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SOME COMMENTS ON EDELSTEIN'S FIXED POINT THEOREMS IN ν-GENERALIZED METRIC SPACES

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Abstract

We study deeply two fixed point theorems in ν-generalized metric spaces. The two theorems are generalizations of the famous, Edelstein's fixed point theorem in compact metric spaces.

1. Introduction

In 1962, Edelstein proved the following, famous fixed point theorem in compact metric spaces.

Theorem 1 (Edelstein [5]). *Let (X, d) be a compact metric space and let T be a mapping on X. Assume*

(1) $$x \neq y \quad \Rightarrow \quad d(Tx, Ty) < d(x, y)$$

for any $x, y \in X$. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

In 2000, Branciari introduced the following, very interesting concept.

Definition 2 (Branciari [2]). Let X be a nonempty set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbf{N}$. Then (X, d) is said to be a *ν-generalized metric space* if the following hold:

- (N1) $d(x, y) = 0 \Leftrightarrow x = y$.
- (N2) $d(x, y) = d(y, x)$.
- (N3) $d(x, y) \leq D(x, u_1, u_2, \dots, u_\nu, y)$ for any $x, u_1, u_2, \dots, u_\nu, y \in X$ such that $x, u_1, u_2, \dots, u_\nu, y$ are all different, where

$$D(x, u_1, u_2, \dots, u_\nu, y) = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y).$$

It is obvious that (X, d) is a metric space iff (X, d) is a 1-generalized metric space. We have studied the topological structure of this space. Indeed, recent studies tell the following:

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- 1- and 3-generalized metric spaces have the compatible topology (see [16]).
- For any $\nu \in \mathbf{N}\setminus\{1,3\}$, there exists a ν-generalized metric space which does not have the compatible topology (see [10, 16]).
- All ν-generalized metric spaces have the strongly compatible topology (see [15]).
- All ν-generalized metric spaces have the strongest sequentially compatible topology (see [13]).

Several fixed point theorems in ν-generalized metric spaces have been proved. See [10, 12] and others. Theorem 1 is also extended to ν-generalized metric spaces. It is interesting that there are two generalizations of Theorem 1.

THEOREM 3 (Theorem 3.2 in [11]). *Let (X,d) be a compact ν-generalized metric space. Let T be a mapping on X satisfying* (1) *for any $x, y \in X$. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.*

THEOREM 4 (Theorem 3.4 in [17]). *Let (X,d) be a ν-generalized metric space such that X is compact in the strong sense. Let T be a mapping on X satisfying* (1) *for any $x, y \in X$. Then T has a unique fixed point z. Moreover for all $x \in X$, $\{T^n x\}$ converges to z in the strong sense.*

In this paper, we study the above two theorems deeply.

2. Preliminaries

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers. For an arbitrary set A, we also denote by $\#A$ the cardinal number of A. We define a subset $A^{(k)}$ of A^k as follows: $(x_1, x_2, \dots, x_k) \in A^{(k)}$ iff $(x_1, x_2, \dots, x_k) \in A^k$ and $x_1, x_2, \dots, x_k$ are all different. For a real number t, we denote by $[t]$ the maximum integer not exceeding t.

In this section, we give some preliminaries.

DEFINITION 5. Let (X,d) be a ν-generalized metric space and let $\{x_n\}$ be a sequence in X and let $x \in X$.

(i) $\{x_n\}$ is said to be *Cauchy* [2] if $\lim_n \sup_{m>n} d(x_n, x_m) = 0$.

(ii) $\{x_n\}$ is said to *converge* to x [2] if $\lim_n d(x_n, x) = 0$.

(iii) $\{x_n\}$ is said to *converge only* to x [1] if

$$\lim_{n\to\infty} d(x_n, x) = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} d(x_n, y) > 0$$

for any $y \in X\setminus\{x\}$.

(iv) $\{x_n\}$ is said to *converge exclusively* to x [17] if

$$\lim_{n\to\infty} d(x_n, x) = 0 \qquad \text{and} \qquad \liminf_{n\to\infty} d(x_n, y) > 0$$

for any $y \in X\setminus\{x\}$.

(v) $\{x_n\}$ is said to *converge to x in the strong sense* [17] if $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x.

Remark. We know the following.

- (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) holds; see Proposition 2.3 (ii) in [17].

Definition 6. Let (X, d) be a ν-generalized metric space.

- X is said to be *compact* [17] if for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z \in X$.
- X is said to be *compact in the strong sense* [17] if for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z \in X$ in the strong sense.
- d is said to be *sequentially continuous* if $\lim_n d(x_n, y_n) = d(x, y)$ provided $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively.
- X is said to be *Hausdorff* [9] if $\lim_n d(x_n, x) = \lim_n d(x_n, y) = 0$ implies $x = y$.

Let (X, d) be a ν-generalized metric space. In the case where $\# X \geq \nu + 3$ holds, we define a function δ from $X^{(\nu+3)}$ into $[0, \infty)$ by

$$\delta(x; u_1, \ldots, u_{\nu+2}) = \max\left\{ d(x, u_{\sigma(1)}) + \sum_{j=1}^{\nu+1} d(u_{\sigma(j)}, u_{\sigma(j+1)}) : \sigma \in S_{\nu+2} \right\}$$

for $(x, u_1, \ldots, u_{\nu+2}) \in X^{(\nu+3)}$, where $S_{\nu+2}$ is the permutation group consisting of all bijective mappings on $\{1, 2, \ldots, \nu+2\}$. Define a function η from X into $[0, \infty)$ by

$$\eta(x) = \inf\{\delta(x; u_1, \ldots, u_{\nu+2}) : (x, u_1, \ldots, u_{\nu+2}) \in X^{(\nu+3)}\}$$

for $x \in X$. In the other case, where $\# X < \nu + 3$ holds, we define $\eta(x) = \infty$ for all $x \in X$.

