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S-ROCK methods for stochastic delay differential equations with one fixed delay

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Abstract

We propose stabilized explicit stochastic Runge–Kutta methods of strong order one half for Itô stochastic delay differential equations with one fixed delay. The family of the methods is constructed by embedding Runge–Kutta–Chebyshev methods of order one for ordinary differential equations. The values of a damping parameter of the methods are determined appropriately in order to obtain excellent mean square stability properties. Numerical experiments are carried out to confirm their order of convergence and stability properties.

1 Introduction

While one generalization of ordinary differential equations (ODEs) is delay differential equations (DDEs), the stochastic generalization of ODEs is stochastic differential equations (SDEs). The both classes are used for modeling in many fields such as biology, economics, and neuroscience, and numerical methods for them are proposed and studied by many researchers [9, 18, 25]. A further generalization that emerges by mixing both classes is stochastic delay differential equations (SDDEs). SDDEs can deal with more general situations in applications to the real-world (see [14, 27, 22] and references therein). In addition, as SDDEs rarely have analytical explicit solutions, numerical methods for SDDEs attract attention of researchers [11, 12, 24, 28]. Here we are concerned with numerically stabilized explicit methods for some type of SDDEs.

It has been customary to treat the numerical solution of stiff ODEs by implicit methods. However, there are a few classes of stabilized explicit methods. One such class is the class of Runge–Kutta–Chebyshev (RKC) methods, which are well suited to solving stiff problems whose eigenvalues lie near the negative real axis [3, 26]. The class has been recently extended to cope with stiff SDEs [1, 2, 4, 19, 20]. These approaches are important because implicit methods lead to solving a large nonlinear system of equations when the dimension of SDEs is large. For example, Abdulle and Li [2] have developed a family of explicit stochastic orthogonal Runge–Kutta–Chebyshev (SROCK) methods with extended mean square (MS) stability regions. The methods are of strong order one half for non-commutative Itô SDEs. We will extend an idea used in SROCK methods later.

In the case of SDDEs, the issues to derive numerical methods are much more complicated. Nevertheless, Küchler and Platen [21] have derived a strong first order Taylor method for SDDEs as well as a family of stochastic theta methods including the explicit and implicit Euler–Maruyama (EM) methods for SDDEs. Baker and Buckwar [7] and Buckwar [10] have given an important theorem for the strong order of convergence of explicit one-step methods. Mao [23] have carefully investigated relationships between MS exponential stability properties of the explicit EM method and the solution of not only SDDEs with one fixed delay but also SDDEs with one variable delay. Huang, Gan, and Wang [17] have analyzed the asymptotic MS stability of the stochastic theta methods when the methods are applied to a scalar test equation with real coefficients and one fixed delay. Wang, Gan, and Wang [27] have analyzed the MS exponential stability of the stochastic theta methods for SDDEs with one variable delay and the asymptotic MS stability of the methods for SDDEs with one fixed delay. Hu, Mohammed, and Yan [16] have derived the Milstein method for SDDEs with several delays, which is of strong order one, whereas the EM method is of strong order one half. Recently, in [14] there has been an attempt to extend SROCK methods for SDDEs. However, the attempt is only for a specific linear SDDE, not for general problems, and does not give favorable MS stability properties as the values of a damping factor of the SROCK methods have not been determined appropriately.

In the present paper we shall propose a family of SROCK methods for SDDEs with one fixed delay and determine the values of a damping factor of the methods on the basis of MS stability analysis. In Section 2 we will briefly introduce RKC methods of order one for ODEs. In Section 3 we will introduce the EM method for SDDEs and a useful theorem when we consider explicit one-step methods for the strong approximation. In Section 4 we will derive our SROCK methods, and in Section 5 we will give their stability analysis. In Section 6 we will present numerical results and in Section 7 our concluding remarks.

2 First order RKC methods for ODEs

For the autonomous d-dimensional ODE

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t \in [0, T], \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{2. 1}$$

van der Houwen and Sommeijer [26] have constructed the RKC method

$$\begin{aligned} \boldsymbol{K}_0 &= \boldsymbol{y}_n, \qquad \boldsymbol{K}_1 = \boldsymbol{y}_n + \frac{h}{s^2}\boldsymbol{f}(\boldsymbol{K}_0), \\ \boldsymbol{K}_i &= 2\frac{h}{s^2}\boldsymbol{f}(\boldsymbol{K}_{i-1}) + 2\boldsymbol{K}_{i-1} - \boldsymbol{K}_{i-2}, \quad i = 2, 3, \ldots, s, \\ \boldsymbol{y}_{n+1} &= \boldsymbol{K}_s, \end{aligned} \tag{2. 2}$$

where $\boldsymbol{y}_n$ denotes a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (2. 1) for an equidistant grid point $t_n = nh$ ($n = 1, 2, \ldots, N$) with step size $h = T/N < 1$ (N is a natural number). Regardless of $s \geq 1$, the method gives first order approximations to the solution of (2. 1).

When a one-step method is applied to the scalar test equation

$$y'(t) = \lambda y(t), \quad t \geq 0, \quad y(0) = y_0, \tag{2. 3}$$

where $\Re(\lambda) \leq 0$ and $y_0 \neq 0$, it is expressed as $y_{n+1} = R(h\lambda)y_n$ in general. Then $R(z)$ is called its stability function, and $\{z \mid |R(z)| \leq 1, \ z \in \mathbb{C}\}$ is called its stability region.

The stability function of (2. 2) is given by

$$R(z) = T_s\left(1 + \frac{z}{s^2}\right), \tag{2. 4}$$

where $T_k(x)$ is the Chebyshev polynomial of degree k defined by $T_k(\cos\theta) = \cos(k\theta)$ or by the three term recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), \quad k = 2, 3, \ldots, s.$$

For a given s, (2. 4) has the maximal stability region along the negative real axis $[-2s^2, 0]$. However, it has the drawback that the stability region reduces to a single point at $s-1$ intermediate points in $[-2s^2, 0]$.

