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PAPER *Special Section on Discrete Mathematics and Its Applications*

An Approximation Algorithm for the Maximum Induced Matching Problem on C_5-Free Regular Graphs

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SUMMARY In this paper, we investigate the maximum induced matching problem (MaxIM) on C_5-free d-regular graphs. The previously known best approximation ratio for MaxIM on C_5-free d-regular graphs is $\left(\frac{3d}{4} - \frac{1}{8} + \frac{3}{16d-8}\right)$. In this paper, we design a $\left(\frac{2d}{3} + \frac{1}{3}\right)$-approximation algorithm, whose approximation ratio is strictly smaller/better than the previous one when $d \geq 6$.

key words: *induced matching problem, C_5-free regular graph, approximation algorithm*

1. Introduction

Let $G = (V, E)$ be a simple unweighted graph, where V and E are the set of vertices and the set of edges, respectively. Two edges are called *adjacent* if they have a common vertex. A *matching* in the graph G is a subset of edges, no two of which are adjacent. A matching M is *induced* if no two vertices belonging to different edges of M are adjacent. In other words, an induced matching M in G is formed by the edges of a 1-regular induced subgraph of G. An induced matching is often called the *strong matching* [5], [7].

The Maximum Induced Matching problem (MaxIM) is that of finding an induced matching of maximum cardinality in an input graph. The MaxIM problem was originally introduced by Stockmeyer and Vazirani [14] as a variant of the Maximum Matching problem and motivated as the Risk-Free Marriage problem. Induced matchings have applications in the areas of concurrent transmission of messages in wireless ad hoc networks [1], secure communication channels in broadcast networks [6], communication network testing [14], and many other fields. Thus, MaxIM has received much attention in recent years.

The MaxIM problem is generally intractable. Stockmeyer and Vazirani [14], and Cameron [2] independently proved that MaxIM is NP-hard. Also, it remains NP-hard for several graph classes such as planar graphs of vertex degree at most four [9], bipartite graphs of vertex degree at most three [11], [13], line graphs, chair-free graphs, Hamiltonian graphs [10], and d-regular graphs for $d \geq 3$ [3].

In this paper, we focus only on d-regular graphs as input and consider the approximability of MaxIM on d-regular graphs. Zito [15] proved that a natural greedy strategy yields an approximation algorithm for MaxIM on d-regular graphs with approximation ratio $d - \frac{1}{2} + \frac{1}{4d-2}$. Then, Duckworth, Manlove, and Zito [3] improved the approximation ratio slightly into $\frac{n(d-1)}{n-2}$, i.e., asymptotically $d - 1$ for d-regular graphs of n vertices. Subsequently, Gotthilf and Lewenstein [8] provided a $\left(\frac{3d}{4} + 0.15\right)$-approximation algorithm for MaxIM on d-regular graphs by combining a greedy approach with a local search.

For subclasses of d-regular graphs, several better approximation algorithms are known. Rautenbach [12] designed a $(0.7084d + 0.425)$-approximation algorithm for MaxIM on $\{C_3, C_5\}$-free d-regular graphs. Fürst, Leichter, and Rautenbach [4] provided approximation algorithms for the following three subclasses of d-regular graphs: a $\left(\frac{9d}{16} + \frac{33}{80}\right)$-approximation algorithm for C_4-free d-regular graphs, a $\left(\frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}\right)$-approximation algorithm for $\{C_3, C_4\}$-free d-regular graphs, and a $\left(\frac{3d}{4} - \frac{1}{8} + \frac{3}{16d-8}\right)$-approximation algorithm for C_5-free d-regular graphs.

The goal of this paper is to improve the previously best known $\left(\frac{3d}{4} - \frac{1}{8} + \frac{3}{16d-8}\right)$-approximation algorithm for C_5-free d-regular graphs [4], and we design a $\left(\frac{2d}{3} + \frac{1}{3}\right)$-approximation algorithm, whose approximation ratio is strictly smaller/better than the previously best one when $d \geq 6$. It is important to note that our approximation algorithm works also for $\{C_3, C_5\}$-free d-regular graphs, i.e., MaxIM on $\{C_3, C_5\}$-free d-regular graphs can be better (than [12]) approximated within an approximation ratio of $\left(\frac{2d}{3} + \frac{1}{3}\right)$ for $d \geq 3$.

Related work. The inapproximability results on MaxIM for graph subclasses are also known. Duckworth, Manlove, and Zito [3] proved that for any $\varepsilon > 0$, it is NP-hard to approximate MaxIM on graphs of maximum degree three within $\frac{475}{474} - \varepsilon$, 3-regular graphs within $\frac{2375}{2374} - \varepsilon$, and bipartite graphs of maximum degree three within $\frac{6600}{6659} - \varepsilon$.

On the other hand, polynomial-time algorithms for MaxIM have been developed, for example, for chordal graphs, interval graphs [2], trees [5], circular-arc graphs [7], trapezoid graphs, k-interval-dimension graphs, and cocomparability graphs [6].

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2. Preliminaries

Let $G = (V, E)$ be a simple, unweighted, and undirected graph, where V and E denote the set of vertices and the set of edges, respectively. $V(G)$ and $E(G)$ also denote the vertex set and the edge set of G, respectively. Throughout the paper, let $n = |V|$ and $m = |E|$ for any given graph. Let $G[V']$ denote a vertex-induced subgraph of $G = (V, E)$, consisting of a subset $V' \subseteq V$ and all the edges connecting pairs of vertices in V'. Also, let $G[E']$ denote an edge-induced subgraph of $G = (V, E)$, consisting of a subset $E' \subseteq E$ and the vertices that are endpoints of edges in E'. Let H be a set of graphs. A graph is *H-free* if it does not contain any graph in H as a vertex-induced subgraph.