Proposition 7 (Proposition 4.1 in [11]). *Let (X, d) be a ν-generalized metric space and let $\lambda \in \mathbf{N}$ such that λ is divisible by ν. Then (X, d) is a λ-generalized metric space.*

Lemma 8 (Proposition 2.7 in [17], Proposition 13 in [15]). *Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y in the strong sense, respectively. Then*

$$d(x, y) = \lim_{n\to\infty} d(x_n, y_n) \tag{2}$$

holds.

Lemma 9 (Lemma 5 in [15]). *Let (X, d) be a ν-generalized metric space. For any $(x, y, z) \in X^3$.*

$$d(x, z) \leq d(x, y) + d(y, z) + 2\eta(x)$$

and

$$d(x,z) \le d(x,y) + d(y,z) + 2\eta(y)$$

hold.

REMARK. We assume $\#X \ge \nu + 3$ in Lemma 5 in [15]. However, in this paper, we do not have to assume $\#X \ge \nu + 3$ because we have defined $\eta(x) = \infty$ in the other case.

LEMMA 10. *Let (X,d) be a ν-generalized metric space. Let $(u_1, \dots, u_n) \in X^n$, where $n \ge 3$. Assume $\eta(u_k) = 0$ for some $k \in \{1, \dots, n\}$. Then*

$$d(u_1, u_n) \le D(u_1, \dots, u_n)$$

holds.

PROOF. The conclusion follows from Lemma 9. □

LEMMA 11 (Lemma 8 in [15]). *Let (X,d) be a ν-generalized metric space. Let $\{x_\alpha : \alpha \in E\}$ be a Cauchy net in X such that for any $\alpha \in E$, there exists $\beta \ge \alpha$ satisfying $x_\alpha \ne x_\beta$. Then the following hold:*

(i) $\lim_\alpha \eta(x_\alpha) = 0$.

(ii) *If* $\lim_\alpha d(x, x_\alpha) = 0$ *for some* $x \in X$, *then* $\eta(x) = 0$.

REMARK. In Lemma 8 in [15], we assume $\#X \ge \nu + 3$. However, from its proof, $\#X = \infty$ holds. So we do not have to assume $\#X \ge \nu + 3$.

LEMMA 12. *Let (X,d) be a ν-generalized metric space. Let $\{x_n\}$ be a Cauchy sequence in X such that for any $n \in \mathbf{N}$, there exists $m > n$ satisfying $x_m \ne x_n$. Then the following hold:*

(i) $\lim_n \eta(x_n) = 0$.

(ii) *If* $\lim_n d(x_n, x) = 0$ *for some* $x \in X$, *then* $\eta(x) = 0$.

PROOF. By Lemma 11, we obtain the desired result. □

3. Lemmas

In this section, we prove some lemmas.

LEMMA 13. *Let (X,d) be a ν-generalized metric space. Let $\{x_n\}$ be a sequence in X converging to z. Then the following hold:*

(i) *If $\{x_n\}$ is Cauchy, then $\{x_n\}$ converges exclusively to z.*

(ii) *If $\{x_n\}$ converges to z, $\lim_n d(x_n, x_{n+1}) = 0$ and $x_n \ne z$ for any $n \in \mathbf{N}$, then $\{x_n\}$ is Cauchy, that is, $\{x_n\}$ converges to z in the strong sense.*

(iii) *If X is Hausdorff, then $\{x_n\}$ converges exclusively to z.*
(iv) $\#\{n \in \mathbf{N} : x_n = x\} < \infty$ *for any* $x \in X \setminus \{z\}$.
(v) *If* $\lim_n d(x_n, w) = 0$ *for some* $w \in X \setminus \{z\}$, *then* $\#\{n \in \mathbf{N} : x_n = x\} < \infty$ *for any* $x \in X$.

Proof. We have proved (i) and (ii); see Proposition 2.3 in [17].

Let us prove (iii). Let $w \in X$ satisfy $\liminf_n d(x_n, w) = 0$. Then there exists a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbf{N}$ satisfying $\lim_n d(x_{f(n)}, w) = 0$. Since $\lim_n d(x_{f(n)}, z) = 0$ holds, $\{x_{f(n)}\}$ converges to w and z. Since X is Hausdorff, we have $w = z$.

We next show (iv). Arguing by contradiction, we assume that there exists $w \in X \setminus \{z\}$ satisfying $\#\{n \in \mathbf{N} : x_n = w\} = \infty$. Then we have

$$\liminf_{n \to \infty} d(x_n, z) \geq d(w, z) > 0,$$

which implies a contradiction. Therefore we have shown (iv).

Noting $(X \setminus \{z\}) \cup (X \setminus \{w\}) = X$, we obtain (v) from (iv). □

Lemma 14. *Let (X, d) be a ν-generalized metric space and let T be a mapping on X. Define a sequence $\{x_n\}$ in X by $x_1 \in X$ and $x_{n+1} = Tx_n$. Assume the following:*
(i) *$\{x_n\}$ converges to z.*
(ii) $\lim_n d(x_n, x_{n+1}) = 0$.
Then $\{x_n\}$ is Cauchy. That is, $\{x_n\}$ converges to z in the strong sense.

Proof. We consider the following two cases:
- There exist $k, \ell \in \mathbf{N}$ satisfying $k < \ell$ and $x_k = x_\ell$.
- x_n $(n \in \mathbf{N})$ are all different.

