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# ON COMPLEMENTARY DUALS—BOTH FIXED POINTS—

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**Abstract**

We consider a quadratic minimization (primal) problem with both fixed endpoints and its associated maximization (dual) problem from a view point of complementarity. We focus on a pair of linear terms, which generates the respective quadratic functions (sums of squares) through an elementary inequality. We show that a complementary identity plays a fundamental part in establishing a dual relation between primal and dual. The identity produces the pair with an equality condition. The condition turns out to be a linear system of $2n$-equation in $2n$-variable. The system yields a couple of solutions, one is a minimum solution and the other is a maximum one. In the $n$-variable pair, both the solutions turn out to be complementary. The optimal solution is characterized by the backward Fibonacci sequence. The duality is enhanced through conjugate function. The solution is also given by dynamic programming. Thus Fibonacci complementary duality is established through the complementary identity approach.

## 1. Introduction

R. Bellman has already analyzed a wide class of dynamic optimization problems—linear, quadratic and nonlinear—[2, 3, 4, 5, 6, 8, 9, 10, 11, 12]. Recently a duality in quadratic programming without constraint has been established through several approaches such as (i) Lagrangean method, (ii) plus-minus method, (iii) inequality method and others [1, 7, 18, 19, 20]. This duality applies partly to the class [18, 19]. It also lights up a duality theory for quadratic problems in [2, 3, 4, 5, 6].

In this paper we propose a new approach, which is called (iv) complementary method. We show that the complementary method analyzes a dual relation in a clear way. It produces a pair of $n$-variable minimization problem (primal) and $n$-variable maximization problem (dual) with an equality condition—a linear system of $2n$-equation on $2n$-variable—. The condition splits into two systems of $n$-equation on $n$-variable. One system yields a minimum solution, while the other does a maximum solution. Thus the complementary approach compresses the duality-analysis into a linear equation problem.

An identity

$$(\mathrm{CI}_n) \quad (c - x_1)\mu_1 + x_1(\mu_1 - \mu_2) + \sum_{k=2}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})]$$

$$+ (x_{n-1} - x_n)\mu_n + (x_n - d)\mu_n = c\mu_1 - d\mu_n$$

is called *complementary*. We show that the complementary identity plays a fundamental part in analyzing a duality between primal and dual. This complementarity with an *elementary inequality* with equality condition

$$\text{(EI)} \quad 2xy \le x^2 + y^2 \qquad \text{on } R^2;\ x = y$$

produces the pair with an equality condition. The condition turns out to be a linear system of $2n$-equation in $2n$-variable. The system yields a couple of solutions, one is a minimum solution and the other is a maximum one. The solution is characterized by the Fibonacci sequence [13, 15, 21, 24].

Section 2 gives a complementary identity in the form of the first three (1-variable, 2-variable and 3-variable) and of a general $n$-variable. Section 3 presents the first three pairs of primal and dual. Each pair is accompanied with an equality condition. This is a linear system of $2k$-equation on $2k$-variable where $k = 1, 2, 3$. The linear system splits into two systems of $k$-equation on $k$-variable, whose solutions yield a minimum solution of primal and a maximum solution of dual, respectively. Section 4 solves a general $n$-variable pair. Section 5 shows a duality between $n$-variable pair. In Section 6, we discuss the duality through conjugate function [14, 16, 22]. Section 7 solves both three-variable pair and $n$-variable pair through dynamic programming [2, 17, 23].

## 2. Complementary identities

First we present three elementary complementary identities, which take a fundamental role in analyzing respective pairs of primal and dual.

Let two real constants $c$, $d$ be given. First we consider a pair of one-variable $x$ and $\lambda$. Then an identity

$$(\mathrm{C}_1) \quad (c - x)\lambda + (x - d)\lambda = c\lambda - d\lambda$$

holds true.

Second we consider a pair of two-variable $(x, y)$ and $(\lambda, \mu)$. Then an identity

$$(\mathrm{C}_2) \quad (c - x)\lambda + x(\lambda - \mu) + (x - y)\mu + (y - d)\mu = c\lambda - d\mu$$

holds true.