For a vertex v in a graph G, the *open neighborhood* of v in G is $N_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v in G is denoted by $deg_G(v) = |N_G(v)|$. A graph G is *d-regular* if all the vertices in G have degree d. Throughout the paper, we assume that $d \geq 3$ since MaxIM on 1-regular and 2-regular graphs can be solved in polynomial time.

A (simple) *path* P_k with k vertices $v_1, v_2, \cdots, v_k$ is represented as a sequence $\langle v_1, v_1, \cdots, v_k \rangle$ of those k vertices where $\{v_i, v_{i+1}\}$ is an edge in P_k for each $i = 1, 2, \cdots, k-1$. The *length* of the path P is the number of edges in P, i.e., the length of P_k with k vertices is $k - 1$. A *cycle* C_k with k vertices is similarly written as $C_k = \langle v_1, v_2, \cdots, v_k, v_1 \rangle$.

For a pair of vertices v and v' in G, the *distance* between v and v' is the length of a shortest path from v to v', which is denoted by $dist_G(v, v')$. For the path $P = \{v_1, v_2, v_3, v_4, v_5, \cdots, v_k\}$ of length $k - 1$, for example, $dist_P(v_1, v_1) = 0$, $dist_P(v_1, v_2) = 1$, $dist_P(v_1, v_3) = 2$ and so on. If $dist_G(v, v') = \ell$ for two vertices v and v', then v' is called a distance-ℓ vertex of v. Let $DV_\ell(v)$ be a set of distance-ℓ vertices of v. Similarly, for a pair of edges e and e' in $E(G)$, we define the distance $dist_G(e, e')$ between two edges e and e': The *line graph* $L(G)$ of G is the graph whose vertices are the edges of G, and in which two vertices are adjacent only if they share an incident vertex as edges of G. Then, the distance $dist_G(e, e')$ between two edges e and e' in G is defined as $dist_{L(G)}(e, e')$ between two vertices e and e' in $L(G)$, i.e., the length of a shortest path from e to e' in the line graph $L(G)$ of G. For example, for $P = \langle v_1, v_2, v_3, v_4, v_5, \cdots, v_k \rangle$ again, $dist_P(\{v_1, v_2\}, \{v_1, v_2\}) = 0$, $dist_P(\{v_1, v_2\}, \{v_2, v_3\}) = 1$, $dist_P(\{v_1, v_2\}, \{v_3, v_4\}) = 2$, and so on. If $dist_G(e, e') = \ell$ for two edges e and e', then e' is called a distance-ℓ edge of e. Let $DE_\ell(e)$ be a set of distance-ℓ edges of e. Furthermore, we define the distance between an edge e and a vertex v as the length of a shortest path from one endpoint of e to v, i.e., $dist_G(e, v) = \min\{dist_G(v_e, v), dist_G(v'_e, v)\}$ for $e = \{v_e, v'_e\}$. For example, $dist_P(\{v_2, v_3\}, v_1) = 1$, $dist_P(\{v_2, v_3\}, v_4) = 1$, $dist_P(\{v_2, v_3\}, v_5) = 2$, and so on.

We say that an edge $e \in E(G)$ is *in conflict with* another edge $e' \in E(G)$ if $dist_G(e, e') \leq 2$ and the edge $e \in E(G)$ is called a *conflict edge* of $e' \in E(G)$. Then, for an edge e of a graph G, let

$$C_G(e) = \{e' \in E(G) \mid dist_G(e, e') \leq 2\} = \{e\} \cup DE_1(e) \cup DE_2(e).$$

be the set of all the conflict edges of e. Also, the set of all the conflict edges of a set $E' \subseteq E(G)$ is defined as follows:

$$C_G(E') = \bigcup_{e \in E'} C_G(e).$$

For a subset $E' \subseteq E(G)$ of edges and an edge e in G, let

$$PC_G(E', e) = C_G(e) \setminus \bigcup_{e' \in E' \setminus \{e\}} C_G(e')$$

be the set of edges that are in conflict with e but *not* in conflict with every $e' \in E' \setminus \{e\}$. The edge in $PC_G(E', e)$ is called a *private conflict edge* of e to the set E'. For example, for the graph G shown in Fig. 1, the conflict edges of e are e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, and e. Also, the private conflict edges of e to the set $M = \{e, f, f', f''\}$ are e_2, e_5, e_7, and e.

Let $OPT(G)$ be an optimal induced matching on the input G. We say that an algorithm ALG is a *σ-approximation algorithm* for MaxIM or that ALG's *approximation ratio* is at most σ if $|OPT(G)| \leq \sigma \cdot |ALG(G)|$ holds for any input G, where $ALG(G)$ is an induced matching returned by ALG.

3. Approximation Algorithm

In this section we design a $\left(\frac{2d}{3} + \frac{1}{3}\right)$-approximation algorithm for MaxIM on C_5-free d-regular graphs.

3.1 Algorithm

Here is an outline of our approximation algorithm for an input C_5-free d-regular graph G, which mainly consists of two steps. (i) In the first step, the algorithm initially finds a *maximal* induced matching M by iteratively picking an edge e into the induced matching M, and eliminating all the edges in $C_G(e)$ from the candidates of the solution. (ii) In the second step, the algorithm tries to find a larger induced matching from the temporally obtained induced matching M by a "small modification" as follows: Let M be the set of induced matching edges currently obtained. The algorithm picks one edge e from M. Then, if there exist (at least) two edges e' and e'' in $PC_G(M, e) \setminus \{e\}$ such that $dist_G(e', e'') > 2$, then the algorithm updates the "old" induced matching M to the "new" $M = (M \setminus \{e\}) \cup \{e', e''\}$. If there does not exist such an edge e in M, then the algorithm tries to find an edge e_{min} from $PC_G(M, e)$ such that $|C_G(e_{min})|$ is the minimum among $|C_G(e')|$ for every $e' \in PC_G(M, e)$. If the algorithm finds e_{min}, then it swaps e and e_{min}, i.e., updates $M = (M \setminus \{e\}) \cup \{e_{min}\}$.

