Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 66, 2019, pp. 1–14

COMMENTS ON SOME EXISTENCE THEOREMS OF BEST PROXIMITY POINTS FOR CONTRACTIVE-TYPE MAPPINGS

Misako Kikkawa and Tomonari Suzuki

Abstract

In 2010, Sadiq Basha proved two existence theorems of best proximity points for contractive-type mappings. The purpose of this paper is to clarify the mathematical structure of these theorems.

1. Introduction

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers and by $\mathbf{R}$ the set of all real numbers. We let (X,d) be a metric space and let A and B be nonempty subsets of X. Let T be a mapping from A into B and let S be a mapping from B into A. Define $d(A,B) \in \mathbf{R}$ and a function d^* from $X \times X$ into $[0,\infty)$ by

$$d(A,B) = \inf\{d(x,u) : x \in A,\ u \in B\}$$

and

$$d^*(a,b) = d(a,b) - d(A,B)$$

for any $a,b \in X$.

A point $x \in A$ is said to be a *best proximity point* of T if $d^*(x,Tx) = 0$ holds. Also, a point $u \in B$ is said to be a *best proximity point* of S if $d^*(Su,u) = 0$ holds. In the case where $A \cap B \neq \emptyset$, it is obvious that $d(A,B) = 0$ holds. Hence $x \in A$ is a fixed point of T iff x is a best proximity point of T. In the other case, where $A \cap B = \emptyset$, best proximity points of T are minimizers of the problem: $\min\{d(x,Tx) : x \in A\}$. Similarly for $y \in B$.

We human beings have studied the existence of best proximity points; see [3, 4, 5, 8, 10, 11] and others. In 2013, Sadiq Basha, Shahzad and Jeyaraj in [7] proved two existence theorems of best proximity points for Kannan-type and Chatterjea-type mappings. Very recently, in [9], the mathematical structure of these theorems were clarified.

2010 *Mathematics Subject Classification.* Primary 54H25, Secondary 47J25, 90C30.

Key words and phrases. Best proximity point, fixed point, contraction.

The second author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

In 2010, Sadiq Basha [6] proved two existence theorems, Theorems 2 and 7 below, of best proximity points for contractive-type mappings. Motivated by the results in [9], in this paper, we clarify the mathematical structure of these theorems.

2. Banach contraction principle

The fixed point theorem for contractions is referred to as the *Banach contraction principle*. The proof of this is easy and well known. However, for the sake of completeness, we give a proof.

THEOREM 1 ([1, 2]). *Let (Y,d) be a metric space and let U be a contraction on Y, that is, there exists $r \in [0,1)$ satisfying*

$$d(Ua,Ub) \le rd(a,b) \tag{1}$$

for all $a,b \in Y$. Then the following hold:

(i) *$\{U^n a\}$ is a Cauchy sequence for all $a \in Y$.*
(ii) *U has at most one fixed point.*
(iii) *If Y is complete, then U has a unique fixed point.*
(iv) *If U has a fixed point c, then $\{U^n a\}$ converges to c for any $a \in Y$.*

PROOF. Fix $a \in Y$. We first show (i). We have

$$\sum_{j=1}^{\infty} d(U^j a, U^{j+1} a) \le \sum_{j=1}^{\infty} r^j d(a,Ua) = \frac{r}{1-r} d(a,Ua) < \infty.$$

So, a standard argument shows that $\{U^n a\}$ is a Cauchy sequence.

In order to show (ii), we let $c, c' \in Y$ be fixed points of U. Then we have

$$d(c,c') = d(Uc,Uc') \le rd(c,c').$$

Since $r < 1$, we have $d(c,c') = 0$. Thus, (ii) holds.

We next show (iii). By (i), we note that $\{U^n a\}$ is Cauchy. Since Y is complete, $\{U^n a\}$ converges to some $c \in Y$. We have

$$d(c,Uc) = \lim_{n\to\infty} d(U^n a, Uc) \le \lim_{n\to\infty} rd(U^{n-1}a, c) = 0.$$

Hence $Uc = c$ holds, thus, c is a fixed point of U.