Third we consider three-variable case. Let us divide two 3-dimensional vectors

$$(x, y, z), \qquad (\lambda, \mu, \nu)$$

into 6-dimensional ones

$$\begin{array}{cccccc} (c - x, & x, & x - y, & y, & y - z, & z - d), \\ (\lambda, & \lambda - \mu, & \mu, & \mu - \nu, & \nu, & \nu), \end{array}$$

respectively. It turns out that the *inner product* is $c\lambda - d\nu$. Thus an identity

$$(\mathrm{C}_3) \quad (c-x)\lambda + x(\lambda-\mu) + (x-y)\mu + y(\mu-\nu) + (y-z)\nu + (z-d)\nu = c\lambda - d\nu$$

holds true.

Finally we consider $n$-variable case. Let us divide two $n$-dimensional vectors

$$(x_1, x_2, x_3, \ldots, x_k, \ldots, x_n),$$
$$(\mu_1, \mu_2, \mu_3, \ldots, \mu_k, \ldots, \mu_n)$$

into $2n$-dimensional ones

$$(c-x_1, x_1, x_1-x_2, x_2, x_2-x_3, \ldots,$$
$$x_{k-1}, x_{k-1}-x_k, x_k, \ldots, x_{n-1}, x_{n-1}-x_n, x_n-d),$$
$$(\mu_1, \mu_1-\mu_2, \mu_2, \mu_2-\mu_3, \mu_3, \ldots,$$
$$\mu_{k-1}-\mu_k, \mu_k, \mu_k-\mu_{k+1}, \ldots, \mu_{n-1}-\mu_n, \mu_n, \mu_n),$$

respectively. Then we make an inner product of resulting two ones. It turns out to be $c\mu_1 - d\mu_n$:

$$(\mathrm{C}_n) \quad (c-x_1)\mu_1 + x_1(\mu_1-\mu_2) + \sum_{k=2}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})]$$
$$+ (x_{n-1}-x_n)\mu_n + (x_n-d)\mu_n = c\mu_1 - d\mu_n.$$

This identity is called *complementary.*

## 3. Three pairs

In this paper as a pair of *primal*[1] and *dual*, we take $n$-variable optimization problems.

We present the first three pairs as follows. The first pair is

$$(\mathrm{P}_1) \quad \begin{array}{ll} \text{minimize} & (c-x)^2 + (x-d)^2 \\ \text{subject to} & \text{(i)} \quad x \in R^1 \end{array}$$

$$(\mathrm{D}_1) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - (\lambda^2 + \lambda^2) - 2d\lambda \\ \text{subject to} & \text{(i)} \quad \lambda \in R^1. \end{array}$$

[1] Two nouns *primal* and *dual* mean *primal problem* and *dual problem*, respectively.

The second is

$$(\mathrm{P}_2) \quad \begin{array}{ll} \text{minimize} & (c-x)^2 + x^2 + (x-y)^2 + (y-d)^2 \\ \text{subject to} & \text{(i)} \quad (x, y) \in R^2 \end{array}$$

$$(\mathrm{D}_2) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + \mu^2] - 2d\mu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu) \in R^2. \end{array}$$

The third is

$$(\mathrm{P}_3) \quad \begin{array}{ll} \text{minimize} & (c-x)^2 + x^2 + (x-y)^2 + y^2 + (y-z)^2 + (z-d)^2 \\ \text{subject to} & \text{(i)} \quad (x, y, z) \in R^3 \end{array}$$

$$(\mathrm{D}_3) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 + \nu^2] - 2d\nu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu, \nu) \in R^3. \end{array}$$

**3.1.** $(\mathrm{P}_1)$ **vs** $(\mathrm{D}_1)$

Let us consider the first pair of $(\mathrm{P}_1)$ and $(\mathrm{D}_1)$. Then it turns out that both are dual to each other. It holds that

$$2c\lambda - (\lambda^2 + \lambda^2) - 2d\lambda \le (c-x)^2 + (x-d)^2$$

for any feasible pair $(x; \lambda)$. An equality condition is

$$(1) \qquad (\mathrm{EC}_1) \quad c - x = \lambda, \qquad x - d = \lambda.$$

The equality condition $(\mathrm{EC}_1)$ is a linear system of 2-equation on 2-variable $(x, \lambda)$.