The following is a description of our algorithm ALG, where let M be the induced matching obtained by ALG:

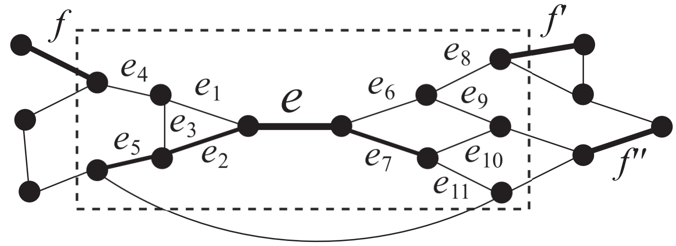


Fig. 1 Edges e_1, e_2, $\cdots$, e_{11} and e in the dotted-line rectangle are conflict edges of e. If $M = \{e, f, f', f''\}$, then the private conflict edges of e to M are e_2, e_5, e_7 and e.

Algorithm `ALG`

Input: A C_5-free d-regular graph $G = (V, E)$.
Output: An induced matching M of G.
Initialization: Set $M = \emptyset$, and obtain $C_G(e)$ and $|C_G(e)|$ for every edge $e \in E$.
Step 1. /* Find an initial maximal set M of induced matching edges. */
If $C_G(M) = E$, then go to **Step 2**; otherwise, arbitrarily select an edge e from $E \setminus C_G(M)$, set $M = M \cup \{e\}$ and repeat **Step 1**.
Step 2. /* Find a larger set M of induced matching edges */
Obtain $PC_G(M, e)$ for every $e \in M$.

(i) If there exists an edge e such that the size of a *maximal* induced matching $MAX(e)$ in $PC_G(M, e) \setminus \{e\}$ is at least two, then set $M = (M \setminus \{e\}) \cup MAX(e)$ and repeat **Step 2**.
(ii) If there exists a pair of edges $e \in M$ and $e' \in PC_G(M, e)$ such that $|C_G(e)| > |C_G(e')|$ and $|C_G(e')|$ is the minimum among $|C_G(e'')|$ for every $e'' \in PC_G(M, e)$, then set $M = (M\setminus\{e\})\cup\{e'\}$ and repeat **Step 2**.
(iii) Otherwise, go to **Termination**.

Termination. Output the solution M and halt.

[End of `ALG`]

Here is a detailed implementation of **Step 2**(i): Suppose that $PC_G(M, e)$ has k edges and let $PC_G(M, e) = \{e, e_1, e_2, \cdots, e_{k-1}\}$. Also, for each $1 \leq i \leq k-1$, let $MAX(e, e_i)$ be a maximal induced matching which is obtained by first selecting e_i from $PC_G(M, e) \setminus \{e\}$ and then selecting induced matching edges from $(PC_G(M, e) \setminus \{e\}) \setminus C_G(e_i)$ if such induced matching edges exist. In **Step 2**(i), `ALG` first obtains $k-1$ maximal induced matchings $MAX(e, e_1)$ through $MAX(e, e_{k-1})$, and then finds the set of maximum cardinality among those $k-1$ sets as $MAX(e)$. One can see that if there exists at least one maximal matching which has at least two induced matching edges, then `ALG` surely finds it in polynomial time.

Now we show the feasibility of the induced matching M output by `ALG`. One can see that if an edge e is selected into M, then all the edges in $C_G(e)$ are eliminated from candidates of the solution. Moreover, we can verify that each edge in $PC_G(M, e)$ is not in conflict with any edge in M except the edge e. Thus, the distance of any two edges in M is at least three and thus all the edges in the output M are induced matching edges. That is, `ALG` can always output a feasible induced matching M.

Next, we bound the running time of `ALG`: Clearly, **Initialization** and **Step 1** can be executed in $O(m^2)$ time. In each execution of **Step 2**(i), the number of induced matching edges in M is incremented at least by one. Hence the total number of executions of **Step 2**(i) is at most $O(m)$. Each iteration of **Step 2**(i) can be done in $O(m^2)$. Therefore, the total computational complexity of **Step 2**(i) is $O(m^3)$. As for **Step 2**(ii), if $|M| = i$ at some time point, then `ALG` has to check i private conflict edge sets, $PC_G(M, e_1)$ through $PC_G(M, e_i)$, in **Step 2**(ii). That is, the total number of executions of **Step 2**(ii) is at most $O(m^2)$. **Step 2**(ii) can be implemented in $O(m)$ time. Hence the total comutational complexity of **Step 2**(ii) is again $O(m^3)$. In the beginning of each iteration of **Step 2** we need $O(m^2)$ time to obtain $PC_G(M, e)$ for every $e \in M$. Since the iteration of **Step 2** is bounded from above by $O(m^2)$, the time complexity of **Step 2** is $O(m^4)$. Therefore, `ALG` runs in $O(m^4)$.

We make a detailed observation on **Step 2**: From the maximality of M, $\bigcup_{e \in M} C_G(e) = E(G)$ holds after **Step 1**. Now suppose that in some iteration of **Step 2**(i), `ALG` finds an edge e_1 such that a maximal induced matching $MAX(e_1)$ in $PC_G(M, e_1)$ has at least two induced matching edges. At this moment, $\bigcup_{e \in M \setminus \{e_1\}} C_G(e) = E(G) \setminus PC_G(M, e_1)$ holds since all the edges in $PC_G(M, e_1)$ are in conflict only with e_1. Moreover, from the maximality of $MAX(e_1)$, $PC_G(M, e_1) \subseteq \bigcup_{e' \in MAX(e_1)} C_G(e')$ must hold. Since `ALG` obtains a new temporal solution M' by setting $M' = (M \setminus \{e_1\}) \cup MAX(e_1)$ in **Step 2**(i), $\bigcup_{e \in M'} C_G(e) = E(G)$ is satisfied again for M'. Note that **Step 2**(ii) guarantees that when M is eventually output by `ALG`, $|C_G(e)| \leq |C_G(e')|$ must hold for every edge $e' \in PC_G(M, e)$. Therefore, from the termination condition of `ALG`, the following should be remarked:

Remark 1. When `ALG` terminates and outputs an induced matching M for an input graph G, the following three properties must be satisfied:

1. As for every private conflict edge set $PC_G(M, e)$ of e to M, any two edges in $PC_G(M, e)$ must be in conflict

with each other;

2. For every edge $e' \in PC_G(M, e)$, $|C_G(e)| \leq |C_G(e')|$ holds; and
3. $\bigcup_{e \in M} C_G(e) = E(G)$ holds, *i.e.*, M must be a maximal set of induced matching edges.