In order to overcome it, a damping parameter η has been introduced. Then the RKC method with η can be written as

$$\begin{aligned} \boldsymbol{K}_0 &= \boldsymbol{y}_n, \qquad \boldsymbol{K}_1 = \boldsymbol{y}_n + h\frac{\omega_1}{\omega_0}\boldsymbol{f}(\boldsymbol{K}_0), \\ \boldsymbol{K}_i &= 2\frac{T_{i-1}(\omega_0)}{T_i(\omega_0)}\big(h\omega_1\boldsymbol{f}(\boldsymbol{K}_{i-1}) + \omega_0\boldsymbol{K}_{i-1}\big) \\ &\quad - \frac{T_{i-2}(\omega_0)}{T_i(\omega_0)}\boldsymbol{K}_{i-2}, \quad i = 2, 3, \ldots, s, \\ \boldsymbol{y}_{n+1} &= \boldsymbol{K}_s, \end{aligned} \tag{2. 5}$$

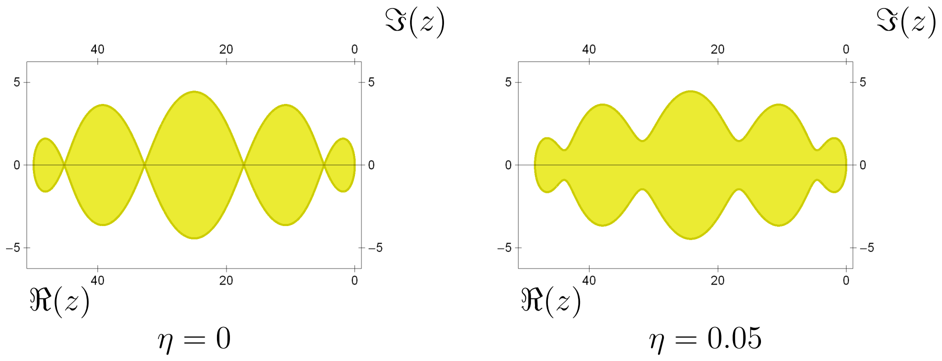


Figure 1: Stability region for $s = 5$ and $\eta = 0,\, 0.05$

where

$$\omega_0 = 1 + \frac{\eta}{s^2}, \qquad \omega_1 = \frac{T_s(\omega_0)}{T_s'(\omega_0)}.$$

Its stability function is given by

$$R(z) = P_s(z) \stackrel{\text{def}}{=} \frac{T_s(\omega_0 + \omega_1 z)}{T_s(\omega_0)}. \tag{2. 6}$$

Here, note that if $\eta = 0$, then (2. 6) leads to (2. 4). In Figure 1 we can see that (2. 6) can have a strip included in the stability region at the cost that the stability interval is slightly shortened. We will refer to $P_s(z)$ in later sections.

Incidentally, when $\boldsymbol{f}$ is Lipschitz continuous, by the formulation of (2. 5) we obtain

$$\|\boldsymbol{K}_s - \boldsymbol{y}_n - h\boldsymbol{f}(\boldsymbol{y}_n)\| \le Ch^2 \tag{2. 7}$$

for a given s and sufficiently small $h > 0$, where C is a constant independent of h.

3 Explicit EM method for SDDEs

We consider the autonomous d-dimensional SDDE

$$\begin{aligned} &\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{f}(\boldsymbol{y}(t), \boldsymbol{y}(t-\tau))\mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t), \boldsymbol{y}(t-\tau))\mathrm{d}W_j(t), \quad t \in [0, T], \\ &\boldsymbol{y}(0) = \boldsymbol{\Psi}(t), \quad t \in [-\tau, 0], \end{aligned} \tag{3. 1}$$

where $\tau > 0$ is a constant, $W_j(t)$, $j = 1, 2, \ldots, m$, are scalar Wiener processes, and $\boldsymbol{\Psi}$ is continuous on $[-\tau, 0]$ and independent of $W_j(t) - W_j(0)$, $j = 1, 2, \ldots, m$, for $t > 0$ and satisfies $E[\sup_{t\in[-\tau,0]} \|\boldsymbol{\Psi}(t)\|^2] < \infty$. We assume the following global Lipschitz condition to ensure that the SDDE has exactly one global solution on $[-\tau, T]$ [10, 21]: there exists a constant $L > 0$ such that

$$\|\boldsymbol{f}(\boldsymbol{\zeta}, \boldsymbol{\eta}) - \boldsymbol{f}(\boldsymbol{\nu}, \boldsymbol{\xi})\| + \sum_{j=1}^{m} \|\boldsymbol{g}_j(\boldsymbol{\zeta}, \boldsymbol{\eta}) - \boldsymbol{g}_j(\boldsymbol{\nu}, \boldsymbol{\xi})\| \le L(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\| + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|) \tag{3. 2}$$

for all $\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\nu}, \boldsymbol{\xi} \in \mathbb{R}^d$. Note that the following linear growth condition holds from (3. 2) since (3. 1) is autonomous [6, p. 113]: there exists a constant $K > 0$ such that

$$\|\boldsymbol{f}(\boldsymbol{\zeta}, \boldsymbol{\eta})\| + \sum_{j=1}^{m} \|\boldsymbol{g}_j(\boldsymbol{\zeta}, \boldsymbol{\eta})\| \leq K(1 + \|\boldsymbol{\zeta}\| + \|\boldsymbol{\eta}\|)$$

for all $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d$.

The definition of strong convergence of order q is given as follows [10, 21]. Suppose that discrete approximations $\boldsymbol{y}_n$, $n = 1, 2, \ldots, N$, are given by a numerical method. Then, we say that the method is of strong (global) order q if there exist positive constants C (independent of h) and δ_0 such that

$$(E[\|\boldsymbol{y}(T) - \boldsymbol{y}_N\|^2])^{1/2} \leq Ch^q, \quad \forall h \in (0, \delta_0).$$

Throughout the present paper, we assume that $\tau = Mh$ holds for τ and $h = T/N$ (M is a natural number). The explicit EM method for (3. 1) is given as follows [21]:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n, \boldsymbol{y}_{n-M})h + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}_n, \boldsymbol{y}_{n-M}) \triangle W_j. \tag{3. 3}$$

Here, $\boldsymbol{y}_{n-M}$, $n = 0, 1, \ldots, M$, are defined as $\boldsymbol{\Psi}(t_n - \tau)$, whereas they are defined by the above formulation for $n \geq M + 1$. In addition,

$$\triangle W_j = \triangle W_{j,n} \stackrel{\text{def}}{=} W_j(t_{n+1}) - W_j(t_n).$$

In what follows, for simplicity we will use the notation $\triangle W_j$ without indicating the dependence on n. The EM method is of strong order one half [10, 21].