In order to prove (iv), we let $c \in Y$ be a fixed point of U. We have

$$\lim_{n\to\infty} d(U^n a, c) = \lim_{n\to\infty} d(U^n a, U^n c) \le \lim_{n\to\infty} r^n d(a,c) = 0.$$

Thus, (iv) holds. □

3. Theorem 3.1 in [6]

In this section, we study Theorem 3.1 in [6], which is Theorem 2 in this paper. We begin with the notations and definitions that appear in the statement of Theorem 2.

Define two subsets A_0 and B_0 of A and B, respectively, by

$$A_0 = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

B is said to be *approximatively compact* with respect to A if every sequence $\{y_n\}$ in B satisfying the condition that $d(x, y_n) \to d(x, B)$ for some $x \in A$ has a convergent subsequence. T is said to be a *proximal contraction* if there exists $r \in [0,1)$ such that

$$d(u, Tx) + d(Tx, Ty) + d(Ty, v) \le r d(x, y) \tag{2}$$

whenever x and y are distinct elements in A satisfying the condition that

$$d(u, Tx) = d(A,B) \qquad \text{and} \qquad d(v, Ty) = d(A,B) \tag{3}$$

for some $u, v \in A$.

THEOREM 2 (Theorem 3.1 in [6]). *Assume the following:*
(a) *X is complete and A and B are closed.*
(b) *B is approximatively compact with respect to A.*
(c) *A_0 and B_0 are nonempty.*
(d) *$T(A_0) \subset B_0$.*
(e) *T is a proximal contraction.*

Then the following hold:
(i) *There exists a unique best proximity point z in A of T.*
(ii) *For each fixed $x_0 \in A_0$, there is a sequence $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ in A such that $d(x_{n+1}, Tx_n) = d(A,B)$ for every $n \in \mathbf{N} \cup \{0\}$, where at least one of the x_n's is the same as z, or the sequence $\{x_n\}$ converges to z.*

It is important to confirm the following fact.

LEMMA 3. *Assume* (c) *and* (d) *of Theorem* 2. *Then the following hold:*
(i) *For every $x \in A_0$, there exists $u \in A_0$ satisfying $d(u, Tx) = d(A,B)$.*
(ii) *For each fixed $x_0 \in A_0$, there is a sequence $\{x_n\}$ in A_0 such that $d(x_{n+1}, Tx_n) = d(A,B)$ for every $n \in \mathbf{N} \cup \{0\}$.*
(iii) *If $x \in A_0$ and $u \in A$ satisfy $d(u, Tx) = d(A,B)$, then $u \in A_0$ holds.*
(iv) *If a sequence $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ in A satisfies $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A,B)$ for $n \in \mathbf{N} \cup \{0\}$, then $x_n \in A_0$ holds for all $n \in \mathbf{N}$.*

PROOF. (i), (iii) and (iv) obviously hold. (ii) follows from (i). □

We give a slight improvement of Theorem 2.

THEOREM 4. *Assume* (c)–(e) *of Theorem* 2. *Assume additionally* (a) *of Theorem* 2 *in the case where* $d(A,B) = 0$. *Then the following hold:*

(i) *There exists a unique best proximity point* z *in* A *of* T.

(ii) *If a sequence* $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ *in* A *satisfies* $x_0 \in A_0$ *and* $d(x_{n+1}, Tx_n) = d(A,B)$ *for* $n \in \mathbf{N} \cup \{0\}$, *then* $\{x_n\}$ *converges to* z.

Considering two cases of $d(A,B) > 0$ and $d(A,B) = 0$, we will prove Theorem 4.

LEMMA 5. *Assume* $d(A,B) > 0$ *and* (c)–(e) *of Theorem* 2. *Then the following hold:*

(i) *If a sequence* $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ *in* A *satisfies* $x_0 \in A_0$ *and* $d(x_{n+1}, Tx_n) = d(A,B)$ *for* $n \in \mathbf{N} \cup \{0\}$, *then there exists* $\nu \in \mathbf{N}$ *satisfying* $x_{\nu+1} = x_\nu$.

(ii) *There exists a unique element* $z \in A$ *satisfying* $d(z, Tz) = d(A,B)$.

(iii) *If* $d(x, Tz) = d(A,B)$ *for* $x \in A$, *then* $x = z$ *holds.*

(iv) *If a sequence* $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ *in* A *satisfies* $x_0 \in A_0$ *and* $d(x_{n+1}, Tx_n) = d(A,B)$ *for* $n \in \mathbf{N} \cup \{0\}$, *then there exists* $\nu \in \mathbf{N}$ *satisfying* $x_n = z$ *for all* $n \geq \nu$.