Let $(x, \lambda)$ be a solution. Then both sides become a common value with five expressions:

$$\begin{aligned} & (c-x)^2 + (x-d)^2 \\ & = c(c-x) - d(x-d) \\ (5\mathrm{V}_1) \quad & = 2c\lambda - (\lambda^2 + \lambda^2) - 2d\lambda \\ & = \lambda^2 + \lambda^2 \\ & = c\lambda - d\lambda. \end{aligned}$$

The primal $(\mathrm{P}_1)$ has a minimum value

$$\begin{aligned} m_1 &= (c-x)^2 + (x-d)^2 \\ &= c(c-x) - d(x-d) \end{aligned}$$

at $x$, while the dual $(D_1)$ has a maximum value

$$\begin{aligned} M_1 &= 2c\lambda - 2\lambda^2 - 2d\lambda \\ &= 2\lambda^2 \\ &= c\lambda - d\lambda \end{aligned}$$

at $\lambda$.

Lemma 3.1. $(EC_1)$ *has indeed a unique solution:*

$$x = \frac{1}{2}(c+d) \tag{2}$$

$$\lambda = \frac{1}{2}(c-d). \tag{3}$$

Proof. From $(EC_1)$, we have a pair of linear systems of 1-variable on 1-equation:

$$(EQ_1) \quad 2x = c + d \qquad 2\lambda = c - d.$$

The left system has a solution (2), while the right has a solution (3). □

The primal $(P_1)$ has a minimum value

$$m_1 = c(c - \hat{x}) - d(\hat{x} - d) = \frac{1}{2}(c^2 - 2cd + d^2)$$

at a path (point)

$$\hat{x} = \frac{1}{2}(c+d).$$

The dual $(D_1)$ has a maximum value

$$M_1 = c\lambda^* - d\mu^* = \frac{1}{2}(c^2 - 2cd + d^2)$$

at a path

$$\lambda^* = \frac{1}{2}(c-d).$$

### 3.2. $(P_2)$ vs $(D_2)$

Let us consider the second pair. Then $(P_2)$ and $(D_2)$ are dual to each other. It holds that

$$2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + \mu^2] - 2d\mu$$
$$\leq (c - x)^2 + x^2 + (x - y)^2 + (y - d)^2$$

for any feasible pair $(x, y; \lambda, \mu)$. An equality condition is

$$(\mathrm{EC}_2) \quad \begin{array}{ll} c - x = \lambda, & x = \lambda - \mu \\ x - y = \mu, & y - d = \mu. \end{array}$$

The equality condition $(\mathrm{EC}_2)$ is a linear system of 4-equation on 4-variable $(x, y, \lambda, \mu)$.

Let $(x, y, \lambda, \mu)$ be a solution of $(\mathrm{EC}_2)$. Then both sides become a common value with five expressions:

$$\begin{aligned} & (c - x)^2 + x^2 + (x - y)^2 + (y - d)^2 \\ & = c(c - x) - d(y - d) \\ (5\mathrm{V}_2) \quad & = 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + \mu^2] - 2d\mu \\ & = \lambda^2 + (\lambda - \mu)^2 + \mu^2 + \mu^2 \\ & = c\lambda - d\mu. \end{aligned}$$

The primal $(\mathrm{P}_2)$ has a minimum value

$$\begin{aligned} m_2 & = (c - x)^2 + x^2 + (x - y)^2 + (y - d)^2 \\ & = c(c - x) - d(y - d) \end{aligned}$$

at $(x, y)$, while the dual $(\mathrm{D}_2)$ has a maximum value

$$\begin{aligned} M_2 & = 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + \mu^2] - 2d\mu \\ & = \lambda^2 + (\lambda - \mu)^2 + \mu^2 + \mu^2 \\ & = c\lambda - d\mu \end{aligned}$$

at $(\lambda, \mu)$.