In the first case, we have $x_k = x_\ell = T^{\ell - k} x_k$ and hence

$$0 = \lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{m \to \infty} d(x_{m(\ell-k)+k}, x_{m(\ell-k)+k+1}) = d(x_k, x_{k+1}).$$

So $Tx_k = x_k$ holds. Therefore we have $x_n = x_k = z$ for any $n \in \mathbf{N}$ with $n \geq k$. Thus, $\{x_n\}$ is Cauchy. In the second case, we have $x_n \neq z$ for sufficiently large $n \in \mathbf{N}$. So by Lemma 13 (ii), $\{x_n\}$ is Cauchy. □

Lemma 15. *Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ be a Cauchy sequence in X. Put*

$$A = \{y \in X : \lim_n d(x_n, y) = 0\}. \tag{3}$$

Then the following hold:
(i) *If* $\liminf_n d(x_n, z) = 0$ *holds for some* $z \in X$, *then* $z \in A$.
(ii) $\#A \leq 1$.

Proof. We have proved (i). See Lemma 12 in [12].

Arguing by contradiction, we assume $\#A \geq 2$. Since $\{x_n\}$ is Cauchy, by Lemma 13 (i), $\{x_n\}$ converges exclusively to more than one point. This is a contradiction. Therefore we have shown $\#A \leq 1$. □

Lemma 16. *Let (X, d) be a ν-generalized metric space, where ν is odd. Let $\{x_n\}$ be a sequence in X. Put A by* (3). *Then $\#A \leq \max\{1, (\nu-1)/2\}$ holds.*

Proof. In the case where $\nu = 1$, the conclusion obviously holds. So we assume $\nu \geq 3$. Put $\kappa = (\nu-1)/2$. Arguing by contradiction, we assume $\#A > \kappa$. Let $(y_1, \dots, y_{\kappa+1}) \in A^{(\kappa+1)}$. Then since $\#A \geq 2$ holds, by Lemma 13 (v), we have $\#\{n \in \mathbf{N} : x_n = x\} < \infty$, which implies $\#\{x_n : n \in \mathbf{N}\} = \infty$. Fix $\varepsilon > 0$. Then there exists $\mu \in \mathbf{N}$ satisfying

$$x_n \notin \{y_1, \dots, y_{\kappa+1}\},$$
$$\max\{d(y_j, x_n) : j = 1, \dots, \kappa+1\} < \varepsilon$$

for any $n \geq \mu$. Fix $m, n \in \mathbf{N}$ with $\mu < n < m$ and $x_m \neq x_n$. Then we have

$$d(x_m, x_n) \leq D(x_m, y_1, x_{\ell_1}, y_2, \dots, x_{\ell_\kappa}, y_{\kappa+1}, x_n) < (\nu+1)\varepsilon,$$

where $\min\{\ell_1, \dots, \ell_\kappa\} \geq \mu$ and $(x_m, x_n, x_{\ell_1}, \dots, x_{\ell_\kappa}, y_1, \dots, y_{\kappa+1}) \in X^{(\nu+2)}$. This implies that $\{x_n\}$ is Cauchy. By Lemma 15 (ii), we obtain $\#A \leq 1$, which implies a contradiction. Therefore we have shown $\#A \leq \kappa$. □

Lemma 17. *Let (X, d) be a ν-generalized metric space. Assume $\nu \in \{1, 3\}$. Then X is Hausdorff.*

Proof. Suppose that $\{x_n\}$ converges to x and y. We put A by (3). Then by Lemma 16, we have $\#A \leq 1$. Since $x, y \in A$ holds, we obtain $x = y$. □

Lemma 18. *Let (X, d) be a ν-generalized metric space, where ν is even. Let $\{x_n\}$ be a sequence in X. Put A by* (3) *and assume $\#A \geq \nu/2 + 1$. Then $d(x, y) = d(x, z)$ holds for any $(x, y, z) \in A^{(3)}$.*

Proof. From the assumption, $\#A \geq 2$ holds. By Lemma 13 (v), $\#\{n \in \mathbf{N} : x_n = x\} < \infty$ holds for any $x \in X$. Taking a subsequence, we may assume that x_n $(n \in \mathbf{N})$ are all different. Fix $(y_0, \dots, y_{\nu/2}) \in A^{(\nu/2+1)}$. Then we have

$$\begin{aligned} d(y_0, y_{\nu/2}) &\leq \liminf_{n\to\infty} D(y_0, x_n, x_{n+1}, y_1, x_{n+2}, \dots, x_{n+\nu/2}, y_{\nu/2}) \\ &= \liminf_{n\to\infty} d(x_n, x_{n+1}) \\ &\leq \limsup_{n\to\infty} d(x_n, x_{n+1}) \end{aligned}$$

$$\le \limsup_{n\to\infty} D(x_n, y_0, y_{v/2}, x_{n+2}, y_1, x_{n+3}, y_2, \ldots, x_{n+v/2}, y_{v/2-1}, x_{n+1})$$

$$= d(y_0, y_{v/2}).$$

Since $(y_0, y_{v/2}) \in A^{(2)}$ is arbitrary, we obtain

$$d(x, y) = d(x, z) = \lim_{n\to\infty} d(x_n, x_{n+1})$$

for any $(x, y, z) \in A^{(3)}$. □

LEMMA 19. *Let (X, d) be a v-generalized metric space. Let $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(i) *$\{x_n\}$ is Cauchy.*

(ii) *$\lim_n d(x_{f(n)}, x_{g(n)}) = 0$ for any subsequences $\{f(n)\}$ and $\{g(n)\}$ of $\{n\}$ in $\mathbf{N}$.*

PROOF. Obvious. □

4. Compactness

In this section, we study compactness and strong compactness.

PROPOSITION 20. *Let (X, d) be a v-generalized metric space. Then the following are equivalent:*

(i) *X is compact in the strong sense.*

(ii) *X is compact and d is sequentially continuous.*

PROOF. We first prove (i) ⇒ (ii). It is obvious that X is compact. In order to prove the sequential continuity of d, suppose that $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively. We consider the following two cases:

- $\#\{n \in \mathbf{N} : x_n \neq x\} < \infty$ and $\#\{n \in \mathbf{N} : y_n \neq y\} < \infty$.
- $\#\{n \in \mathbf{N} : x_n \neq x\} = \infty$ or $\#\{n \in \mathbf{N} : y_n \neq y\} = \infty$.