When we consider strong approximations for (3. 1) by an explicit one-step method with an increment function $\boldsymbol{\phi}$

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \boldsymbol{\phi}(\boldsymbol{y}_n, \boldsymbol{y}_{n-M}, h, \{\triangle W_j\}_{j=1}^m), \tag{3. 4}$$

the following theorem is very useful [10].

Theorem 3.1 *In addition to (3. 2), suppose that the following conditions hold:*

(1) *there exist positive constants C_1, C_2 (independent of h) such that*

$$\begin{aligned} &\left\| E[\boldsymbol{\phi}(\boldsymbol{\zeta}, \boldsymbol{\eta}, h, \{\triangle W_j\}_{j=1}^m) - \boldsymbol{\phi}(\boldsymbol{\nu}, \boldsymbol{\xi}, h, \{\triangle W_j\}_{j=1}^m)] \right\| \\ &\leq C_1 h(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\| + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|), \end{aligned} \tag{3. 5}$$

$$\begin{aligned} &E\left[\|\boldsymbol{\phi}(\boldsymbol{\zeta}, \boldsymbol{\eta}, h, \{\triangle W_j\}_{j=1}^m) - \boldsymbol{\phi}(\boldsymbol{\nu}, \boldsymbol{\xi}, h, \{\triangle W_j\}_{j=1}^m)\|^2\right] \\ &\leq C_2 h(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\|^2 + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|^2) \end{aligned} \tag{3. 6}$$

for all $\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\nu}, \boldsymbol{\xi} \in \mathbb{R}^d$ and sufficiently small $h > 0$;

(2) *there exist positive constants C_1, C_2 (independent of h) and δ_0 such that*

$$\|E[\boldsymbol{y}(t_{n+1}) - \boldsymbol{y}_{n+1}]\| \leq C_1 h^{p_1}, \qquad \forall h \in (0, \delta_0), \tag{3. 7}$$

$$\left(E\left[\|\boldsymbol{y}(t_{n+1}) - \boldsymbol{y}_{n+1}\|^2\right]\right)^{1/2} \leq C_2 h^{p_2}, \qquad \forall h \in (0, \delta_0) \tag{3. 8}$$

when $\boldsymbol{y}(t_n) = \boldsymbol{y}_n$, where $p_2 \geq 1/2$ and $p_1 \geq p_2 + 1/2$.

Then, (3. 4) is of strong order $p_2 - 1/2$ for (3. 1).

Buckwar [10] has proved a scalar version of this theorem, which means $d = m = 1$. The proof can be obviously extended to the multidimensional case $d, m \geq 2$.

4 SROCK methods for SDDEs

We introduced (2. 5) as a stabilized explicit method for ODEs. Taking it into account, we propose our new explicit method

$$
\begin{aligned}
&\boldsymbol{K}_0 = \boldsymbol{y}_n, \qquad \boldsymbol{K}_1 = \boldsymbol{y}_n + h\frac{\omega_1}{\omega_0}\boldsymbol{f}(\boldsymbol{K}_0, \boldsymbol{y}^*_{n+1-M}),\\
&\boldsymbol{K}_j = 2\frac{T_{j-1}(\omega_0)}{T_j(\omega_0)}\left(h\omega_1\boldsymbol{f}(\boldsymbol{K}_{j-1}, \boldsymbol{y}^*_{n+1-M}) + \omega_0\boldsymbol{K}_{j-1}\right)\\
&\qquad -\frac{T_{j-2}(\omega_0)}{T_j(\omega_0)}\boldsymbol{K}_{j-2}, \quad j = 2, 3, \dots, s,\\
&\boldsymbol{y}^*_{n+1} = \boldsymbol{K}_s, \qquad \boldsymbol{y}_{n+1} = \boldsymbol{y}^*_{n+1} + \sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{y}^*_{n+1}, \boldsymbol{y}^*_{n+1-M})\triangle W_j
\end{aligned}
\tag{4. 1}
$$

for (3. 1). Here, note that $\boldsymbol{y}^*_{n+1-M}$, $n = -1, 0, \dots, M-1$, are defined as $\boldsymbol{\Psi}(t_{n+1} - \tau)$, whereas they are defined by the above formulation for $n \geq M$.

Theorem 4.1 *Suppose that (3. 1) satisfies (3. 2). Then, (4. 1) is of strong order one half for (3. 1).*

Proof. For given $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n-M}$, let us denote by $\tilde{\boldsymbol{y}}_{n+1}$ the approximation obtained by the EM method. Then, we have

$$
\begin{aligned}
&\|E[\boldsymbol{y}(t_{n+1}) - \boldsymbol{y}_{n+1}]\| \leq \|E[\boldsymbol{y}(t_{n+1}) - \tilde{\boldsymbol{y}}_{n+1}]\| + \|E[\tilde{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}]\|,\\
&E\left[\|\boldsymbol{y}(t_{n+1}) - \boldsymbol{y}_{n+1}\|^2\right] \leq 2E\left[\|\boldsymbol{y}(t_{n+1}) - \tilde{\boldsymbol{y}}_{n+1}\|^2\right] + 2E\left[\|\tilde{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\|^2\right].
\end{aligned}
$$

As it is known in [10] that the EM method satisfies (3. 7) and (3. 8) for $p_1 = 2$ and $p_2 = 1$, we can concentrate on the estimates of the second terms in the right-hand side of the above inequalities.