LEMMA 3.2. *The system* $(\mathrm{EC}_2)$ *has indeed a unique solution:*

$$(x, y) = \frac{1}{5}(2c + d, c + 3d) \tag{4}$$

$$(\lambda, \mu) = \frac{1}{5}(3c - d, c - 2d). \tag{5}$$

PROOF. From $(EC_2)$, we have a pair of linear systems of 2-variable on 2-equation:

$$(EQ_2)\quad \begin{matrix} 3x - y = c & 2\lambda - \mu = c \\ -x + 2y = d & -\lambda + 3\mu = -d. \end{matrix}$$

The left system has a solution (4), while the right has a solution (5). □

The primal $(P_2)$ has a minimum value

$$m_2 = c(c - \hat{x}) - d(\hat{y} - d) = \frac{1}{5}(3c^2 - 2cd + 2d^2)$$

at a path

$$(\hat{x}, \hat{y}) = \frac{1}{5}(2c + d, c + 3d).$$

The dual $(D_2)$ has a maximum value

$$M_2 = c\lambda^* - d\mu^* = \frac{1}{5}(3c^2 - 2cd + 2d^2)$$

at a path

$$(\lambda^*, \mu^*) = \frac{1}{5}(3c - d, c - 2d).$$

### 3.3. $(P_3)$ vs $(D_3)$

Let us consider the third pair. Then $(P_3)$ and $(D_3)$ are dual to each other. It holds that

$$\begin{aligned} &2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + 2\nu^2] - 2d\nu \\ &\quad \le (c - x)^2 + x^2 + (x - y)^2 + y^2 + (y - z)^2 + (z - d)^2 \end{aligned}$$

for any feasible pair $(x, y, z, \lambda, \mu, \nu)$. An equality condition is

$$(EC_3)\quad \begin{matrix} c - x = \lambda, & x = \lambda - \mu \\ x - y = \mu, & y = \mu - \nu \\ y - z = \nu, & z - d = \nu. \end{matrix}$$

The equality condition $(EC_3)$ is a linear system of 6-equation on 6-variable.

Let $(x, y, z, \lambda, \mu, \nu)$ be a solution of $(EC_3)$. Then both sides become a common value with five expressions:

$$\begin{aligned} & (c-x)^2+x^2+(x-y)^2+y^2+(y-z)^2+(z-d)^2 \\ & = c(c-x)-d(z-d) \\ (5\mathrm{V}_3) \quad & = 2c\lambda-[\lambda^2+(\lambda-\mu)^2+\mu^2+(\mu-\nu)^2+2\nu^2]-2d\nu \\ & = \lambda^2+(\lambda-\mu)^2+\mu^2+(\mu-\nu)^2+2\nu^2 \\ & = c\lambda-d\nu. \end{aligned}$$

The primal ($\mathrm{P}_3$) has a minimum value

$$\begin{aligned} m_3 &= (c-x)^2+x^2+(x-y)^2+y^2+(y-z)^2+(z-d)^2 \\ &= c(c-x)-d(z-d) \end{aligned}$$

at $(x, y, z)$, while the dual ($\mathrm{D}_3$) has a maximum value

$$\begin{aligned} M_3 &= 2c\lambda-[\lambda^2+(\lambda-\mu)^2+\mu^2+(\mu-\nu)^2+2\nu^2]-2d\nu \\ &= \lambda^2+(\lambda-\mu)^2+\mu^2+(\mu-\nu)^2+2\nu^2 \\ &= c\lambda-d\nu \end{aligned}$$

at $(\lambda, \mu, \nu)$.

LEMMA 3.3. *The system* ($\mathrm{EC}_3$) *has indeed a unique solution:*

$$(x, y, z) = \frac{1}{13}(5c+d, 2c+3d, c+8d) \tag{6}$$

$$(\lambda, \mu, \nu) = \frac{1}{13}(8c-d, 3c-2d, c-5d). \tag{7}$$

PROOF. From ($\mathrm{EC}_3$), we have a pair of linear systems of 3-variable on 3-equation:

$$(\mathrm{EQ}_3) \quad \begin{aligned} 3x-y&=c \\ -x+3y-z&=0 \\ -y+2z&=d \end{aligned} \qquad \begin{aligned} 2\lambda-\mu&=c \\ -\lambda+3\mu-\nu&=0 \\ -\mu+3\nu&=-d. \end{aligned}$$