In the first case, we have $x_n = x$ and $y_n = y$ for sufficiently large $n \in \mathbf{N}$. So (2) obviously holds. In the second case, without loss of generality, we may assume $\#\{n \in \mathbf{N} : x_n \neq x\} = \infty$. Using Lemma 13 (iv), we can choose a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbf{N}$ such that $x_{f(n)}$ $(n \in \mathbf{N})$ are all different. From (i), without loss of generality, we may assume $\{x_{f(n)}\}$ is Cauchy. By Lemma 12, we have $\eta(x) = 0$. We have by Lemma 10

$$\limsup_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} D(x_n, x, y, y_n)$$

$$= d(x, y)$$

$$\le \liminf_{n\to\infty} D(x, x_n, y_n, y) = \liminf_{n\to\infty} d(x_n, y_n).$$

Hence (2) holds. We have shown (ii).

Let us prove (ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a sequence in X. Then there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z \in X$. Let $\{g(n)\}$ and $\{h(n)\}$ be arbitrary subsequences of $\{n\}$ in $\mathbf{N}$. Then we have by (ii)

$$\lim_{n\to\infty} d(x_{f\circ g(n)}, x_{f\circ h(n)}) = d(x, x) = 0.$$

We have obtained Lemma 19 (ii). By Lemma 19, $\{x_{f(n)}\}$ is Cauchy. □

Lemma 21. *Let (X, d) be a v-generalized metric space. If X is compact, then X is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence. Since X is compact, there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbf{N}$ such that $\{x_{f(n)}\}$ converges to some z. By Lemma 15 (i), $\{x_n\}$ converges to z. □

5. Contractive conditions

In this section, we state known results concerning contractive conditions.

Definition 22. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Let T be a mapping on X.

(1) T is said to be an *Edelstein contraction* [5] if $d(Tx, Ty) < d(x, y)$ for any $x, y \in X$ with $d(Tx, Ty) > 0$.

(2) T is said to be a *CJM contraction* [4, 7, 8] if the following hold:
  (2-i) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \le \varepsilon$.
  (2-ii) T is an Edelstein contraction.

(3) T is said to be a *Browder contraction* [3] if there exists a function φ from $[0, \infty)$ into itself satisfying the following:
  (3-i) φ is nondecreasing and right continuous.
  (3-ii) $\varphi(t) < t$ for any $t \in (0, \infty)$.
  (3-iii) $d(Tx, Ty) \le \varphi \circ d(x, y)$ for all $x, y \in X$.

In order to study the Browder and Boyd-Wong contractive conditions, Hegedüs and Szilágyi in [6] considered subsets of $[0, \infty)^2$.

Definition 23 (see [14]). Let Q be a subset of $[0, \infty)^2$.

(1) Q is said to be *Edelstein* if $u > 0$ implies $u < t$ for any $(t, u) \in Q$.

(2) Q is said to be *CJM* if the following hold:
  (2-i) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $u \le \varepsilon$ holds for any $(t, u) \in Q$ with $t < \varepsilon + \delta$.
  (2-ii) Q is Edelstein.

(3) Q is said to be a *Browder* if there exists a function φ from $[0, \infty)$ into itself satisfying the following:

(3-i) φ is nondecreasing and right continuous.
(3-ii) $\varphi(t) < t$ for any $t \in (0, \infty)$.
(3-iii) $u \le \varphi(t)$ for any $(t, u) \in Q$.

(4) Q is said to satisfy *Condition* $C(0, 0, 0)$ if the following hold:
(4-i) Q is Edelstein.
(4-ii) There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $\tau < u_n$ and $\lim_n t_n = \lim_n u_n = \tau$.

(5) Q is said to satisfy *Condition* $C(1, 1, 2)$ if the following hold:
(5-i) Q is Edelstein.
(5-ii) There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\lim_n t_n = \lim_n u_n = \tau$.

We know the following:

Proposition 24 (see [14]). *Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Let T be a mapping on X. Define a subset Q of $[0, \infty)^2$ by*

$$Q = \{(d(x, y), d(Tx, Ty)) : x, y \in X\}. \tag{4}$$

Then the following hold:
(i) *T is an Edelstein contraction $\Leftrightarrow$ Q is Edelstein.*
(ii) *T is a CJM contraction $\Leftrightarrow$ Q is CJM $\Leftrightarrow$ Q satisfies Condition $C(0, 0, 0)$.*
(iii) *T is a Browder contraction $\Leftrightarrow$ Q is Browder $\Leftrightarrow$ Q satisfies Condition $C(1, 1, 2)$.*

Theorem 25 (Theorem 13 in [10]). *Let (X, d) be a complete ν-generalized metric space and let T be a CJM contraction on X. Then T has a unique fixed point z. Moreover for all $x \in X$, $\{T^n x\}$ converges to z in the strong sense.*

6. Theorem 4

In order to clarify the mathematical structure of Theorem 4, we give two proofs of Theorem 4. We first give a proof by using Theorem 25.

Proof of Theorem 4 by Theorem 25. Using (1), we first note that T is nonexpansive. That is,

$$d(Tx, Ty) \le d(x, y) \tag{5}$$

hold for all $x, y \in X$. By Proposition 20, we next note that X is compact and d is sequentially continuous. So by Lemma 21, X is complete. Define a subset Q of $[0, \infty)^2$ by (4).