Similarly to (2. 7), as $\|\boldsymbol{y}^*_{n+1} - \boldsymbol{y}_n - h\boldsymbol{f}(\boldsymbol{y}_n, \boldsymbol{y}^*_{n+1-M})\| \leq C_1h^2$ by the formulation of (4. 1), we have

$$
\|\boldsymbol{y}^*_{n+1} - \boldsymbol{y}_n - h\boldsymbol{f}(\boldsymbol{y}_n, \boldsymbol{y}_{n-M})\| \leq C_2h^2, \tag{4. 2}
$$

where C_1, C_2 are (generic) constants independent of h. From this and (3. 2), we obtain

$$
\|E[\tilde{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}]\| = \|E[\boldsymbol{y}_n + h\boldsymbol{f}(\boldsymbol{y}_n, \boldsymbol{y}_{n-M}) - \boldsymbol{y}^*_{n+1}]\| \leq C_2h^2
$$

and

$$
\begin{aligned}
&E\left[\|\tilde{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\|^2\right]\\
&\leq 2E\left[\|\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n, \boldsymbol{y}_{n-M})h - \boldsymbol{y}^*_{n+1}\|^2\right]\\
&\quad + 2E\left[\|\sum_{j=1}^{m}\left(\boldsymbol{g}_j(\boldsymbol{y}_n, \boldsymbol{y}_{n-M}) - \boldsymbol{g}_j(\boldsymbol{y}^*_{n+1}, \boldsymbol{y}^*_{n+1-M})\right)\triangle W_j\|^2\right]\\
&\leq 2C_2^2h^4 + 2\sum_{j=1}^{m}\left\|\boldsymbol{g}_j(\boldsymbol{y}_n, \boldsymbol{y}_{n-M}) - \boldsymbol{g}_j(\boldsymbol{y}^*_{n+1}, \boldsymbol{y}^*_{n+1-M})\right\|^2 h\\
&\leq 2C_2^2h^4 + 2L^2\left(\left\|\boldsymbol{y}_n - \boldsymbol{y}^*_{n+1}\right\| + \left\|\boldsymbol{y}_{n-M} - \boldsymbol{y}^*_{n+1-M}\right\|\right)^2 h\\
&\leq 2C_2^2h^4 + 8L^2C_3^2h^3,
\end{aligned}
$$

where C_2, C_3 and L are positive constants. These imply that there exists a constant $C > 0$ such that

$$\|E[\tilde{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}]\| \leq Ch^2, \qquad \left(E\left[\|\tilde{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\|^2\right]\right)^{1/2} \leq Ch^{3/2}$$

for sufficient small $h > 0$. Thus, (4. 1) satisfies (3. 7) and (3. 8) for $p_1 = 2$ and $p_2 = 1$.

Incidentally, for (4. 1) we have

$$\begin{aligned} &\boldsymbol{\phi}(\boldsymbol{y}_n, \boldsymbol{y}_{n-M}, h, \{\triangle W_j\}_{j=1}^m) \\ &= \boldsymbol{K}_s(\boldsymbol{y}_n, \boldsymbol{y}^*_{n+1-M}(\boldsymbol{y}_{n-M})) - \boldsymbol{y}_n \\ &\quad + \sum_{j=1}^m \boldsymbol{g}_j(\boldsymbol{K}_s(\boldsymbol{y}_n, \boldsymbol{y}^*_{n+1-M}(\boldsymbol{y}_{n-M})), \boldsymbol{y}^*_{n+1-M}(\boldsymbol{y}_{n-M}))\triangle W_j, \end{aligned} \tag{4. 3}$$

where $\boldsymbol{K}_s(\boldsymbol{y}_n, \boldsymbol{y}^*_{n+1-M}(\boldsymbol{y}_{n-M})) = \boldsymbol{K}_s$ in (4. 1), which indicates that $\boldsymbol{K}_s$ depends on $\boldsymbol{y}_n$ and $\boldsymbol{y}^*_{n+1-M}$, whereas $\boldsymbol{y}^*_{n+1-M}$ depends on $\boldsymbol{y}_{n-M}$. Let us denote $\boldsymbol{K}_s$ by $\boldsymbol{\eta}^*, \boldsymbol{\xi}^*$ for $\boldsymbol{K}_0 = \boldsymbol{\eta}, \boldsymbol{\xi}$ in (4. 1), respectively. From (3. 2), (4. 2) and (4. 3) we have

$$\begin{aligned} &\left\|E\left[\boldsymbol{\phi}(\boldsymbol{\zeta}, \boldsymbol{\eta}, h, \{\triangle W_j\}_{j=1}^m) - \boldsymbol{\phi}(\boldsymbol{\nu}, \boldsymbol{\xi}, h, \{\triangle W_j\}_{j=1}^m)\right]\right\| \\ &= \|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\| \\ &\leq \|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - h\boldsymbol{f}(\boldsymbol{\zeta}, \boldsymbol{\eta})\| + \|\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu} - h\boldsymbol{f}(\boldsymbol{\nu}, \boldsymbol{\xi})\| \\ &\quad + \|\boldsymbol{f}(\boldsymbol{\zeta}, \boldsymbol{\eta}) - \boldsymbol{f}(\boldsymbol{\nu}, \boldsymbol{\xi})\|\, h \\ &\leq 2C_2h^2 + Lh\left(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\| + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|\right) \end{aligned}$$