3.2 Approximation Ratio

In this section, we investigate the approximation ratio of the algorithm ALG. Now suppose that given a graph $G = (V, E)$, ALG finally outputs a set M of induced matching edges, and $|ALG(G)| = |M|$. Note that the output M by ALG cannot be enlarged by picking other two or more edges from $PC_G(M, e)$ if edge e is in M. We can obtain the following relationship between $|C_G(e)|$ and $|PC_G(M, e)|$:

Lemma 1. For any maximal set M of induced matching edges in a graph $G = (V, E)$, the following inequality is satisfied:

$$\sum_{e \in M} (|C_G(e)| - |PC_G(M, e)|) \geq 2(|E| - \sum_{e \in M} |PC_G(M, e)|).$$

Proof. Consider an edge e in a subset M of edges, the conflict edge set $C_G(e)$ of e, and the private conflict edge set $PC_G(M, e)$ of e to M. From the definitions, we know

$$\bigcup_{e \in M} (C_G(e) \setminus PC_G(M, e)) = E \setminus \left(\bigcup_{e \in M} PC_G(M, e) \right).$$

Since the private conflict edge sets are independent, the following equality holds:

$$\left| E \setminus \left(\bigcup_{e \in M} PC_G(M, e) \right) \right| = |E| - \sum_{e \in M} |PC_G(M, e)|.$$

Recall that every edge in $C_G(e) \setminus PC_G(M, e)$ must be included in at least one different conflict edge set, say, $C_G(e')$ of $e' \in M$ for $e' \neq e$. Therefore, the inequality holds. □

Now we can estimate the maximum number Γ_d of conflict edges of an edge e in d-regular graphs, which was shown in [3]:

Proposition 1 (Theorem 3.1 in [3]). For any edge e in a d-regular graph G, the number $|C_G(e)|$ of conflict edges is at most $2d^2 - 2d + 1$.

Let Γ_d be the upper bound of $|C_G(e)|$ of conflict edges over all of the edges $e \in E(G)$. One can see that the number $|C_G(e)|$ of conflict edges of the edge e gets much smaller than $2d^2 - 2d + 1$ if an edge e' in $C_G(e)$ is in a short cycle, for example, C_3 or C_4. Indeed, the following results are known [8]:

Proposition 2 (Lemmas 4 and 6 in [8]). If a cycle C_3 of length three contains an edge e in $C_G(e)$ of a d-regular graph G, then the cycle C_3 decreases the upper bound Γ_d of $|C_G(e)|$ by at least d. Moreover, if a cycle C_4 of length four contains an edge e in $C_G(e)$, then the cycle C_4 decreases the upper bound Γ_d by at least one.

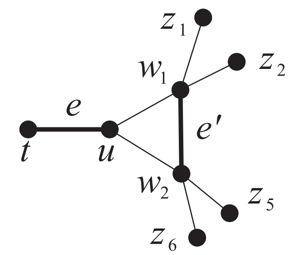


Fig. 2 An edge $e = \{t, u\}$ owns a triangle edge $e' = \{w_1, w_2\}$.

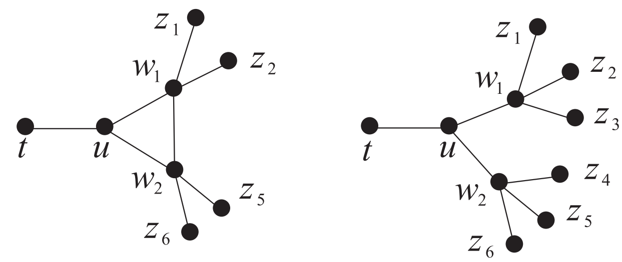


Fig. 3 Since an edge $e = \{t, u\}$ owns a triangle edge $e' = \{w_1, w_2\}$, $e' = \{w_1, w_2\}$ decreases the upper bound Γ_d of $|C_G(e)|$ by at least one.

Take a look at an edge $e = \{t, u\}$ illustrated in Figure 2. If two neighbor vertices, w_1 and w_2, of the edge e are connected by an edge $e' = \{w_1, w_2\}$, then e' is called the *triangle edge* of e, and we say that e owns the triangle edge e' or e' is the triangle edge of e. Then, we can obtain Lemma 2:

Lemma 2. If an edge e in a graph G owns a triangle edge e', then e' decreases the upper bound Γ_d of $|C_G(e)|$ by at least one.

Proof. This lemma can be obtained by a simple observation on two graphs illustrated in Fig. 3. The right graph does not have any triangle edge but the left one has one triangle edge $e' = \{w_1, w_2\}$. That is, we can think that two edges $\{w_1, z_3\}$ and $\{w_2, z_4\}$ in the right graph are replaced with one triangle edge $\{w_1, w_2\}$, or two edges are combined into one edge. Therefore, the value of Γ_d must decrease by at least one, because of the triangle edge e'. □

Now consider an edge $e = \{t, u\}$ in the solution M and the private conflict edges of e to M, $PC_G(M, e)$. Then, let $U_G(e) = (\{e' \mid dist_G(e', u) \leq 1\} \cap PC_G(M, e)) \setminus \{e\}$ and $T_G(e) = (\{e' \mid dist_G(e', t) \leq 1\} \cap PC_G(M, e)) \setminus \{e\}$. Roughly speaking, $U_G(e)$ and $T_G(e)$ are the "u-side" subset and the "t-side" subset of edges in $PC_G(M, e)$, respectively. Note that $PC_G(M, e) = U_G(e) \cup T_G(e) \cup \{e\}$ and $U_G(e) \cap T_G(e)$ may be non-empty. Moreover, let $U_G^0(e) = \{e' \in U_G(e) \mid dist_G(e', u) = 0\}$, $U_G^1(e) = U_G(e) \setminus U_G^0(e)$, $T_G^0(e) = \{e' \in T_G(e) \mid dist_G(e', t) = 0\}$, and $T_G^1(e) = T_G(e) \setminus T_G^0(e)$.