We will show that Q satisfies Condition $C(1, 1, 2)$. By Proposition 24, Q is Edelstein. Arguing by contradiction, we suppose that $\{(t_n, u_n)\}$ is a sequence in Q converging to (τ, τ) for some $\tau \in (0, \infty)$. We can choose $(x_n, y_n) \in X^2$ satisfying $t_n =$

$d(x_n, y_n)$ and $u_n = d(Tx_n, Ty_n)$. Since X is compact in the strong sense, there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbf{N}$ such that $\{x_{f(n)}\}$ and $\{y_{f(n)}\}$ converge to some x and y, respectively. We have by (5)

$$\lim_{n\to\infty} d(Tx_n, Tx) \le \lim_{n\to\infty} d(x_n, x) = 0.$$

So, $\{Tx_n\}$ converges to Tx. Similarly $\{Ty_n\}$ converges to Ty. Since d is sequentially continuous, we have

$$d(x, y) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} t_n = \tau > 0$$

and

$$d(Tx, Ty) = \lim_{n\to\infty} d(Tx_n, Ty_n) = \lim_{n\to\infty} u_n = \tau > 0.$$

This contradicts (1). Thus, we have shown that Q satisfies Condition $\mathrm{C}(1, 1, 2)$.

In particular, Q satisfies Condition $\mathrm{C}(0, 0, 0)$. By Proposition 24, T is a CJM contraction. So by Theorem 25, we obtain the desired result. □

We next give another proof of Theorem 4, by using Theorem 3. Before proving it, we need some preliminaries.

Lemma 26. *Let (X, d) be a ν-generalized metric space. Let T be a mapping on X. Assume the following:*

(i) *There exists $z \in X$ such that $\{T^n x\}$ converges to z for all $x \in X$.*

(ii) *There exists $u \in X$ satisfying $T^n u \ne z$ for $n \in \mathbf{N}$ and $\lim_n d(T^n u, T^{n+1} u) = 0$.*

Then $\{T^n x\}$ converges to z in the strong sense for all $x \in X$.

Proof. Arguing by contradiction, we assume that there exist $k, \ell \in \mathbf{N}$ satisfying $k < \ell$ and $T^k u = T^\ell u$. As in the proof of Lemma 14, we can prove $T^n u = T^k u = z$ for any $n \in \mathbf{N}$ with $n \ge k$. This contradicts (ii). Therefore we have shown that $T^n u$ $(n \in \mathbf{N})$ are all different.

By Lemma 14, $\{T^n u\}$ is Cauchy. So by Lemma 12, we obtain $\eta(z) = 0$. Fix $x \in X$. Then we have by Lemma 9

$$\lim_{n\to\infty} \sup_{m>n} d(T^n x, T^m x) \le \lim_{n\to\infty} \sup_{m>n} (d(T^m x, z) + d(T^n x, z) + \eta(z)) = 0.$$

Therefore we have shown that $\{T^n x\}$ is Cauchy. □

Proposition 27. *Let (X, d) be a compact ν-generalized metric space. Let T be a mapping on X satisfying (1) for any $x, y \in X$. Let z be a unique fixed point of T. Assume that there exists $u \in X$ satisfying $T^n u \ne z$ for any $n \in \mathbf{N}$ and $\lim_n d(T^n u, T^{n+1} u) = 0$. Then for all $x \in X$, $\{T^n x\}$ converges to z in the strong sense.*

Proof. By Theorem 3, we obtain Lemma 26 (i). From the assumption, Lemma 26 (ii) holds. So by Lemma 26, we obtain the desired result. □

Proof of Theorem 4 by Theorem 3. Since X is compact in the strong sense, X is compact. Therefore all the assumptions of Theorem 3 hold. By Theorem 3, T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$. We consider the following two cases:

- $\#\{T^n x : n \in \mathbf{N}\} < \infty$ for all $x \in X$.
- $\#\{T^n u : n \in \mathbf{N}\} = \infty$ for some $u \in X$.

In the first case, the conclusion obviously holds. In the second case, we note $T^n u \neq z$ for all $n \in \mathbf{N}$. Since X is compact in the strong sense, there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbf{N}$ such that $\{T^{f(n)} u\}$ and $\{T^{f(n)+1} u\}$ converge to some v and w in the strong sense, respectively. By Lemma 13 (i), $\{T^{f(n)} u\}$ and $\{T^{f(n)+1} u\}$ converges exclusively to v and w, respectively. Since $\{T^{f(n)} u\}$ and $\{T^{f(n)+1} u\}$ converges to z, we obtain $v = w = z$. By Lemma 8, we have

$$\lim_{n \to \infty} d(T^{f(n)} u, T^{f(n)+1} u) = d(z, z) = 0.$$

Since $\{d(T^n u, T^{n+1} u)\}$ is nonincreasing, we have $\lim_n d(T^n u, T^{n+1} u) = 0$. By Proposition 27, we obtain the desired result. □

7. Theorem 3

In this section, we study Theorem 3. Indeed we prove finer results than Theorem 3, depending on ν.

Theorem 28. *Let (X, d) be a compact ν-generalized metric space, where $\nu \in \{1, 3\}$. Let T be a mapping on X satisfying* (1) *for any $x, y \in X$. Then the following hold:*

(i) *T has a unique fixed point z.*

(ii) *$\{T^n x\}$ converges exclusively to z for all $x \in X$.*

Proof. By Theorem 3, T has a unique fixed point z. Thus, (i) holds.

By Theorem 3 again, for any $x \in X$, $\{T^n x\}$ converges to z. By Lemma 17, we note that X is Hausdorff. So by Lemma 13 (iii), $\{T^n x\}$ converges exclusively to z. □

Theorem 29. *Let (X, d) be a compact 2-generalized metric space. Let T be a mapping on X satisfying* (1) *for any $x, y \in X$. Then the following hold:*

(i) *T has a unique fixed point z.*

(ii) *$\{T^n x\}$ converges to z for all $x \in X$.*

(iii) *If $\{T^n x\}$ converges to y, then $Ty = z$ holds.*

Proof. By Theorem 3, (i) and (ii) hold.