and

$$\begin{aligned} &E\left[\|\boldsymbol{\phi}(\boldsymbol{\zeta}, \boldsymbol{\eta}, h, \{\triangle W_j\}_{j=1}^m) - \boldsymbol{\phi}(\boldsymbol{\nu}, \boldsymbol{\xi}, h, \{\triangle W_j\}_{j=1}^m)\|^2\right] \\ &= E\left[\left\|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu}) \right.\right. \\ &\qquad \left.\left. + \sum_{j=1}^m \left\{\boldsymbol{g}_j(\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*), \boldsymbol{\eta}^*) - \boldsymbol{g}_j(\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*), \boldsymbol{\xi}^*)\right\} \triangle W_j\right\|^2\right] \\ &= \|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\|^2 \\ &\quad + \sum_{j=1}^m \left\|\boldsymbol{g}_j(\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*), \boldsymbol{\eta}^*) - \boldsymbol{g}_j(\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*), \boldsymbol{\xi}^*)\right\|^2 h \\ &\leq \|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\|^2 \\ &\quad + L^2\left(\|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*)\| + \|\boldsymbol{\eta}^* - \boldsymbol{\xi}^*\|\right)^2 h \\ &\leq \|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\|^2 \\ &\quad + 2L^2\|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*)\|^2 h + 2L^2\|\boldsymbol{\eta}^* - \boldsymbol{\xi}^*\|^2 h \\ &\leq \|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\|^2 \\ &\quad + 2L^2\left(\|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\| + \|\boldsymbol{\zeta} - \boldsymbol{\nu}\|\right)^2 h \\ &\quad + 2L^2\left(\|\boldsymbol{\eta}^* - \boldsymbol{\eta}\| + \|\boldsymbol{\xi}^* - \boldsymbol{\xi}\| + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|\right)^2 h \\ &\leq (4L^2h + 1)\|\boldsymbol{K}_s(\boldsymbol{\zeta}, \boldsymbol{\eta}^*) - \boldsymbol{\zeta} - (\boldsymbol{K}_s(\boldsymbol{\nu}, \boldsymbol{\xi}^*) - \boldsymbol{\nu})\|^2 + 4L^2\|\boldsymbol{\zeta} - \boldsymbol{\nu}\|^2 h \\ &\quad + 6L^2\left(\|\boldsymbol{\eta}^* - \boldsymbol{\eta}\|^2 + \|\boldsymbol{\xi}^* - \boldsymbol{\xi}\|^2 + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|^2\right) h \\ &\leq (4L^2h + 1)\left(2C_2h^2 + Lh\left(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\| + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|\right)\right)^2 \\ &\quad + 4L^2\|\boldsymbol{\zeta} - \boldsymbol{\nu}\|^2 h + 6L^2\left(2C_3^2h^2 + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|^2\right) h, \end{aligned}$$

where C_2, C_3 and L are positive constants. These imply that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\left\| E[\boldsymbol{\phi}(\boldsymbol{\zeta}, \boldsymbol{\eta}, h, \{\triangle W_j\}_{j=1}^m) - \boldsymbol{\phi}(\boldsymbol{\nu}, \boldsymbol{\xi}, h, \{\triangle W_j\}_{j=1}^m)] \right\| \\
\le\; & Ch(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\| + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|), \\
& E\left[\|\boldsymbol{\phi}(\boldsymbol{\zeta}, \boldsymbol{\eta}, h, \{\triangle W_j\}_{j=1}^m) - \boldsymbol{\phi}(\boldsymbol{\nu}, \boldsymbol{\xi}, h, \{\triangle W_j\}_{j=1}^m)\|^2 \right] \\
\le\; & Ch(\|\boldsymbol{\zeta} - \boldsymbol{\nu}\|^2 + \|\boldsymbol{\eta} - \boldsymbol{\xi}\|^2)
\end{aligned}$$

for sufficient small $h > 0$. Consequently, (4. 1) is of strong order one half by Theorem 3.1. $\square$

5 MS stability analysis

Taking a stochastic feedback control system into account, Guo, Qiu, and Mitsui [14] have dealt with a multidimensional linear test SDDE with a scalar Wiener process ($d \ge 2$, $m = 1$) in which the drift term depends on $\boldsymbol{y}(t)$ only and the diffusion term depends on $\boldsymbol{y}(t-\tau)$ only. In this section we shall deal with a similar test SDDE in the one dimensional case ($d = m = 1$) to determine the value of η appropriately.

Let us consider the scalar linear test equation

$$\begin{aligned}
&\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sigma y(t-\tau)\mathrm{d}W(t), \qquad t \ge 0, \\
&y(0) = \Psi(t), \quad t \in [-\tau, 0],
\end{aligned} \tag{5. 1}$$

where $E[|\Psi(t)|^2]$ is continuous on $[-\tau, 0]$ and where $\lambda, \sigma \in \mathbb{C}$ satisfy

$$2\Re(\lambda) + |\sigma|^2 < 0. \tag{5. 2}$$

When we apply Itô's theorem to $|y(t)|^2$ and take expectations on both sides of the obtained equation, we have

$$\mathrm{d}v(t) = 2\Re(\lambda)v(t)\mathrm{d}t + |\sigma|^2 v(t-\tau)\mathrm{d}t,$$

where $v(t) = E[|y(t)|^2]$. Thus, the continuity of $E[|\Psi(t)|^2]$ and (5. 2) mean that $\lim_{t\to\infty} v(t) = 0$ holds [5, 8], that is, the solution of (5. 1) is (asymptotically) MS-stable.

When applied to (5. 1), (4. 1) is expressed as

$$y_{n+1} = P_s(h\lambda)y_n + \sigma P_s(h\lambda)y_{n-M}\triangle W$$

since $y_{n+1}^* = P_s(h\lambda)y_n$ from (2. 5) and (2. 6). This yields

$$E\left[|y_{n+1}|^2\right] = |P_s(p)|^2 E\left[|y_n|^2\right] + q|P_s(p)|^2 E\left[|y_{n-M}|^2\right]$$

and its characteristic equation is given by

$$\xi = |P_s(p)|^2 + q|P_s(p)|^2 \xi^{-M}, \tag{5. 3}$$

where $p = h\lambda$ and $q = h|\sigma|^2$. Thus, if we require that (4. 1) is MS-stable ($\lim_{n\to\infty} E\left[|y_n|^2\right] = 0$) when p, q and M are given, then all the roots of (5. 3) must satisfy $|\xi| < 1$.