PROOF. In order to prove (i), we let $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ be a sequence in A satisfying $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A,B)$ for $n \in \mathbf{N} \cup \{0\}$. By Lemma 3 (iv), we note $x_n \in A_0$ for all $n \in \mathbf{N}$. Arguing by contradiction, we assume $x_{n+1} \neq x_n$ for all $n \in \mathbf{N}$. Then since T is a proximal contraction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) + d(Tx_n, x_{n+1}) \\ &\leq r d(x_{n-1}, x_n) \leq \cdots \leq r^n d(x_0, x_1). \end{aligned}$$

Hence

$$\lim_{n \to \infty} (d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) + d(Tx_n, x_{n+1})) = 0$$

holds. So we obtain

$$0 < d(A,B) = \lim_{n \to \infty} d(Tx_n, x_{n+1}) = 0.$$

This is a contradiction. Therefore there exists $\nu \in \mathbf{N}$ satisfying $x_{\nu+1} = x_\nu$. We put $z = x_\nu$.

We next show (ii). Arguing by contradiction, we assume that there exists an element w of A satisfying

$$w \neq z \qquad \text{and} \qquad d(w, Tw) = d(A,B).$$

Since T is a proximal contraction, we have

$$d(w,z) \leq d(w, Tw) + d(Tw, Tz) + d(Tz, z) \leq r d(w,z).$$

Since $r \in [0, 1)$, we obtain $d(w, z) = 0$ and hence $w = z$. This is a contradiction. Therefore we have shown (ii).

In order to show (iii), suppose $d(x, Tz) = d(A, B)$ for some $x \in A$. Arguing by contradiction, we assume $x \neq z$. Then we have $x \in A_0$ and hence $Tx \in B_0$. So there exists $u \in A_0$ satisfying $d(u, Tx) = d(A, B)$. Since T is a proximal contraction, we have

$$\begin{aligned}2d(A, B) &\leq 2d(A, B) + d(Tz, Tx)\\ &= d(x, Tz) + d(Tz, Tx) + d(Tx, u)\\ &\leq rd(z, x)\\ &\leq r(d(z, Tz) + d(Tz, x))\\ &= 2rd(A, B).\end{aligned}$$

Hence, $d(A, B) = 0$ holds. This is a contradiction. Therefore we obtain (iii).

In order to prove (iv), we let $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ be a sequence in A satisfying $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \in \mathbf{N} \cup \{0\}$. From (i), there exists $v \in \mathbf{N}$ satisfying $x_v = z$. By (iii), we have $x_{v+1} = z$. Thus, we obtain $x_n = z$ for all $n \in \mathbf{N}$ with $n \geq v$. □

Lemma 6. *Assume* $d(A, B) = 0$, (a) *and* (c)–(e) *of Theorem* 2. *Then the following hold:*

- (i) *$A_0 = B_0 = A \cap B$ holds.*
- (ii) *A_0 is complete.*
- (iii) *The restriction U of T to A_0 is a contraction on A_0.*
- (iv) *There exists a unique element $z \in A_0$ satisfying $Uz = z$.*
- (v) *z is a unique element of A satisfying $d(z, Tz) = d(A, B)$.*
- (vi) *If a sequence $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ in A satisfies $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \in \mathbf{N} \cup \{0\}$, then $x_n = U^n x_0$ holds for all $n \in \mathbf{N}$ and $\{x_n\}$ converges to z.*

Proof. (i) obviously holds.

We next show (ii). Since A and B are closed, A_0 is closed. Since X is complete, A_0 is complete.

In order to prove (iii), we let U be the restriction of T to A_0. Fix $x, y \in A_0$. It is obvious that $Ux = Tx \in B_0 = A_0$ holds. So U is a mapping on A_0. Put $u = Tx$ and $v = Ty$. Then

$$d(u, Tx) = d(v, Ty) = 0 = d(A, B)$$

holds. In the case where $x \neq y$, since T is a proximal contraction, we have

$$d(Ux, Uy) = d(u, Tx) + d(Tx, Ty) + d(Ty, v) \leq rd(x, y).$$

It the other case, where $x = y$, it is obvious that $d(Ux, Uy) = 0 \leq rd(x, y)$ holds. Therefore we have shown that U is a contraction on A_0.