We put

$$A = \{u \in X : \lim_n d(T^n x, u) = 0\}. \tag{6}$$

In order to prove (iii), suppose $\lim_n d(T^n x, y) = 0$. We consider the following two cases:

- $y = z$.
- $y \neq z$.

In the first case, we have $Ty = Tz = z$. In the second case, arguing by contradiction, we assume $Ty \neq z$. Then we note $Ty \neq y$ because z is a unique fixed point of T and $y \neq z$ holds. Using (5), we have

$$\lim_{n\to\infty} d(T^n x, Ty) \le \lim_{n\to\infty} d(T^{n-1}x, y) = 0.$$

Thus, $Ty \in A$ holds. Therefore we have $(z, y, Ty) \in A^{(3)}$. By Lemma 18, we have

$$d(z, Ty) = d(z, y).$$

On the other hand, we have by (1)

$$d(z, Ty) = d(Tz, Ty) < d(z, y),$$

which implies a contradiction. Therefore we obtain $Ty = z$. □

Theorem 30. *Let (X, d) be a compact ν-generalized metric space, where $\nu \ge 4$ holds. Let T be a mapping on X satisfying* (1) *for any $x, y \in X$. Then the following hold:*

(i) *T has a unique fixed point z.*
(ii) *$\{T^n x\}$ converges to z for all $x \in X$.*
(iii) *If $\{T^n x\}$ converges to y, then $T^{[\nu/2]-1} y = z$ holds.*

Proof. By Theorem 3, (i) and (ii) hold.

Fix $x \in X$ and put A by (6). We will show that A is T-invariant. Indeed, let $y \in A$. Then we have by (5)

$$\lim_{n\to\infty} d(T^n x, Ty) \le \lim_{n\to\infty} d(T^{n-1}x, y) = 0.$$

So $Ty \in A$ holds. Therefore we have shown that A is T-invariant. We consider the following three cases:

- ν is odd.
- ν is even and $\#A \le \nu/2$ holds.
- ν is even and $\#A \ge \nu/2 + 1$ holds.

In the first case, by Lemma 16, we note

$$\#A \le (\nu - 1)/2 = [\nu/2] = ([\nu/2] - 1) + 1.$$

So we obtain $T^{[\nu/2]-1}y = z$ for any $y \in A$. In the second case, we note

$$\#A \le \nu/2 = [\nu/2] = ([\nu/2] - 1) + 1.$$

We also obtain $T^{[\nu/2]-1}y = z$ for any $y \in A$. In the third case, arguing by contradiction, we assume $Ty \neq z$ for some $y \in A$. It is obvious that $y \neq z$ holds. Thus, $(z, y, Ty) \in A^{(3)}$ holds. By Lemma 18, we have

$$d(z, Ty) = d(z, y).$$

On the other hand, we have

$$d(z, Ty) = d(Tz, Ty) < d(z, y),$$

which implies a contradiction. Therefore $Ty = z$ for any $y \in A$. We obtain

$$T^{[\nu/2]-1}y = T^{[\nu/2]-2}z = z$$

for all $y \in A$, where T^0 is the identity mapping on X. □

We prove the following lemma, which is useful when we show that we cannot prove Theorem 3 by Theorem 25. See also Section 8.

Lemma 31. *Let (X, d) be a ν-generalized metric space. Let T be a mapping on X satisfying (1) for any $x, y \in X$. Assume that there exists $u \in X$ satisfying*

$$\lim_{n\to\infty} d(T^n u, T^{n+1}u) > 0.$$

Then T is not a CJM contraction.

Proof. We first note that $\{d(T^n u, T^{n+1}u)\}$ is strictly decreasing. Put $\tau := \lim_n d(T^n u, T^{n+1}u) > 0$. Define a subset Q of $[0, \infty)^2$ by (4). Define a sequence $\{(t_n, u_n)\}$ by

$$t_n = d(T^n u, T^{n+1}u) \qquad \text{and} \qquad u_n = d(T^{n+1}u, T^{n+2}u).$$

We have

$$\tau < u_n < t_n \qquad \text{and} \qquad \lim_{n\to\infty} t_n = \lim_{n\to\infty} u_n = \tau.$$

Therefore Q does not satisfy Condition $C(0, 0, 0)$. By Proposition 24, T is not a CJM contraction. □

8. Examples

We finally give examples which are strongly connected with the results in Section 7.

Lemma 32. *Let $\nu \in \mathbf{N}$ be an even positive integer. Let X be a nonempty set and let A and B be subsets of X with $A \cup B = X$, $A \cap B = \varnothing$ and $\#A \le \nu/2$. Let S be a mapping from X into a metric space (Y, ρ). Let M be a positive real number and let f be a function from $X \times X$ into $[0, 3M]$ satisfying the following:*

$$f(x, x) = 0. \tag{7}$$

$$x \ne y \wedge Sx = Sy \Rightarrow f(x, y) > 0. \tag{8}$$

$$f(x, y) = f(y, x). \tag{9}$$

$$(x, y) \in B^{(2)} \Rightarrow f(x, y) = M. \tag{10}$$

$$(x, y) \in (A \times B) \cup (B \times A) \Rightarrow f(x, y) \le M. \tag{11}$$

Define a function d from $X \times X$ into $[0, \infty)$ by

$$d(x, y) = \rho(Sx, Sy) + f(x, y). \tag{12}$$

Then (X, d) is a ν-generalized metric space.

Proof. We can prove (N1) and (N2) by (7)–(9). In order to prove (N3), we fix $(u_0, \dots, u_{\nu+1}) \in X^{(\nu+2)}$. We consider the following two cases:

- $(u_0, u_{\nu+1}) \in A^{(2)}$.
- Otherwise.