If $q|P_s(p)|^2 = 0$, then $|P_s(p)| < 1$ must be satisfied since (5. 3) has a solution $\xi = |P_s(p)|^2$. On the other hand, if $q|P_s(p)|^2 \ne 0$, we can rewrite (5. 3) as $\varphi(\xi) = 0$. Here,

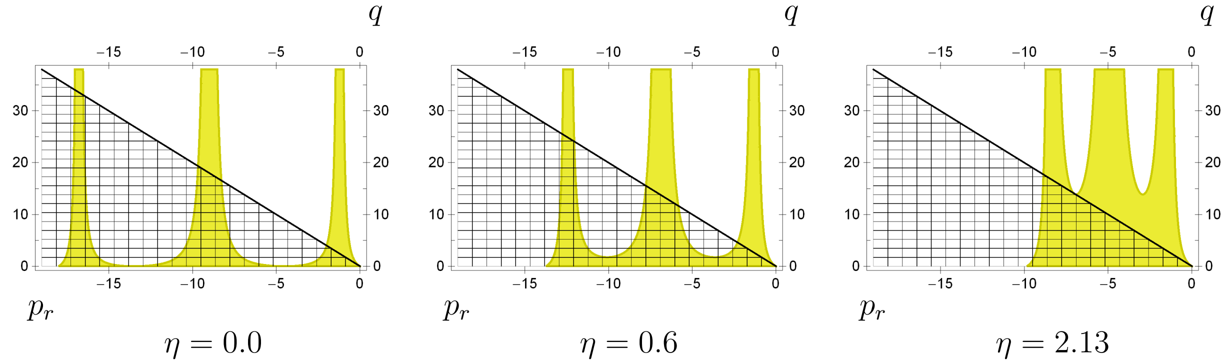


Figure 2: Profile of the MS stability domain of the SROCK method with $s = 3$ when $p_i = 0$ and $\eta = 0.0$, 0.6, and 2.13

Table 1: Optimal values of η

s	η	$\tilde{l}_s^{(\eta)}$	s	η	$\tilde{l}_s^{(\eta)}$	s	η	$\tilde{l}_s^{(\eta)}$
2	1.36	4.74	3	2.13	8.85	6	3.61	27.2
13	5.62	101.40	26	7.84	342.17	53	10.56	1223.45
104	13.55	4156.32	150	15.35	8121.76	200	16.85	13781.80

$\varphi(\xi) \stackrel{\text{def}}{=} \xi^{M+1} - |P_s(p)|^2\xi^M - q|P_s(p)|^2$. This function has a zero point on $(0, 1)$ if $\varphi(1) > 0$, whereas it has a zero point on $[1, \infty)$ if $\varphi(1) \le 0$. Thus, by Cauchy's theorem all the roots of (5. 3) satisfy $|\xi| < 1$ if and only if $\varphi(1) > 0$. Consequently, for any fixed M, the MS stability region of (4. 1) is given by

$$\{(p, q) \mid \hat{R}(p, q) < 1\},$$

where $\hat{R}(p, q) = (1 + q)|P_s(p)|^2$.

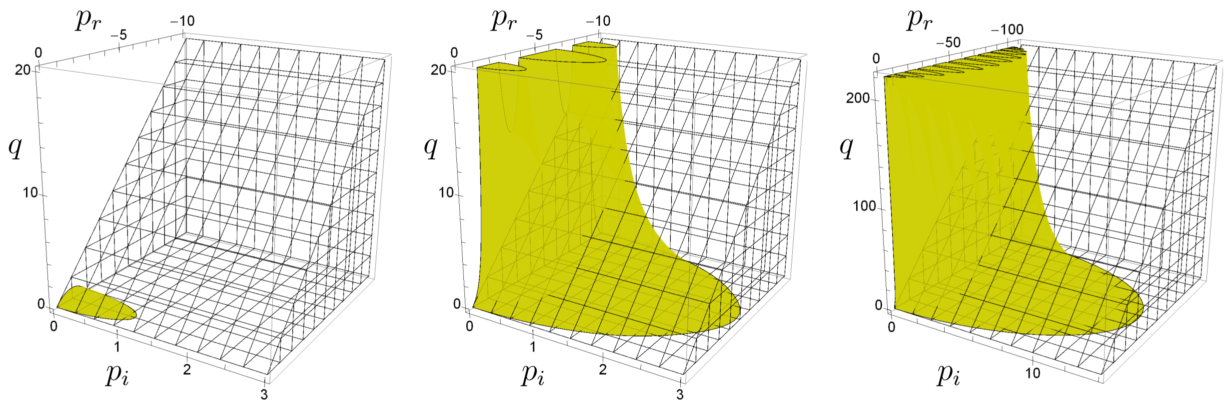
Let us denote $\Re(p)$ and $\Im(p)$ by p_r and p_i, respectively. When $p_i = 0$ and $\eta = 0$, 0.6, and 2.13, the profile of the MS stability domain of (4. 1) with $s = 3$ is given in Figure 2. In the figure the colored part indicates the profile of the MS stability domain when $p_i = 0$, whereas the area enclosed by the mesh indicates the region in which the solution of the test SDDE is MS-stable. Let us consider $\tilde{l}_s^{(\eta)} > 0$ such that for all $p_r \ge -\tilde{l}_s^{(\eta)}$, the profile of the MS stability domain of (4. 1) includes the region where (5. 2) is satisfied. From the figure, since $\tilde{l}_3^{(0.0)}$ and $\tilde{l}_3^{(0.6)}$ are much smaller than $\tilde{l}_3^{(2.13)}$, we can say that the SROCK methods with $\eta = 0.0$ and 0.6 have poorer stability properties than the case of $\eta = 2.13$. Thus, unless the value of η is determined appropriately, even SROCK methods cannot be stabilized well. See also [14].

As $P_s(z)$ is explicitly given in (2. 6), we can arrange the value of η. When we determine the values of $\tilde{l}_s^{(\eta)}$ as large as possible, we obtain Table 1 by numerical calculations. For some other stage numbers, see the appendix.

Incidentally, when (3. 3) is applied to (5. 1), we obtain

$$E\left[|y_{n+1}|^2\right] = |1 + p|^2 E\left[|y_n|^2\right] + qE\left[|y_{n-M}|^2\right]$$

as $y_{n+1} = (1 + h\lambda)y_n + \sigma y_{n-M}\triangle W$. Thus, the MS stability region of (3. 3) is given by $\{(p, q) \mid \hat{R}(p, q) < 1\}$, where $\hat{R}(p, q) = |1 + p|^2 + q$. In the end, we show the MS stability



The EM method $\qquad s = 3,\ \eta = 2.13 \qquad s = 13,\ \eta = 5.62$

Figure 3: MS stability domain of the EM method and the SROCK methods with $s = 3$ and $s = 13$

domain of (3. 3) and (4. 1). As the domain is symmetrical with respect to the plane $p_i = 0$, we plot it only for $p_i \geq 0$. In Figure 3, it is indicated with the colored part. The other part enclosed by mesh indicates the domain in which the solution of the test SDDE is MS-stable. We can see that the SROCK method has a much larger stability domain than the explicit EM method even when $s = 3$, and it is extended along the negative axis of p_r as s increases.