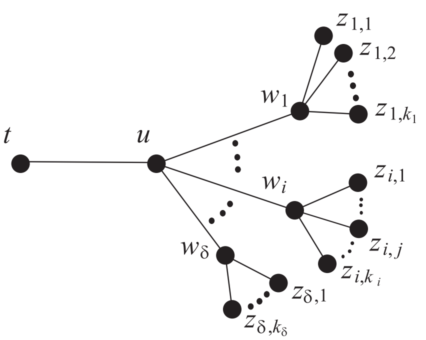


Fig. 4 $W_G(e) = V(G[U_G(e)]) \cap DV_1(u) = \{w_1, w_2, \cdots, w_\delta\}$ where w_i has k_i neighbors, $z_{i,1}$ through z_{i,k_i}.

From now on, let $|PC_G(M, e)| = \beta$. Without loss of generality, we assume that $|U_G(e)| \geq |T_G(e)|$ holds in the following. Then, we obtain the following lemma, which is quite trivial but plays a key role to estimate the approximation ratio of ALG:

Lemma 3. For each $e \in M$, $|U^1_G(e)| \geq \frac{\beta-1}{2} - (d-1)$ holds.

Proof. Clearly $|U^0_G(e)| \leq d - 1$ holds. Since $|U_G(e) \cup T_G(e)| = \beta - 1$ and $|U_G(e)| \geq |T_G(e)|$ by the assumptions, $|U_G(e)| \geq \frac{\beta-1}{2}$ is satisfied. Hence, we can obtain $|U^1_G(e)| = |U_G(e) \setminus U^0_G(e)| \geq \frac{\beta-1}{2} - (d-1)$. □

See Fig. 4. Let $W_G(e) = V(G[U_G(e)]) \cap DV_1(u) = \{w_1, w_2, \cdots, w_\delta\}$ be a set of δ neighbor vertices of u, where $\delta \leq |DV_1(u)| - 1$ holds (where "−1" comes from the edge $\{t, u\}$). Then, we define $U^1_G(e, w_i) = \{(w_i, v) \mid v \in DV_1(w_i)\} \cap U^1_G(e)$ for each $w_i \in W_G(e)$. Without loss of generality, we assume that $|U^1_G(e, w_1)| \geq |U^1_G(e, w_i)|$ for each $i = 2, \cdots, \delta$. Now, we consider the case where $|U^1_G(e, w_1)| \leq 1$ holds. Then, we obtain the following lemma:

Lemma 4. Suppose that $|U^1_G(e, w_1)| \leq 1$ and the algorithm ALG outputs a solution M. Then $|PC_G(M, e)| \leq 4d - 3$ and $|C_G(e)| + |PC_G(M, e)| \leq 2d^2 + 2d - 2$ hold for every induced matching edge $e \in M$.

Proof. From the definition, $PC_G(M, e) = \{e\} \cup U_G(e) \cup T_G(e)$. Then, by the assumption $|U_G(e)| \geq |T_G(e)|$, the following inequality holds:

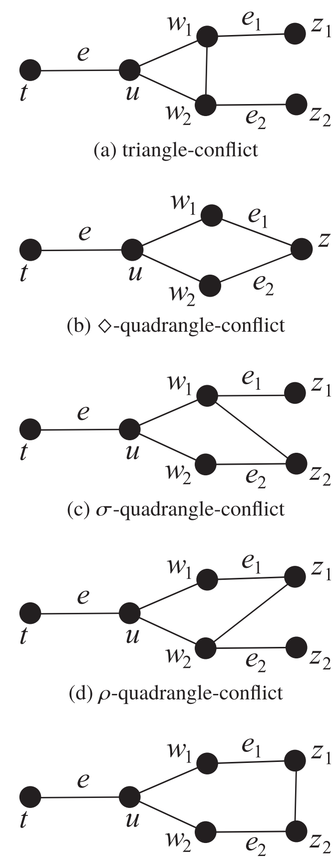
$$\begin{aligned} |PC_G(M, e)| &\leq 1 + |U_G(e)| + |T_G(e)| \\ &\leq 1 + 2|U_G(e)|. \end{aligned}$$

For a d-regular graph G, $|U^0_G(e)| \leq d - 1$ holds. The assumption $|U^1_G(e, w_1)| \leq 1$ means that $|U^1_G(e, w_i)| \leq 1$ holds for each i, $2 \leq i \leq \delta$. It follows that $|U^1_G(e)| \leq d - 1$ and $|U_G(e)| = |U^0_G(e)| + |U^1_G(e)| \leq 2(d-1)$. Therefore, $|PC_G(M, e)| \leq 1 + 4(d-1) = 4d - 3$ holds.

Since $|C_G(e)| \leq 2d^2 - 2d + 1$ as shown in Proposition 1, the inequality

$$\begin{aligned} |C_G(e)| + |PC(M, e)| &\leq (2d^2 - 2d + 1) + (4d - 3) \\ &= 2d^2 + 2d - 2 \end{aligned}$$

is obtained. □



(a) triangle-conflict

(b) ◇-quadrangle-conflict

(c) σ-quadrangle-conflict

(d) ρ-quadrangle-conflict

(e) pentagon-conflict

Fig. 5 Five types of conflicts of two edges e_1 and e_2 in $U^1_G(e)$.

