1) = 24, \quad \pi_4^*(s_1) = a_1, a_2, \quad v^4(s_2) = 24, \quad \pi_4^*(s_2) = a_1$$

and

$$v^1(s_1) = 28, \quad \pi_1^*(s_1) = a_1.$$

Thus, the optimal value is $v^3(s_1) = 28$ and an optimal state-decision sequence is given by

$$\begin{aligned}
x_1 = s_1 \quad &\to \quad u_1^* = \pi_1^*(s_1) = a_1 \\
&\to \quad x_4 = f_4(s_1, a_1) = s_2 \quad \to \quad u_4^* = \pi_4^*(s_2) = a_1 \\
&\to \quad x_6 = f_6(s_2, a_1) = s_1 \quad \to \quad u_6^* = \pi_6^*(s_1) = a_1 \\
x_2 = s_2 \quad &\to \quad u_2^* = \pi_2^*(s_2; s_1, a_1) = a_2 \\
&\to \quad x_5 = f_5(s_2, a_2) = s_2 \quad \to \quad u_5^* = \pi_5^*(s_2; s_1, a_1) = a_1 \\
x_3 = s_1 \quad &\to \quad u_3^* = \pi_3^*(s_1; s_2, a_1, s_1, a_1) = a_1 \\
&\to \quad x_7 = f_7(s_1, a_1, s_2, a_1) = s_1 \quad \to \quad u_7^* = \pi_7^*(s_1; s_1, a_1) = a_2 \\
&\to \quad x_8 = f_8(s_1, a_1, s_1, a_2) = s_1.
\end{aligned}$$

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