(iv) follows from Theorem 1.

We next show (v). We have

$$d(z, Tz) = d(z, Uz) = 0 = d(A, B).$$

Arguing by contradiction, we assume that there exists an element w of A satisfying

$$w \neq z \qquad \text{and} \qquad d(w, Tw) = d(A, B).$$

Then we have $w \in A_0$. Hence w is a fixed point of U. This is a contradiction. Therefore we have shown (v).

In order to prove (vi), we let $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ be a sequence in A satisfying $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \in \mathbf{N} \cup \{0\}$. By Lemma 3 (iv), we note that $\{x_n\}$ is a sequence in A_0. We have

$$d(x_{n+1}, Ux_n) = d(x_{n+1}, Tx_n) = d(A, B) = 0$$

for $n \in \mathbf{N}$. Thus, we obtain $x_n = U^n x_0$. By Theorem 1, $\{x_n\}$ converges to z. □

4. Theorem 3.3 in [6]

In this section, we study Theorem 3.3 in [6], which is Theorem 7 in this paper.

THEOREM 7 (Theorem 3.3 in [6]). *Assume the following:*

(a) *X is complete and A and B are closed.*

(b) *S is nonexpansive, that is, $d(Su, Sv) \le d(u, v)$ for any $u, v \in B$.*

(c) *T is a contraction with contraction constant r.*

(d) *If $(x, y) \in A \times B$ satisfies $d(A, B) < d(x, y)$, then $d(Sy, Tx) < d(x, y)$ holds.*

Define a sequence $\{a_n\}_{n \in \mathbf{N} \cup \{0\}}$ by $a_0 \in A$, $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Sa_{2n+1}$ for $n \in \mathbf{N} \cup \{0\}$. Then the following hold:

(i) *There exist $z \in A$ and $w \in B$ satisfying $d(z, Tz) = d(A, B)$, $d(Sw, w) = d(A, B)$ and $d(z, w) = d(A, B)$.*

(ii) *$\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to some best proximity points in A and B of T and S, respectively.*

(iii) *If $x, y \in A$ are best proximity points in A of T, then*

$$d(x, y) \le \frac{2}{1-r} d(A, B)$$

holds.

We give a slight improvement of Theorem 7.

THEOREM 8. *Assume the following:*

(a) *Either A or B is complete.*

(b) *S is nonexpansive.*

(c) *T is a contraction with contraction constant r.*
(d) $d^*(x, Tx) > 0$ *implies* $d^*(STx, Tx) \neq d^*(x, Tx)$.

Define a sequence $\{a_n\}_{n \in \mathbf{N} \cup \{0\}}$ *by* $a_0 \in A$, $a_{2n+1} = Ta_{2n}$ *and* $a_{2n+2} = Sa_{2n+1}$ *for* $n \in \mathbf{N} \cup \{0\}$. *Then the following hold:*

(i) *ST and TS are contractions on A and B, respectively.*
(ii) *ST and TS have unique fixed points* $z \in A$ *and* $w \in B$, *respectively.*
(iii) *z and w are best proximity points in A and B of T and S, respectively, which satisfy* $Tz = w$ *and* $Sw = z$.
(iv) $\{a_{2n}\}$ *and* $\{a_{2n+1}\}$ *converge to z and w, respectively.*
(v) *If* $x, y \in A$ *are best proximity points of T, then*

$$d(x, y) \leq \frac{2}{1 - r} d(A, B)$$

holds.
(vi) *If* $x \in A$ *is a best proximity point of T, then*

$$d(z, x) \leq \frac{2}{1 - r} d(A, B) \qquad \text{and} \qquad d(x, w) \leq \frac{1 + r}{1 - r} d(A, B)$$

hold.

Remark. It is obvious that (a) of Theorem 8 is weaker than (a) of Theorem 7. It is also obvious that (d) of Theorem 8 is weaker than (d) of Theorem 7.

Proof. We first show (i). For $x, y \in A$ and $u, v \in B$, we have

$$d(STx, STy) \leq d(Tx, Ty) \leq rd(x, y)$$

and

$$d(TSu, TSv) \leq rd(Su, Sv) \leq rd(u, v),$$

thus, ST and TS are contractions with contraction constant r.