Next, suppose that $|U^1_G(e, w_1)| \geq 2$ holds. We first depict all possible conflict ways of an edge of $U^1_G(e, w_1)$ and another edge of $U^1_G(e, w_i)$, where $i \neq 1$.

Recall that any two edges in $PC_G(M, e)$ (and thus any two edges in $U^1_G(e)$) are in conflict with each other to the solution M of ALG. There are five types of conflicts of two edges, say, e_1 and e_2, in $U^1_G(e)$ as follows: (a) triangle-conflict, (b) ◇-quadrangle-conflict, (c) σ-quadrangle-conflict, (d) ρ-quadrangle-conflict, and (e) pentagon-conflict. See Fig. 5 and consider two edges $e_1 = \{w_1, z_1\}$ and $e_2 = \{w_2, z_2\}$ in $U^1_G(e)$. (a) If e_1 is in conflict with e_2 since there exists the edge $\{w_1, w_2\}$ as shown in Fig. 5(a), then we say that e_1 and e_2 are in *triangle-conflict* with each other by the edge $\{w_1, w_2\}$. (b) See Fig. 5(b). If e_1 and e_2 are incident to a common vertex z and $U^1_G(e)$ does not have the edge $\{w_1, w_2\}$, then we say that e_1 and e_2 are in *◇-quadrangle-conflict* with each other. Note that if the graph shown in Fig. 5(b) has the edge $\{w_1, w_2\}$, then we regard the conflict of e_1 and e_2 as the triangle conflict caused by $\{w_1, w_2\}$. (c) If there exists the edge $\{w_1, z_2\}$ but does not exist the edge $\{w_1, w_2\}$ as shown in Fig. 5(c), then we say that e_1 and e_2 are in *σ-quadrangle-conflict* with each other by $\{w_1, z_2\}$. (d) If there exists the edge $\{w_2, z_1\}$ but does not exist the edge $\{w_1, w_2\}$ as shown in Fig. 5(d), then we say that e_1 and e_2 are in *ρ-quadrangle-conflict* with each other by $\{w_2, z_1\}$. (e) See Fig. 5(e). If there exists the edge $\{z_1, z_2\}$ but does not exist the edge $\{w_1, w_2\}$, then we say that e_1 and e_2 are in *pentagon-conflict* with each other by $\{z_1, z_2\}$. Recall, however, that all the input graphs are now C_5-free. It follows that the induced cycle $\langle u, w_1, z_1, z_2, w_2, u\rangle$ of length 5 must have at least one edge inside of it. For example, the graph has the edge $\{w_1, z_2\}$, then we regard the conflict of e_1 and e_2 as the σ-quadrangle-conflict caused by $\{w_1, z_2\}$. Therefore, we do not need to take the pentagon-conflict into account.

In the following, we slightly change the previous definition of *triangle edges*. (We call the previously defined triangle edge the *original triangle edge* in the following.) An edge in $U^1_G(e)$ is called a *triangle edge* of the edge e if its one endpoints is w_i and the other is w_j in $W_G(e) \setminus \{w_i\}$, where $w_i \neq w_1$, $w_j \neq w_1$, and $w_i \neq w_j$. That is, for example, an edge $\{w_1, w_3\}$ is not regarded as a triangle edge since its one endpoint is w_1. Let $TE_G(e)$ be the set of triangle edges. Then, we define as follows:

$$A_G(e) = U^1_G(e) \setminus (U^1_G(e, w_1) \cup TE_G(e)).$$

Every edge e_2 in $A_G(e)$ is in conflict with every edge e_1 in $U^1_G(e, w_1)$, and $|U^1_G(e, w_1)| \geq |U^1_G(e, w_i)|$ from the definition. Then, all the edges in $A_G(e)$ are divided into the following two sets, the sets of *triangle-conflict edges* and *quadrangle-conflict edges*.

Triangle-Conflict edge: If an edge e' in $A_G(e)$ is in triangle-conflict with an edge in $U^1_G(e, w_1)$, then we say that e' is a *triangle-conflict* edge. Let $TC_G(e)$ be the set of triangle-conflict edges.

Quadrangle-Conflict edge: If an edge e' in $A_G(e)$ is in $\diamond$-quadrangle, σ-quadrangle, or ρ-quadrangle-conflict with an edge in $U^1_G(e, w_1)$, then we simply say that the edge e' is a *quadrangle-conflict* edge. Let $QC_G(e)$ be the set of quadrangle-conflict edges.

From the definitions, $U^1_G(e) = TC_G(e) \cup QC_G(e) \cup U^1_G(e, w_1) \cup TE_G(e)$ and $TC_G(e) \cap QC_G(e) = \emptyset$ hold.

Recall that we are now assuming that $|U^1_G(e, w_1)| \geq 2$. We take a look at the edge $e' = \{u, w_1\}$ and calculate the cardinality of the set $C_G(e')$ of conflict edges of e'. Note that each edge in $TC_G(e)$ creates one cycle C_3 of length three, which contains e', and each edge in $QC_G(e)$ creates one cycle C_4 of length four, which contains e'. Also, each edge in $TE_G(e)$ must be an original triangle edge of e'. It follows that each edge in $TC_G(e) \cup QC_G(e) \cup TE_G(e)$ causes decrease of the upper bound Γ_d of $|C_G(e')|$ by at least one from Proposition 2 and Lemma 2.

Lemma 5. Suppose that $|U^1_G(e, w_1)| \geq 2$. Also, suppose that the algorithm ALG outputs a solution M. Then, $|C_G(e')| \leq 2d^2 - \frac{\beta}{2} - \frac{1}{2}$ holds, where $e' = \{u, w_1\}$.