6 Numerical experiments

In Section 4 we have proposed the formulation of our SROCK methods with a parameter η, and in Section 5 we have determined its value appropriately. In order to confirm the performance of the SROCK methods, we carry out numerical experiments. In the sequel, we investigate the root mean square error (RMSE) by simulating 2000 independent trajectories for a given h.

The first example is a linear scalar SDDE with a scalar Wiener process [21]:

$$\mathrm{d}y(t) = (\alpha y(t) + \beta y(t-1))\mathrm{d}t + \gamma y(t)\mathrm{d}W(t), \qquad y(s) = y_0 \text{ (w.p.1)}, \tag{6. 1}$$

where $t \in [0, 3/2]$, $s \in [-1, 0]$, and $\alpha, \beta, \gamma \in \mathbb{R}$ are parameters. The solution is expressed as

$$y(t) = \Phi(0,t)y_0 + \beta \int_0^t \Phi(s,t)y(s-1)\mathrm{d}s$$

for $t \in [0, 1]$, and

$$y(t) = \Phi(1,t)y(1) + \beta \int_1^t \Phi(s,t)y(s-1)\mathrm{d}s$$

for $t \in [1, 3/2]$, where $\Phi(s,t) = \exp[(\alpha - (\gamma^2/2))(t-s) + \gamma(W(t) - W(s))]$.

We set $\alpha = -1/2$, $\beta = -1/4$, $\gamma = 1/2$, and $y_0 = 1$ and investigate the RMSEs of the EM method and the SROCK method with $s = 2$ at $t = 3/2$. The results are indicated in Figure 4. The solid and dashed lines denote the SROCK method with $s = 2$ and the EM method, respectively. Here and in what follows, the dotted line is a reference line

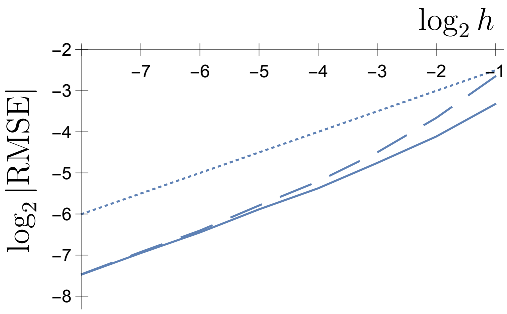


Figure 4: RMSEs of $y(3/2)$. Solid line: SROCK; dashed line: EM; dotted line: reference line with slope 1/2.

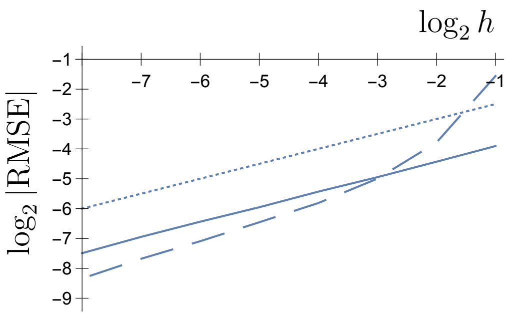


Figure 5: RMSEs of $y(5/2)$. Solid line: SROCK; dashed line: EM; dotted line: reference line with slope 1/2.

with slope 1/2. In the figure, we can confirm the theoretical order of convergence for both methods, especially when h is sufficiently small.

The second example is the Mackey-Glass equation [15] with 2-dimensional multiplicative noise

$$\begin{aligned} &\mathrm{d}y(t) = \left\{ -\alpha y(t) + \frac{\beta y(t-1)}{1 + (y(t-1))^2} \right\} \mathrm{d}t + \gamma_1 y(t) \mathrm{d}W_1(t) + \gamma_2 y(t-1) \mathrm{d}W_2(t), \\ &y(s) = y_0 \ (\text{w.p.1}), \end{aligned} \tag{6. 2}$$

where $t \in [0, 5/2]$, $s \in [-1, 0]$, and $\alpha > 0$, $\beta, \gamma_1, \gamma_2 \in \mathbb{R}$ are parameters. A similar SDDE was considered in [27].

We set $\alpha = 3$, $\beta = 1$, $\gamma_1 = \gamma_2 = 1/2$, and $y_0 = 1$ and investigate the RMSEs of the EM method and the SROCK method with $s = 2$ at $t = 5/2$. As we do not know the exact solution of (6. 2), we seek a numerical solution by the EM method with $h = 2^{-10}$ and use it instead of the exact solution. The results are indicated in Figure 5. The solid and dashed lines denote the SROCK method with $s = 2$ and the EM method, respectively. Both methods show the theoretical order of convergence also for this non-linear SDDE with multidimensional noise.

An advantage of the SROCK methods over the EM method is the stability properties. In order to see this, as a third example, let us consider the following stochastic partial

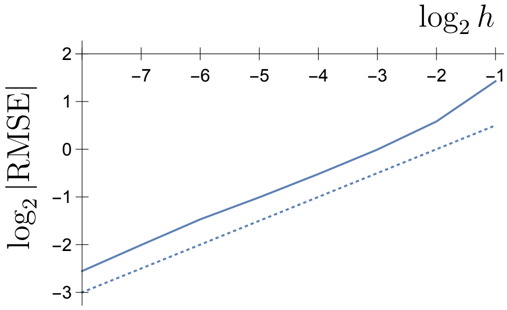


Figure 6: RMSEs of $\boldsymbol{y}(3/2)$. Solid line: SROCK; dotted line: reference line with slope 1/2.

differential equation with delay:

$$\begin{aligned} \mathrm{d}u(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x)\mathrm{d}t + \gamma u(t-1,x)\mathrm{d}W(t), \quad (t,x) \in [0,3/2]\times[0,\pi], \\ u(t,0) &= u(t,\pi) = 0 \text{ (w.p.1)}, \quad t \in [0,3/2], \\ u(t,x) &= \sin(x), \quad (t,x) \in [-1,0]\times[0,\pi], \end{aligned} \tag{6. 3}$$

where $\gamma \in \mathbb{R}$ is a parameter and $W(t)$ is a standard scalar Wiener process. This type of stochastic partial differential equation was dealt with in [13]. The second moment of the solution of (6. 3) is asymptotically stable if $\gamma^2 < 2$. For details, see [13].