The left system has a solution (6), while the right has a solution (7). □

The primal ($\mathrm{P}_3$) has a minimum value

$$m_3 = c(c-\hat{x})-d(\hat{z}-d) = \frac{1}{13}(8c^2-2cd+5d^2)$$

at a path

$$(\hat{x}, \hat{y}, \hat{z}) = \frac{1}{13}(5c + d, 2c + 3d, c + 8d).$$

The dual ($D_3$) has a maximum value

$$M_3 = c\lambda^* - d\nu^* = \frac{1}{13}(8c^2 - 2cd + 5d^2)$$

at a path

$$(\lambda^*, \mu^*, \nu^*) = \frac{1}{13}(8c - d, 3c - 2d, c - 5d).$$

Here we note that the first seven *Fibonacci numbers* appear:

1, 1, 2, 3, 5, 8, 13.

The *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation

$$\text{(Fibo):} \quad x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1, \; x_0 = 0.$$

| $n$ | $\cdots$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | $\cdots$ | $-1$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | $\cdots$ |

Table 1. Fibonacci sequence $\{F_n\}$

## 4. ($P_n$) vs ($D_n$)

Let us now consider the $n$-variable pair, where $n \geq 2$. First we present the $n$-th complementary identity, which takes a fundamental role in analyzing the pair of primal and dual. Let $x = \{x_k\}_0^{n+1}$, $\mu = \{\mu_k\}_1^n$ be a pair of sequences of real number with $x_0 = c$, $x_{n+1} = d$. Then an identity

$$(C_n) \quad c\mu_1 - d\mu_n = \sum_{k=1}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + (x_n - x_{n+1})\mu_n$$

holds true. This identity yields a pair of minimization problem

$$(P_n) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}[(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + (x_n - x_{n+1})^2 \\ \text{subject to} & \text{(i)} \quad x \in R^n, \qquad \text{(ii)} \quad x_0 = c, \; x_{n+1} = d \end{array}$$

and a maximization problem

$$(\mathrm{D}_n)\quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \mu_n^2 - 2d\mu_n \\ \text{subject to} & \text{(i)} \quad \mu \in R^n. \end{array}$$

Then both are dual to each other. It holds that

$$\begin{aligned} &2c\mu_1 - \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \mu_n^2 - 2d\mu_n \\ &\quad \le (c - x_1)^2 + x_1^2 + \sum_{k=2}^{n-1}[(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + (x_n - d)^2 \end{aligned}$$

for any feasible pair $(x, \mu)$. An equality condition is

$$(\mathrm{EC}_n)\quad \begin{array}{lll} c - x_1 = \mu_1, & x_1 = \mu_1 - \mu_2 & \\ x_{k-1} - x_k = \mu_k, & x_k = \mu_k - \mu_{k+1} & 2 \le k \le n-1 \\ x_{n-1} - x_n = \mu_n, & x_n - d = \mu_n. & \end{array}$$

The equality condition $(\mathrm{EC}_n)$ is a linear system of $2n$-equation on $2n$-variable.

Let $(x, \mu)$ be a solution of $(\mathrm{EC}_n)$. Then both sides become a common value with five expressions:

$$\begin{aligned} &(c - x_1)^2 + x_1^2 + \sum_{k=2}^{n-1}[(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + (x_n - d)^2 \\ &= c(c - x_1) - d(x_n - d) \\ (5\mathrm{V}_n)\quad &= 2c\mu_1 - \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - 2\mu_n^2 - 2d\mu_n \\ &= \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] + 2\mu_n^2 \\ &= c\mu_1 - d\mu_n. \end{aligned}$$

The primal $(\mathrm{P}_n)$ has a minimum value

$$\begin{aligned} m_n &= (c - x_1)^2 + x_1^2 + \sum_{k=2}^{n-1}[(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + (x_n - d)^2 \\ &= c(c - x_1) - d(x_n - d) \end{aligned}$$

at $x$, while the dual $(D_n)$ has a maximum value

$$\begin{aligned} M_n &= 2c\mu_1 - \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - 2\mu_n^2 - 2d\mu_n \\ &= \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] + 2\mu_n^2 \\ &= c\mu_1 - d\mu_n \end{aligned}$$

at $\mu$.