Proof. See Fig. 4 again and take a look at triangle-conflict, quadrangle-conflict, and (original) triangle edges in the following:

(i) Suppose that p vertices in $\{w_2, w_3, \cdots, w_\delta\}$ of $\delta - 1$ vertices are endpoints of *triangle-conflict* edges. Then, we can verify that there are p cycles of length three which contain the edge $e' = \{u, w_1\}$. Therefore, by Proposition 2, the value of the upper bound Γ_d of e' is reduced by at least pd. Since each of those p vertices is connected to at most $d - 1$ edges in $TC_G(e)$, $|TC_G(e)| \leq p(d-1) \leq pd$ holds. Namely, we can estimate that each edge in $TC_G(e)$ reduces the value of Γ_d of e' by at least one on average.

(ii) Each edge in $QC_G(e)$ obviously generates one cycle of length four which contains the edge $e' = \{u, w_1\}$. Thus, by Proposition 2, we can also estimate that each edge in $QC_G(e)$ decreases the value of Γ_d of e' by at least one.

(iii) Clearly, each edge in $TE_G(e)$ is a triangle edge of e. Also, it is an *original* triangle edge of $e' = \{u, w_1\}$. Then, by Lemma 2, we can estimate that each edge in $TE_G(e)$ decreases the value of Γ_d of e' by at least one.

Consequently, we can estimate that each edge in $TC_G(e) \cup QC_G(e) \cup TE_G(e)$ decreases the value of Γ_d of e' by at least one. Thus, all the edges in $TC_G(e) \cup QC_G(e) \cup TE_G(e)$ decrease the value of Γ_d of e' by at least $|TC_G(e) \cup QC_G(e) \cup TE_G(e)|$ in total.

Now, recall that $U^1_G(e) = TC_G(e) \cup QC_G(e) \cup U^1_G(e, w_1) \cup TE_G(e)$. Then,

$$\begin{aligned}
&|TC_G(e) \cup QC_G(e) \cup TE_G(e)| \\
&= |U^1_G(e) \setminus U^1_G(e, w_1)| \\
&\geq |U^1_G(e)| - (d-1)
\end{aligned}$$

holds since $|U^1_G(e, w_1)| \leq d - 1$. Furthermore, since $|U^1_G(e)| \geq \frac{\beta-1}{2} - (d-1)$ as shown in Lemma 3, we obtain the following:

$$\begin{aligned}
&|TC_G(e) \cup QC_G(e) \cup TE_G(e)| \\
&\geq |U^1_G(e)| - (d-1) \\
&\geq \left(\frac{\beta - 1}{2} - (d-1)\right) - (d-1) \\
&= \frac{\beta - 1}{2} - 2(d-1).
\end{aligned}$$

Therefore, the upper bound Γ_d of e' decreases by at least $\frac{\beta-1}{2} - 2d + 2$.

From Proposition 1, we obtain the following inequalities:

$$\begin{aligned}
|C_G(e')| &\leq 2d^2 - 2d + 1 - \left(\frac{\beta - 1}{2} - 2d + 2\right) \\
&= 2d^2 - \frac{1}{2} - \frac{\beta}{2}.
\end{aligned}$$

This completes the proof of this lemma. □

From Lemma 5, we can get the following corollary:

Corollary 1. Suppose that $|U^1_G(e, w_1)| \geq 2$ and the algorithm ALG outputs a solution M. Then, $|C_G(e)| \leq 2d^2 - \frac{1}{2} - \frac{\beta}{2}$ for every induced matching edge $e \in M$.

Proof. From Lemma 5, we know that there is an edge e' in $U_G(e)$ of $PC_G(M, e)$ such that $|C_G(e')| \leq 2d^2 - \frac{\beta}{2} - \frac{1}{2}$ for any induced matching edge e. Furthermore, Remark 1 shows that $|C_G(e)| \leq |C_G(e')|$ must be satisfied for e and e'. Therefore, $|C_G(e)| \leq 2d^2 - \frac{1}{2} - \frac{\beta}{2}$ holds. □

The above corollary gives us the following lemma:

Lemma 6. Suppose that $|U^1_G(e, w_1)| \geq 2$ and the algorithm ALG outputs a solution M. Then, $|PC_G(M, e)| \leq \frac{4d^2-1}{3}$, and $|C_G(e)| + |PC_G(M, e)| \leq \frac{8d^2-2}{3}$ hold for every induced matching edge $e \in M$.

Proof. From Corollary 1, we know that for each $e \in M$, $|C_G(e)| \leq 2d^2 - \frac{1}{2} - \frac{\beta}{2}$ holds. From the definitions, $PC_G(M, e) \subseteq C_G(e)$ holds. Therefore, we obtain

$$|PC_G(M, e)| = \beta \leq |C_G(e)| \leq 2d^2 - \frac{\beta}{2} - \frac{1}{2}.$$

That is, $\beta \leq 2d^2 - \frac{\beta}{2} - \frac{1}{2}$ holds and hence β is bounded from above as follows:

$$\beta \leq \frac{4d^2 - 1}{3}. \quad (1)$$

By the definition $|PC_G(M, e)| = \beta$,

$$\begin{aligned} |C_G(e)| + |PC_G(M, e)| &\leq 2d^2 - \frac{\beta}{2} - \frac{1}{2} + \beta \\ &= 2d^2 + \frac{\beta}{2} - \frac{1}{2} \\ &\leq \frac{8d^2 - 2}{3}, \end{aligned}$$

where the last inequality comes from the above (1). This completes the proof of this lemma. □

From Lemmas 4 and 6, we have the following corollary:

Corollary 2. Suppose that a solution M is obtained by the algorithm ALG. Then, $|C_G(e)| + |PC_G(M, e)| \leq \frac{8d^2-2}{3}$ holds for every induced matching edge $e \in M$.