In the first case, since $2 \le \#A \le \nu/2$ holds, we note $\nu \ge 4$. We have

$$\begin{aligned}\#\{&j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in (A \times X) \cup (X \times A)\} \\ &\le 1 + 2(\nu/2 - 2) + 1 = \nu - 2\end{aligned}$$

and hence

$$\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in B^{(2)}\} \ge (\nu + 1) - (\nu - 2) = 3.$$

So we have by (10)

$$\begin{aligned} d(u_0, u_{\nu+1}) &\le 3M + \rho(Su_0, Su_{\nu+1}) \\ &\le 3M + \rho(Su_0, Su_1) + \cdots + \rho(Su_\nu, Su_{\nu+1}) \\ &\le D(u_0, \dots, u_{\nu+1}). \end{aligned}$$

In the second case, since $\#A \le \nu/2$ holds, we have

$$\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in (A \times X) \cup (X \times A)\} \le 2\nu/2 = \nu$$

and hence

$$\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in B^{(2)}\} \geq (\nu + 1) - \nu = 1.$$

So we have by (10) and (11)

$$\begin{aligned} d(u_0, u_{\nu+1}) &\leq M + \rho(Su_0, Su_{\nu+1}) \\ &\leq M + \rho(Su_0, Su_1) + \cdots + \rho(Su_\nu, Su_{\nu+1}) \\ &\leq D(u_0, \dots, u_{\nu+1}). \end{aligned}$$

Thus (N3) holds in all cases. □

Similarly we can prove the following lemma.

LEMMA 33. *Let $\nu \in \mathbf{N}$ be an odd positive integer. Let X be a nonempty set and let A and B be subsets of X with $A \cup B = X$, $A \cap B = \varnothing$ and $\#A \leq (\nu - 1)/2$. Let S be a mapping from X into a metric space (Y, ρ). Let M be a positive real number and let f be a function from $X \times X$ into $[0, 4M]$ satisfying (7)–(10) and the following:*

$$(x, y) \in (A \times B) \cup (B \times A) \Rightarrow f(x, y) \leq 2M. \tag{13}$$

Define a function d from $X \times X$ into $[0, \infty)$ by (12). Then (X, d) is a ν-generalized metric space.

PROOF. We can prove (N1) and (N2) by (7)–(9). In order to prove (N3), we fix $(u_0, \dots, u_{\nu+1}) \in X^{(\nu+2)}$. We consider the following two cases:

- $(u_0, u_{\nu+1}) \in A^{(2)}$.
- Otherwise.

In the first case, since $2 \leq \#A \leq (\nu - 1)/2$ holds, we note $\nu \geq 5$. We have

$$\begin{aligned} &\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in (A \times X) \cup (X \times A)\} \\ &\quad \leq 1 + 2((\nu - 1)/2 - 2) + 1 = \nu - 3 \end{aligned}$$

and hence

$$\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in B^{(2)}\} \geq (\nu + 1) - (\nu - 3) = 4.$$

So we have by (10)

$$\begin{aligned} d(u_0, u_{\nu+1}) &\leq 4M + \rho(Su_0, Su_{\nu+1}) \\ &\leq 4M + \rho(Su_0, Su_1) + \cdots + \rho(Su_\nu, Su_{\nu+1}) \\ &\leq D(u_0, \dots, u_{\nu+1}). \end{aligned}$$

In the second case, since $\#A \le (\nu - 1)/2$ holds, we have

$$\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in (A \times X) \cup (X \times A)\} \le 2(\nu - 1)/2 = \nu - 1$$

and hence

$$\#\{j \in \{0, \dots, \nu\} : (u_j, u_{j+1}) \in B^{(2)}\} \ge (\nu + 1) - (\nu - 1) = 2.$$

So we have by (10) and (13)

$$\begin{aligned} d(u_0, u_{\nu+1}) &\le 2M + \rho(Su_0, Su_{\nu+1}) \\ &\le 2M + \rho(Su_0, Su_1) + \cdots + \rho(Su_\nu, Su_{\nu+1}) \\ &\le D(u_0, \dots, u_{\nu+1}). \end{aligned}$$

Thus (N3) holds in all cases. □

LEMMA 34. *Let $\nu \in \mathbf{N}$. Let X be a nonempty set and let A and B be subsets of X with $A \cap B = \varnothing$. Assume that A consists of at most $(\nu - 1)/2$ elements in the case where ν is odd. Let S be a mapping from X into a metric space (Y, ρ) satisfying $S(A) \cap S(B) = \varnothing$. Let M be a positive real number. Define a function d from $X \times X$ into $[0, \infty)$ by*

$$\begin{aligned} &d(x, x) = 0 \\ &d(x, y) = d(y, x) = \rho(Sx, Sy) \qquad \text{if } (x, y) \in A \times B \\ &d(x, y) = M + \rho(Sx, Sy) \qquad \text{otherwise.} \end{aligned}$$

Then (X, d) is a ν-generalized metric space.

REMARK. See Lemma 4 in [10], Lemmas 4.2 and 4.3 in [11] and Lemma 27 in [12].

PROOF. (N1) and (N2) are obvious. Divide the following three cases:

(a) $\nu = 2$.
(b) ν is even.
(c) ν is odd.

In the case of (a), we fix $(x, y, u, v) \in X^{(4)}$. We further consider the following three cases:

(a-1) $x, v \in A$ and $y, u \in B$.
(a-2) $y, u \in A$ and $x, v \in B$.
(a-3) Otherwise.