LEMMA 4.1. *The system* $(EC_n)$ *has indeed a unique solution:*

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_{n-2} \\ \hat{x}_{n-1} \\ \hat{x}_n \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n-1}c + F_2 d \\ F_{2n-3}c + F_4 d \\ F_{2n-5}c + F_6 d \\ \vdots \\ F_5 c + F_{2n-4}d \\ F_3 c + F_{2n-2}d \\ F_1 c + F_{2n}d \end{pmatrix} \tag{8}$$

$$\mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \\ \vdots \\ \mu_{n-2}^* \\ \mu_{n-1}^* \\ \mu_n^* \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n}c - F_1 d \\ F_{2n-2}c - F_3 d \\ F_{2n-4}c - F_5 d \\ \vdots \\ F_6 c - F_{2n-5}d \\ F_4 c - F_{2n-3}d \\ F_2 c - F_{2n-1}d \end{pmatrix}. \tag{9}$$

PROOF. From $(EC_n)$, we have a pair of linear systems of $n$-variable on $n$-equation:

$$(EQ_n) \qquad \begin{aligned} 3x_1 - x_2 &= c & 2\mu_1 - \mu_2 &= c \\ -x_1 + 3x_2 - x_3 &= 0 & -\mu_1 + 3\mu_2 - \mu_3 &= \mu_4 \\ &\vdots & &\vdots \\ -x_{n-2} + 3x_{2n-1} - x_n &= 0 & -\mu_{n-2} + 3\mu_{n-1} - \mu_n &= 0 \\ -x_{n-1} + 2x_n &= d & -\mu_{n-1} + 3\mu_n &= -d. \end{aligned}$$

The left system has a solution $\hat{x}$ in (8), while the right has a solution $\mu^*$ in (9).

In fact, the left system is written as

$$Ax = b$$

where $x$, $b$ are $n$-vectors:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix}, \qquad b = \begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ d \end{pmatrix}$$

and $A$ is an $n \times n$-matrix:

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Further $A$ has the inverse

$$A^{-1} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n-1} & F_{2n-3} & F_{2n-5} & F_{2n-7} & \cdots & F_7 & F_5 & F_3 & F_2 \\ F_{2n-3} & 3F_{2n-3} & 3F_{2n-5} & 3F_{2n-7} & \cdots & 3F_7 & 3F_5 & 2F_4 & F_4 \\ F_{2n-5} & 3F_{2n-5} & 8F_{2n-5} & 8F_{2n-7} & \cdots & 8F_7 & 8F_5 & 2F_6 & F_6 \\ F_{2n-7} & 3F_{2n-7} & 8F_{2n-7} & 21F_{2n-7} & \cdots & 13F_8 & 5F_8 & 2F_8 & F_8 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_7 & 3F_7 & 8F_7 & 13F_8 & \cdots & 13F_{2n-6} & 5F_{2n-6} & 2F_{2n-6} & F_{2n-6} \\ F_5 & 3F_5 & 8F_5 & 5F_8 & \cdots & 5F_{2n-6} & 5F_{2n-4} & 2F_{2n-4} & F_{2n-4} \\ F_3 & 2F_4 & 2F_6 & 2F_8 & \cdots & 2F_{2n-6} & 2F_{2n-4} & 2F_{2n-2} & F_{2n-2} \\ F_1 & F_4 & F_6 & F_8 & \cdots & F_{2n-6} & F_{2n-4} & F_{2n-2} & F_{2n} \end{pmatrix}.$$

Thus a unique solution $x = A^{-1}b$ is specified in (8).