We next prove (ii). We consider the following two cases:

- A is complete.
- B is complete.

In the first case, by Theorem 1 (iii), ST has a unique fixed point $z \in A$. Since

$$TS(Tz) = T(STz) = Tz,$$

$w := Tz$ is a fixed point of TS. By Theorem 1 (ii), w is a unique fixed point of TS. In the second case, by Theorem 1 (iii), TS has a unique fixed point $w \in B$. Since $STSw = Sw$, $z := Sw$ is a fixed point of ST. By Theorem 1 (ii), z is a unique fixed point of ST.

Let us prove (iii). We have already shown $Tz = w$ and $Sw = z$. It follows from (d) and $STz = z$ that $d^*(STz, Tz) = d^*(z, Tz) = 0$ holds. Thus, z is a best proximity point in A of T. Since

$$0 = d^*(STz, Tz) = d^*(Sw, w),$$

w is a best proximity point in B of S. We have proved (iii).

It is obvious that (iv) follows from Theorem 1 (iv).

Let us prove (v). Let $x, y \in A$ be best proximity points of T. Then we have

$$\begin{aligned} d(x, y) &\le d(x, Tx) + d(Tx, Ty) + d(Ty, y) \\ &\le d(x, Tx) + rd(x, y) + d(y, Ty) \\ &= rd(x, y) + 2d(A, B). \end{aligned}$$

Hence (v) holds.

We finally prove (vi). Let $x \in A$ be a best proximity point of T. Since z is also a best proximity point of T, we have from (v)

$$d(z, x) \le \frac{2}{1 - r} d(A, B).$$

We also have

$$\begin{aligned} d(x, w) &\le d(x, Tx) + d(Tx, w) \\ &= d(x, Tx) + d(Tx, Tz) \\ &\le d(A, B) + rd(x, z) \\ &\le \left(1 + \frac{2r}{1 - r}\right) d(A, B) \\ &= \frac{1 + r}{1 - r} d(A, B). \end{aligned}$$

Thus, (vi) holds. □

The following examples tell that three numbers that appear in (v) and (vi) of Theorem 8 are best possible.

EXAMPLE 9. Let $r \in (0, 1)$ and put $\sigma := 2/(1 - r) \in (2, \infty)$. Define sequences $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ and $\{u_n\}_{n \in \mathbf{N}}$ by

$$x_n = (0, \sigma r^n) \qquad \text{and} \qquad u_n = (1, \sigma r^n).$$

Put $z = (0, 0)$ and $w = (1, 0)$. Define subsets A, B and X of $\mathbf{R}^2$ by

$$A = \{z\} \cup \{x_n : n \in \mathbf{N} \cup \{0\}\},$$
$$B = \{w\} \cup \{u_n : n \in \mathbf{N}\}$$

and $X = A \cup B$. Define mappings T and S by

$$Tx_n = u_{n+1}, \qquad Tz = w,$$
$$Su_n = x_n, \qquad Sw = z.$$

Define a function e from $X \times X$ into $[0, \infty)$ by

$$e(a, b) = \begin{cases} 1 & \text{if } (a, b) = (x_0, u_1) \text{ or } (a, b) = (u_1, x_0) \\ \|a - b\|_1 & \text{otherwise,} \end{cases}$$

where $\|\cdot\|_1$ is the ℓ_1-norm on $\mathbf{R}^2$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$d(a, b) = \min\left\{ \sum_{j=1}^{n} e(a_{j-1}, a_j) : (a_0, \dots, a_n) \in X^{n+1},\ a_0 = a,\ a_n = b \right\}. \tag{4}$$

Then the following hold:

(i) A, B and X are complete.
(ii) S is nonexpansive.
(iii) T is a contraction.
(iv) (d) of Theorem 8 holds.
(v) x_0 and z are best proximity points of T.
(vi) $d(A, B) = 1$.
(vii) $d(z, x_0) = \dfrac{2}{1 - r}$.
(viii) $d(x_0, w) = \dfrac{1 + r}{1 - r}$.

Proof. We first note

$$d(x_0, x) = e(x_0, x) = \|x_0 - x\|_1,$$
$$d(x_0, u) = e(x_0, u_1) + e(u_1, u) = \|x_0 - u\|_1 - 2,$$
$$d(x, y) = e(x, y) = \|x - y\|_1,$$
$$d(u, v) = e(u, v) = \|u - v\|_1$$

for $x, y \in A \backslash \{x_0\}$ and $u, v \in B$; see also Lemma 12 below. So, (vii) and (viii) hold.