Proof. By Lemma 6, we know that for $|U^1_G(e, w_1)| \geq 2$,

$$|C_G(e)| + |PC_G(M, e)| \leq \frac{8d^2 - 2}{3}.$$

From the assumption $d \geq 3$ and Lemma 4, we obtain the following inequality also for $|U^1_G(e, w_1)| \leq 1$:

$$\begin{aligned} |C_G(e)| + |PC_G(M, e)| &\leq 2d^2 + 2d - 2 \\ &\leq \frac{8d^2 - 2}{3}. \end{aligned}$$

This completes the proof of this corollary. □

The following is our main theorem:

Theorem 1. The algorithm ALG is a $\left(\frac{2d}{3} + \frac{1}{3}\right)$-approximation algorithm for MaxIM on C_5-free d-regular graphs, whose running time is $O(m^4)$.

Proof. From Remark 1, the solution for an input C_5-free d-regular graph $G = (V, E)$ satisfies the inequality in Lemma 1, that is, we have obtained

$$\begin{aligned} &\sum_{e \in M} (|C_G(e)| - |PC_G(M, e)|) \\ &\geq 2(|E| - \sum_{e \in M} |PC_G(M, e)|), \end{aligned}$$

or equivalently,

$$\sum_{e \in M} (|C_G(e)| + |PC_G(M, e)|) \geq 2|E|. \quad (2)$$

From Corollary 2 and $|ALG(G)| = |M|$, we obtain:

$$\begin{aligned} &\sum_{e \in M} (|C_G(e)| + |PC_G(M, e)|) \\ &\leq \frac{|ALG(G)|(8d^2 - 2)}{3}. \end{aligned} \quad (3)$$

Suppose that $|V| = n$, and hence $|E| = \frac{nd}{2}$. Then, the above (2) and (3) give the following inequality:

$$\frac{|ALG(G)|(8d^2 - 2)}{3} \geq nd.$$

Thus,

$$|ALG(G)| \geq \frac{3nd}{8d^2 - 2}.$$

It is known [15] that the size $|OPT(G)|$ of an optimal solution is at most $\frac{nd}{4d-2}$. Therefore, the approximation ratio is as follows:

$$\frac{|OPT(G)|}{|ALG(G)|} \leq \frac{2d}{3} + \frac{1}{3}.$$

□

4. Concluding Remarks

In this paper we have considered the approximability of MaxIM on C_5-free d-regular graphs. The previously best known approximation ratio was $(\frac{3d}{4} - \frac{1}{8} + \frac{3}{16d-8})$. In this paper we have provided a $\left(\frac{2d}{3} + \frac{1}{3}\right)$-approximation algorithm ALG. One can verify that the new approximation ratio of ALG is strictly better than the old one when $d \geq 6$. Recall that ALG initially finds a maximal induced matching M in **Step 1**. However, it is important to note that **Step 1** can be replaced with the $\left(\frac{3d}{4} - \frac{1}{8} + \frac{3}{16d-8}\right)$-approximation algorithm as a subroutine. **Step 2** surely finds an induced matching of the same or larger size than the initial induced matching. This implies that the "hybrid" approximation algorithm achieves the approximation ratio of $\min\left\{\frac{3d}{4} - \frac{1}{8} + \frac{3}{16d-8}, \frac{2d}{3} + \frac{1}{3}\right\}$ for MaxIM on C_5-free d-regular graphs for every $d \geq 3$.

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References

[1] H. Balakrishnan, C.L. Barrett, V.S.A. Kumar, M.V. Marathe, and S. Thite, "The distance-2 matching problem and its relationship to the MAC-layer capacity of ad hoc wireless networks," IEEE J. Sel. Areas Commun., vol.22, no.6, pp.1069–1079, 2004.
[2] K. Cameron, "Induced matching," Discrete Appl. Math., vol.24, no.1-3, pp.97–102, 1989.
[3] W. Duckworth, D.F. Manlove, and M. Zito, "On the approximability of the maximum induced matching problem," J. Discrete Algorithmica, vol.3, no.1, pp.79–91, 2005.
[4] M. Fürst, M. Leichter, and D. Rautenbach, "Locally searching for large induced matchings," Theor. Comput. Sci., vol.720, pp.64–72, 2018.
[5] G. Fricke and R. Laskar, "Strong matching on trees," Congressus Numerantium, vol.89, pp.239–243, 1992.
[6] M.C. Golumbic and M. Lewenstein, "New results on induced matchings," Discrete Appl. Math., vol.101, no.1-3, pp.157–167, 2000.
[7] M.C. Golumbic and R.C. Laskar, "Irredundancy in circular arc graphs," Discrete Appl. Math., vol.44, no.1-3, pp.79–89, 1993.
[8] Z. Gotthilf and M. Lewenstein, "Tighter approximations for maximum induced matchings in regular graphs," Proc. 3rd Workshop on Approximation and Online Algorithms (WAOA'05), LNCS3879, pp.270–281, 2005.
[9] C.W. Ko and F.B. Shepherd, "Bipartite domination and simultaneous matroid covers," SIAM J. Discrete Math., vol.16, no.4, pp.517–523, 2003.
[10] D. Kobler and U. Rotics, "Finding maximum induced matchings in subclasses of claw-free and P_5-free graphs, and in graphs with matching and induced matching of equal maximum size," Algorithmica, vol.37, no.4, pp.327–346, 2003.
[11] V.V. Lozin, "On maximum induced matchings in bipartite graphs," Inform. Process. Lett., vol.81, no.1, pp.7–11, 2002.
[12] D. Rautenbach, "Two greedy consequences for maximum induced matchings," Theor. Comput. Sci., vol.602, pp.32–38, 2015.
[13] I. Rusu, "Maximum weight edge-constrained matchings," Discrete Appl. Math., vol.156, no.5, pp.662–672, 2008.
[14] L.J. Stockmeyer and V.V. Vazirani, "NP-completeness of some generalizations of the maximum matching problem," Inform. Process. Lett., vol.15, no.1, pp.14–19, 1982.
[15] M. Zito, "Maximum induced matchings in regular graphs and trees," Proc. the 25th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'95), LNCS1665, pp.89–100, 1999.

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