On the other hand, the right is

$$B\mu = f$$

where $\mu$, $f$ are $n$-vectors:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{n-2} \\ \mu_{n-1} \\ \mu_n \end{pmatrix}, \qquad f = \begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -d \end{pmatrix}$$

and $B$ is an $n \times n$-matrix:

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 3 \end{pmatrix}.$$

Further $B$ has the inverse

$$B^{-1} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n} & F_{2n-2} & F_{2n-4} & F_{2n-6} & \cdots & F_8 & F_6 & F_4 & F_1 \\ F_{2n-2} & 2F_{2n-4} & 2F_{2n-4} & 2F_{2n-6} & \cdots & 2F_8 & 2F_6 & 2F_4 & F_3 \\ F_{2n-4} & 2F_{2n-4} & 5F_{2n-4} & 5F_{2n-6} & \cdots & 21F_5 & 8F_5 & 3F_5 & F_5 \\ F_{2n-6} & 2F_{2n-6} & 5F_{2n-6} & 13F_{2n-6} & \cdots & 21F_7 & 8F_7 & 3F_7 & F_7 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_8 & 2F_8 & 21F_5 & 21F_7 & \cdots & 21F_{2n-7} & 8F_{2n-7} & 3F_{2n-5} & F_{2n-7} \\ F_6 & 2F_6 & 8F_5 & 8F_7 & \cdots & 8F_{2n-7} & 8F_{2n-5} & 3F_{2n-5} & F_{2n-5} \\ F_4 & 2F_4 & 3F_5 & 3F_7 & \cdots & 3F_{2n-7} & 3F_{2n-5} & 3F_{2n-3} & F_{2n-3} \\ F_2 & F_3 & F_5 & F_7 & \cdots & F_{2n-7} & F_{2n-5} & F_{2n-3} & F_{2n-1} \end{pmatrix}.$$

Thus a unique solution $\mu = B^{-1}f$ is specified in (9). □

LEMMA 4.2. *The primal* ($\mathrm{P}_n$) *has a minimum value*

$$m_n = c(c - \hat{x}_1) - d(\hat{x}_n - d) = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$$

*at the path $\hat{x}$. The dual* ($D_n$) *has a maximum value*

$$M_n = c\mu_1^* - d\mu_n^* = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$$

*at the path $\mu^*$.*

## 5. Duality

Now we show that ($P_n$) and ($D_n$) are dual to each other. It turns out that the duality is based upon the complementary identity and an elementary inequality (EI).

Let $x = \{x_k\}_0^{n+1}$, $\mu = \{\mu_k\}_1^n$ be a pair of sequences of real number with $x_0 = c$, $x_{n+1} = d$. Then an identity ($C_n$) holds true. We make it double. Then an identity

$$2c\mu_1 - 2d\mu_n = \sum_{k=1}^{n-1}[2(x_{k-1} - x_k)\mu_k + 2x_k(\mu_k - \mu_{k+1})] + 2(x_{n-1} - x_n)\mu_n + 2(x_n - x_{n+1})\mu_n$$

with the elementary inequality (EI) yields

$$\begin{aligned} 2c\mu_1 - 2d\mu_n \le & \sum_{k=1}^{n-1}[(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + (x_n - x_{n+1})^2 \\ & + \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_n^2 + \mu_n^2. \end{aligned}$$

Thus we have an inequality

$$\begin{aligned} & 2c\mu_1 - \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \mu_n^2 - 2d\mu_n \\ & \quad \le \sum_{k=1}^{n-1}[(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + (x_n - x_{n+1})^2 \end{aligned}$$

for any feasible pair $(x;\mu)$. The sign of equality holds iff

$$(\mathrm{EC}_n) \quad \begin{array}{lll} c - x_1 = \mu_1, & x_1 = \mu_1 - \mu_2 & \\ x_{k-1} - x_k = \mu_k, & x_k = \mu_k - \mu_{k+1} & 2 \le k \le n-1 \\ x_{n-1} - x_n = \mu_n, & x_n - d = \mu_n. & \end{array}$$

Thus we have a pair of minimization problem ($P_n$) and maximization problem ($D_n$). Hence both are dual to each other.

Similarly it is shown that ($P_1$) and ($D_1$) are dual to each other.