In the case of (a-1), we have

$$\begin{aligned} d(x,y) &= \rho(Sx,Sy) \\ &\le \rho(Sx,Su) + \rho(Su,Sv) + \rho(Sv,Sy) \\ &= d(x,u) + d(u,v) + d(v,y). \end{aligned}$$

Similarly we can prove (N3) in the case of (a-2). In the case of (a-3), noting

$$\{(x,u),(u,v),(v,y)\} \cap (X^2 \backslash (A \times B \cup B \times A)) \neq \emptyset,$$

we have

$$\begin{aligned} d(x,y) &\le M + \rho(Sx,Sy) \\ &\le M + \rho(Sx,Su) + \rho(Su,Sv) + \rho(Sv,Sy) \\ &\le d(x,u) + d(u,v) + d(v,y). \end{aligned}$$

Thus we obtain (N3) in the case of (a). Therefore (X,d) is a 2-generalized metric space. By Proposition 7, (X,d) is a ν-generalized metric space for all even positive integers ν. In the case of (c), using Lemma 33, we can prove that (X,d) is a ν-generalized metric space for all odd positive integers ν. □

Now we give two examples. In Section 6, we give a proof of Theorem 4 by Theorem 25. On the other hand, Theorem 3 cannot be proved by Theorem 25 because of the following examples.

Example 35. Let $\mu \in \mathbf{N}$ and $M = \{0,\dots,\mu\}$. Put $A = \{0_0,\dots,0_\mu\}$, $B = \{1/n : n \in \mathbf{N}\}$ and $X = A \cup B$. Define a mapping S from X into $[0,1]$ by $S0_j = 0$ and $S(1/n) = 1/n$. Define a function d from $X \times X$ into $[0,\infty)$ by

$$\begin{aligned} &d(x,x) = 0 \\ &d(x,y) = d(y,x) = |Sx - Sy| && \text{if } (x,y) \in A \times B \\ &d(x,y) = 1 + |Sx - Sy| && \text{otherwise.} \end{aligned}$$

Define a mapping T on X by $T0_j = 0_0$ and $T(1/n) = 1/(n+1)$. Then the following hold:

- (i) (X,d) is a ν-generalized metric space for all even positive integers ν.
- (ii) X is compact.
- (iii) T is an Edelstein contraction.
- (iv) T is not a CJM contraction.
- (v) $\{T^n 1\}$ converges to 0_j for any $j \in M$.
- (vi) 0_j is not a fixed point of T for any $j \in M \backslash \{0\}$.

PROOF. (i) follows from Lemma 34. (ii), (v) and (vi) are obvious. Since $\lim_n d(T^n 1, T^{n+1} 1) = 1$ holds, we obtain (iv) by Lemma 31.

Let us prove (iii). We have

$$d(T(1/n), T(1/m)) = d(1/(n+1), 1/(m+1)) = 1 + \frac{m-n}{(n+1)(m+1)}$$

$$< 1 + \frac{m-n}{nm} = d(1/n, 1/m),$$

$$d(T(1/n), T0_j) = 1/(n+1) < 1/n = d(1/n, 0_j),$$

$$d(T0_k, T0_\ell) = d(0_0, 0_0) = 0 < 1 = d(0_k, 0_\ell)$$

for any $m, n \in \mathbf{N}$ and $j, k, \ell \in M$ with $n < m$ and $k < \ell$. Thus (iii) holds. □

EXAMPLE 36. Let $\nu \in \mathbf{N}$ with $\nu \geq 4$. Put $\mu := [\nu/2] - 1 \in \mathbf{N}$ and $M = \{0, \ldots, \mu\}$. Put $A = \{0_0, \ldots, 0_\mu\}$, $B = \{1/n : n \in \mathbf{N}\}$ and $X = A \cup B$. Define a mapping S from X into $[0, 1]$ by $S0_j = 0$ and $S(1/n) = 1/n$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned} &d(x, x) = 0 \\ &d(x, y) = d(y, x) = |Sx - Sy| && \text{if } (x, y) \in A \times B \\ &d(0_j, 0_k) = \frac{j+1}{j+2} + \frac{k+1}{k+2} && \text{if } (j, k) \in M^{(2)} \\ &d(x, y) = 1 + |Sx - Sy| && \text{otherwise.} \end{aligned}$$

Define a mapping T on X by $T0_j = 0_{\max\{0, j-1\}}$ and $T(1/n) = 1/(n+1)$. Then the following hold:

(i) (X, d) is a ν-generalized metric space.
(ii) X is compact.
(iii) T is an Edelstein contraction.
(iv) T is not a CJM contraction.
(v) $\{T^n 1\}$ converges to 0_j for any $j \in M$.
(vi) $T^{[\nu/2]-2} 0_\mu$ is not a fixed point.

PROOF. (i) follows from Lemmas 32 and 33. (ii), (v) and (vi) are obvious. Since $\lim_n d(T^n 1, T^{n+1} 1) = 1$ holds, we obtain (iv) by Lemma 31.

Let us prove (iii). As in the proof of Example 35, we can prove

$$d(T(1/n), T(1/m)) < d(1/n, 1/m),$$

$$d(T(1/n), T0_j) < d(1/n, 0_j).$$

for any $m, n \in \mathbf{N}$ and $j \in M$ with $n < m$. We have

$$d(T0_0, T0_j) = d(0_0, 0_{j-1}) \leq \frac{1}{2} + \frac{j}{j+1}$$
$$< \frac{1}{2} + \frac{j+1}{j+2} = d(0_0, 0_j)$$

for any $j \in M \backslash \{0\}$. We also have

$$d(T0_j, T0_k) = d(0_{j-1}, 0_{k-1}) = \frac{j}{j+1} + \frac{k}{k+1}$$
$$< \frac{j+1}{j+2} + \frac{k+1}{k+2} = d(0_j, 0_k)$$

for any $(j,k) \in (M \backslash \{0\})^{(2)}$. Thus (iii) holds. □

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