(i) obviously holds.

Since

$$d(Su, Sv) = d(u, v)$$

for any $u, v \in B$, (ii) holds.

Since

$$d(Tx, Ty) = rd(x, y)$$

for any $x, y \in A$, (iii) holds.

Since $d^*(STx, Tx) = 0$ for any $x \in A$, (iv) holds.

(v) and (vi) obviously hold. □

We show that even in the case where $r = 0$, three numbers that appear in (v) and (vi) of Theorem 8 are best possible.

EXAMPLE 10. Put $r = 0$, $\sigma = 2$ and

$$x_0 = (0, 2), \qquad z = (0, 0), \qquad w = (1, 0).$$

Define subsets A, B and X of $\mathbf{R}^2$ by

$$A = \{x_0, z\}, \qquad B = \{w\}, \qquad X = A \cup B.$$

Define mappings T and S by

$$Tx_0 = w, \qquad Tz = w, \qquad Sw = z.$$

Define a function e from $X \times X$ into $[0, \infty)$ by

$$e(a, b) = \begin{cases} 1 & \text{if } (a, b) = (x_0, w) \text{ or } (a, b) = (w, x_0) \\ \|a - b\|_1 & \text{otherwise.} \end{cases}$$

Define a function d from $X \times X$ into $[0, \infty)$ by (4). Then (i)–(viii) of Example 9 hold.

5. Lemma

In this section, we prove one lemma, connected with the underlying metric spaces in Examples 9 and 10. See also Examples 10 and 13 in [9].

We give the definition of metric space, though it is well known. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a *metric space* if the following hold:

(D1) $d(x, x) = 0$

(D2) $d(x, y) = 0 \Rightarrow x = y$

(D3) $d(x, y) = d(y, x)$

(D4) $d(x, z) \le d(x, y) + d(y, z)$

We can easily prove the following.

Lemma 11. *Let X be a nonempty set and let e be a function from $X \times X$ into $[0, \infty)$ satisfying* (D1) *and* (D3) *with $d := e$. Define a function d from $X \times X$ into $[0, \infty)$ by*

$$d(x, y) = \inf\left\{ \sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_0, \dots, u_n) \in X^{n+1},\ u_0 = x,\ u_n = y \right\}.$$

Assume (D2). *Then (X, d) is a metric space.*

We finally prove the following.

Lemma 12. *Let (X, ρ) be a metric space and let A and B be nonempty subsets of X. Put $Y := A \cup B$ and $\ell := \rho(A, B) \in (0, \infty)$. Assume that there exist a subset A_2 of A and mappings Q and R from A_2 into A and B, respectively, satisfying*

$$\rho(a, v) = 2\ell + \rho(a, Qv), \tag{5}$$

$$\rho(a, Rv) = \ell + \rho(a, Qv), \tag{6}$$

$$\rho(v, b) = 3\ell + \rho(Rv, b), \tag{7}$$

$$\rho(Qv, b) = \ell + \rho(Rv, b) \tag{8}$$

for any $a \in A$, $b \in B$, $v \in A_2$ with $a \neq v$. Put $A_1 = A \backslash A_2$. Define a function e from $Y \times Y$ into $[0, \infty)$ by

$$e(v, Rv) = e(Rv, v) = \ell \qquad \text{for all } v \in A_2,$$
$$e(x, y) = \rho(x, y) \qquad \text{otherwise}.$$

Define a function d from $Y \times Y$ into $[0, \infty)$ by

$$d(x, y) = \min\left\{ \sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_0, \dots, u_n) \in Y^{n+1},\ u_0 = x,\ u_n = y \right\}.$$

Then the following hold:

(i) $Qv \in A_1$ *for all* $v \in A_2$.
(ii) $e(x, y) \le e(x, v) + e(v, y)$ *for* $x, y \in Y$ *and* $v \in A_2$.
(iii) $e(x, y) \le e(x, z) + e(z, y)$ *for* $x, y, z \in Y$ *with* $(x, z), (y, z) \notin \mathrm{Gr}(R)$, *where* $\mathrm{Gr}(R)$ *is the graph of* R.
(iv) $e(x, y) \le e(x, z) + e(z, y)$ *for* $x, y \in A_1 \cup B$ *and* $z \in Y$.
(v) $d(x, y) = \rho(x, y)$ *for* $x, y \in A_1 \cup B$.
(vi) $d(u, v) = \rho(u, v)$ *for* $u \in A_1$ *and* $v \in A_2$.
(vii) $d(v, b) = \rho(v, b) - 2\ell = \ell + \rho(Rv, b)$ *for* $v \in A_2$ *and* $b \in B$.
(viii) $d(v, v') = \rho(v, v') - 2\ell = 2\ell + \rho(Rv, Rv')$ *for* $v, v' \in A_2$ *with* $v \neq v'$.
(ix) $d(A, B) = \ell$.
(x) (X, d) *is a metric space.*

REMARK. $A_2 = \varnothing$ is possible. On the other hand, $A_2 = A$ cannot be possible from (i).

PROOF. We first redefine d by

$$d(x,y) = \inf\left\{\sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_0, \dots, u_n) \in Y^{n+1},\ u_0 = x,\ u_n = y\right\}.$$

After showing (viii), we will find that the above infimum is the minimum.

We have by (7)

$$e(v, Rv) = \ell < 3\ell = \rho(v, Rv) \tag{9}$$

for all $v \in A_2$. So we note

$$e(x,y) \le \rho(x,y)$$

for all $x, y \in Y$. It is obvious that

$$e(x,x) = \rho(x,x) = 0 \qquad \text{and} \qquad e(x,y) = e(y,x)$$

hold for all $x \in Y$. Thus, (D1) and (D3) with $d := e$ hold.

We will show (i). Arguing by contradiction, we assume that $Qv \in A_2$ for some $v \in A_2$. Then we have by (7) and (8)

$$3\ell \le 3\ell + \rho(RQv, Rv) = \rho(Qv, Rv) = \ell < 3\ell,$$

which implies a contradiction. Therefore we obtain (i).

In order to show (ii), we let $v \in A_2$. We observe the following:

$$\begin{aligned}
e(v,v) + e(v,Rv) &= e(v,Rv) \\
e(a,v) + e(v,Rv) &= \rho(a,v) + \ell = 3\ell + \rho(a,Qv) \\
&= 2\ell + \rho(a,Rv) \ge \rho(a,Rv) \ge e(a,Rv), \\
e(b,v) + e(v,Rv) &= \rho(b,v) + \ell = 4\ell + \rho(b,Rv) \\
&\ge \rho(b,Rv) = e(b,Rv), \\
e(Rv,v) + e(v,Rv) &= 2\ell \ge 0 = e(Rv,Rv)
\end{aligned}$$

for $a \in A$, $b \in B$, $v \in A_2$ with $a \ne v$ and $b \ne Rv$. So (ii) holds in the case where $Rv \in \{x,y\}$. In the other case, where $Rv \notin \{x,y\}$, we have

$$e(x,y) \le \rho(x,y) \le \rho(x,v) + \rho(v,y) = e(x,v) + e(v,y).$$

We have shown (ii). So we note

$$d(x, y) = \inf\left\{\sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_1, \dots, u_{n-1}) \in (A_1 \cup B)^{n-1},\ u_0 = x,\ u_n = y\right\}.$$

In order to show (iii), we let $x, y, z \in Y$ satisfy $(x, z), (y, z) \notin \mathrm{Gr}(R)$. We have already shown (iii) in the case where $z \in A_2$. So suppose $z \in A_1 \cup B$. Then we have $(z, x), (z, y) \notin \mathrm{Gr}(R)$ and hence

$$e(x, y) \le \rho(x, y) \le \rho(x, z) + \rho(z, y) = e(x, y) + e(z, y).$$

We have shown (iii). In particular, (iv) holds. So we note

$$\begin{aligned} d(x, y) = \min\{&e(x, y), \inf\{e(x, z) + e(z, y) : z \in A_1 \cup B\},\\ &\inf\{e(x, z) + e(z, w) + e(w, y) : z, w \in A_1 \cup B\}\}. \end{aligned}$$