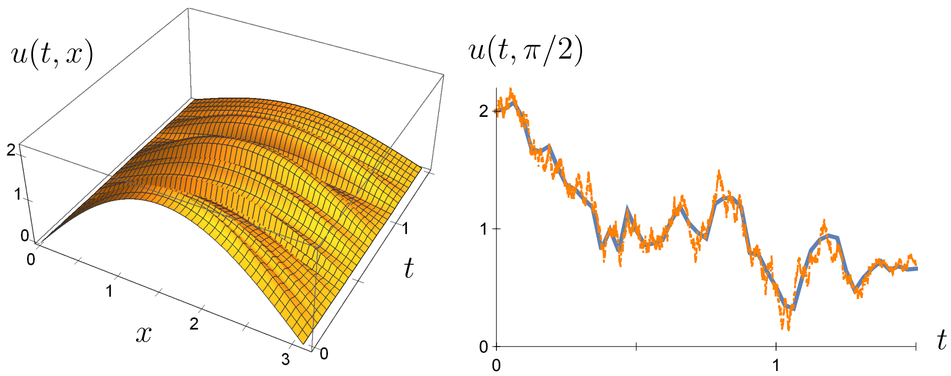
If we discretize the space interval by $N+2$ equidistant points x_i, $i = 0, 1, \ldots, N+1$, and define a vector-valued function by $\boldsymbol{y}(t) \overset{\text{def}}{=} [u(t,x_1)\ u(t,x_2),\ \ldots,\ u(t,x_N)]^\top$, then the application of the central difference scheme to (6. 3) yields the SDDE

$$\begin{aligned} \mathrm{d}\boldsymbol{y}(t) &= A\boldsymbol{y}(t)\mathrm{d}t + \gamma\boldsymbol{y}(t-1)\mathrm{d}W(t), \\ \boldsymbol{y}(0) &= [2\sin(x_1)\ 2\sin(x_1),\ldots,\ 2\sin(x_N)]^\top \quad \text{(w.p.1)}, \end{aligned} \tag{6. 4}$$

where $A \overset{\text{def}}{=} (N+1)^2\,\mathrm{tridiag}(1,-2,1)$. For example, when $N = 127$, the eigenvalues of A are distributed in the interval $(-6.6\times 10^3, -1.0)$. From (5. 2), thus, the solution of (6. 4) is (asymptotically) MS-stable if $\gamma^2 < 2.0$.

We set $N = 127$ and $\gamma = 1/2$ and investigate the performance of the SROCK and EM methods. The EM method requires a very small step size for stability. We can solve the SDDE by the EM method with $h = 2^{-12}$, but cannot with $h = 2^{-i}$, $i = 1, 2, \ldots, 11$. As we do not know the exact solution of (6. 4), we seek a numerical solution by the EM method with $h = 2^{-12}$ and use it instead of the exact solution. In order to solve the SDDE numerically stably with reasonable cost by the SROCK method, we set $s = 95, 63, 45, 30, 20, 14, 9,$ and 6 when $h = 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7},$ and 2^{-8}, respectively.

Figure 6 indicates results over 2000 independent trajectories, whereas Figure 7 indicates results from one trajectory for (6. 4). In Figure 6, the SROCK methods with several values of s show the theoretical order of convergence even for relatively much larger step size than $h = 2^{-12}$, which is required by the EM method. Here, note that as a solution is a vector, the Euclidean norm is used. In Figure 7, the left-hand plot shows an approximation to $u(t,x)$ in (6. 3), which is obtained from one approximate trajectory to the solution of (6. 4) that the SROCK method with $s = 20$ yields for $h = 2^{-5}$. On the other hand,



SROCK
Profile of $u(t,x)$ at $x = \pi/2$

Figure 7: Approximations to $u(t,x)$ in (6. 3). Thick line: SROCK ($s = 20,\ h = 2^{-5}$); dash-dotted line: EM ($h = 2^{-12}$).

the right-hand plot shows the profile of the approximation to $u(t,x)$ at $x = \pi/2$, as well as the profile of another approximation by the EM method. The thick and dash-dotted lines denote the SROCK method and the EM method, respectively. From this, we can see that the SROCK method captures the behaviour of the reference solution by a large step size.

7 Concluding remarks

For d-dimensional Itô SDDEs with an m-dimensional Wiener process and one fixed delay, we have derived the SROCK methods of strong order one half. As the s-stage RKC method with a damping parameter η is embedded in the SROCK method, an optimal damping value of η has been chosen for each s. As a result, all SROCK methods derived in this article have very large MS stability domain, compared with not only the EM method but also another SROCK method proposed in [14].

In the numerical experiments we have confirmed our theoretical results and the advantages of our SROCK methods. The first example was a linear SDDE whose solution can be obtained sufficiently precisely by numerically integrating a term. The second example was a non-linear SDDE with multidimensional noise. In both examples, the SROCK methods clearly showed the theoretical order of convergence. The final example was a high-dimensional stiff SDDE, which comes from a stochastic partial differential equation with delay. This example highlighted the advantages of SROCK methods. The explicit EM method suffered from step size restriction for stability. In general, although implicit methods such as the implicit EM method might be considered as alternatives, they can be computationally expensive for a large system of SDDEs.

Finally, we make the following remarks. In Section 5 we have dealt with (5. 1) as a test equation and obtained a stability function $\hat{R}(p,q)$. Even if there is no delay, that is, even if we replace $\sigma y(t-\tau)$ with $\sigma y(t)$, we have the same stability function. This fact gives us a question. As in [17], when a test equation has $\sigma_1 y(t-\tau) + \sigma_2 y(t)$ in the diffusion term, how is its stability function expressed? Additional analysis to answer it would substantially increase the length of the paper and be beyond the scope of the paper's original intention. However, we will consider this issue in future work.

Acknowledgments

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