Using (ii)–(iv), we will prove (v)–(viii). We can easily prove (v). For $u \in A_1$ and $v \in A_2$, we have

$$e(v, Rv) + e(Rv, u) = \ell + \rho(Rv, u) = 2\ell + \rho(Qv, u) = \rho(v, u) = e(v, u),$$

which implies (vi). For $b \in B$ and $v \in A_2$ with $b \ne Rv$, we have

$$e(v, Rv) + e(Rv, b) = \ell + \rho(Rv, b) = \rho(v, b) - 2\ell = e(v, b) - 2\ell.$$

We also have by (9)

$$e(v, Rv) = \ell = \rho(v, Rv) - 2\ell.$$

These imply (vii). For $v, v' \in A_2$ with $v \ne v'$, we further observe the following.

$$\begin{aligned} e(v, Rv) + e(Rv, Rv') + e(Rv', v') &= 2\ell + \rho(Rv, Rv')\\ &= \rho(Qv, Rv') + \ell = \rho(Qv, Qv') + 2\ell = \rho(v, Qv')\\ &= \rho(v, v') - 2\ell =: \eta,\\ e(v, Rv) + e(Rv, v') = \ell + \rho(Rv, v') &= 4\ell + \rho(Rv, Rv') \ge \eta \qquad (Rv \ne Rv'),\\ e(v, Rv') + e(Rv', v') = \rho(v, Rv') + \ell &= 4\ell + \rho(Rv, Rv') \ge \eta \qquad (Rv \ne Rv'),\\ e(v, v') = \rho(v, v') &\ge \eta. \end{aligned}$$

From these observations, we obtain (viii).

Let us prove (ix). Since $d(x, y) \le e(x, y) \le \rho(x, y)$ holds for $x, y \in Y$, we have $d(A, B) \le \rho(A, B) = \ell$. Fix $(a, b) \in A \times B$. In the case $a \in A_1$, we have by (v)

$$\ell = \rho(A, B) \le \rho(a, b) = d(a, b).$$

In the other case, where $a \in A_2$, we have by (vii)

$$\ell \le \ell + \rho(Ra, b) = d(a, b).$$

Thus, we obtain (ix).

From (v)–(viii), we obtain (D2). So by Lemma 11, (X, d) is a metric space. □

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., **3** (1922), 133–181.

[2] R. Caccioppoli, Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale, Rend. Accad. Naz. Lincei, **11** (1930), 794–799.

[3] C. Di Bari, T. Suzuki and C. Vetro, Best proximity points for cyclic Meir-Keeler contractions, Nonlinear Anal., **69** (2008), 3790–3794. MR2463333

[4] A. A. Eldred, W. A. Kirk and P. Veeramani, Proximal normal structure and relatively nonexpansive mappings, Studia Math., **171** (2005), 283–293. MR2188054

[5] A. A. Eldred and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl., **323** (2006), 1001–1006. MR2260159

[6] S. Sadiq Basha, Extensions of Banach's contraction principle, Numer. Funct. Anal. Optim., **31** (2010), 569–576. MR2682830

[7] S. Sadiq Basha, N. Shahzad and R. Jeyaraj, Best proximity points: approximation and optimization, Optim. Lett., **7** (2013), 145–155. MR3017101

[8] T. Suzuki, The existence of best proximity points with the weak P-property, Fixed Point Theory Appl., 2013, 2013:259. MR3213093

[9] ———, Comments on some recent existence theorems of best proximity points for Kannan-type and Chatterjea-type mappings, Nihonkai Math. J., **28** (2017), 105–116. MR3794319

[10] T. Suzuki, M. Kikkawa and C. Vetro, The existence of best proximity points in metric spaces with the property UC, Nonlinear Anal., **71** (2009), 2918–2926. MR2532818

[11] T. Suzuki and C. Vetro, Three existence theorems for weak contractions of Matkowski type, Int. J. Math. Stat., **6** (2010), 110–120. MR2520397

(M. Kikkawa)
Department of Mathematics
Faculty of Science, Saitama University
Sakura, Saitama 338-8570, *Japan*
E-mail: mi-sa-ko-kikkawa@jupiter.sannet.ne.jp

(T. Suzuki)
Department of Basic Sciences
Faculty of Engineering
Kyushu Institute of Technology
Tobata, Kitakyushu 804-8550, *Japan*
E-mail: suzuki-t@mns.kyutech.ac.jp