## 6. Conjugate dual

Let $f$ be a differentiable convex function on $R^1$. Then a *conjugate function* $f^*$ is defined by

(10) $$f^*(\lambda) = \underset{x \in R^1}{\mathrm{Max}}[\lambda x - f(x)] \qquad \lambda \in R^1.$$

Then it holds that

(11) $$\lambda x \leq f(x) + f^*(\lambda) \qquad (x, \lambda) \in R^2.$$

The sign of equality holds iff

(12) $$f'(x) = \lambda.$$

In the following, we assume that three convex functions $f$, $g$, $h$ are given. We show that three identities generate their respective pairs of primal and dual with equality condition.

The first identity

$$(\mathrm{C}_1) \quad c\lambda - d\lambda = (c - x)\lambda + (x - d)\lambda$$

yields a pair

$$(\mathrm{CP}_1) \quad \begin{array}{ll} \text{minimize} & f(c - x) + h(x - d) \\ \text{subject to} & \text{(i)} \quad x \in R^1 \end{array}$$

$$(\mathrm{CD}_1) \quad \begin{array}{ll} \text{Maximize} & c\lambda - [f^*(\lambda) + h^*(\lambda)] - d\lambda \\ \text{subject to} & \text{(i)} \quad \lambda \in R^1. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_1) \quad f'(c - x) = \lambda, \qquad h'(x - d) = \lambda.$$

The second identity

$$(\mathrm{C}_2) \quad c\lambda - d\mu = (c - x)\lambda + x(\lambda - \mu) + (x - y)\mu + (y - d)\mu$$

yields a pair

$$(\mathrm{CP}_2) \quad \begin{array}{ll} \text{minimize} & f(c - x) + g(x) + f(x - y) + h(y - d) \\ \text{subject to} & \text{(i)} \quad (x, y) \in R^2 \end{array}$$

$$(\mathrm{CD}_2) \quad \begin{array}{ll} \text{Maximize} & c\lambda - [f^*(\lambda) + g^*(\lambda - \mu) + f^*(\mu) + h^*(\mu)] - d\mu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu) \in R^2. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_2)\quad \begin{array}{ll} f'(c-x)=\lambda, & g'(x)=\lambda-\mu \\ f'(x-y)=\mu, & h'(y-d)=\nu. \end{array}$$

The third identity

$$(\mathrm{C}_3)\quad c\lambda - d\nu = (c-x)\lambda + x(\lambda-\mu) + (x-y)\mu + y(\mu-\nu) + (y-z)\nu + (z-d)\nu$$

yields a pair

$$(\mathrm{CP}_3)\quad \begin{array}{ll} \text{minimize} & f(c-x)+g(x)+f(x-y)+g(y)+f(y-z)+h(z-d) \\ \text{subject to} & \text{(i)}\quad (x,y,z)\in R^3 \end{array}$$

$$(\mathrm{CD}_3)\quad \begin{array}{ll} \text{Maximize} & c\lambda - [f^*(\lambda)+g^*(\lambda-\mu)+f^*(\mu)+g^*(\mu-\nu)+f^*(\nu)+h^*(\nu)] - d\nu \\ \text{subject to} & \text{(i)}\quad (\lambda,\mu,\nu)\in R^3. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_3)\quad \begin{array}{ll} f'(c-x)=\lambda, & g'(x)=\lambda-\mu \\ f'(x-y)=\mu, & g'(y)=\mu-\nu \\ f'(y-z)=\nu, & h'(z-d)=\nu. \end{array}$$

Finally the $n$-th identity

$$(\mathrm{C}_n)\quad c\mu_1 - d\mu_n = \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + (x_n-x_{n+1})\mu_n$$

yields a pair

$$(\mathrm{CP}_n)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}[f(x_{k-1}-x_k)+g(x_k)] + f(x_{n-1}-x_n) + h(x_n-x_{n+1}) \\ \text{subject to} & \text{(i)}\quad x\in R^n, \qquad \text{(ii)}\quad x_0=c,\ x_{n+1}=d \end{array}$$

$$(\mathrm{CD}_n)\quad \begin{array}{ll} \text{Maximize} & c\mu_1 - \displaystyle\sum_{k=1}^{n-1}[f^*(\mu_k)+g^*(\mu_k-\mu_{k+1})] - f^*(\mu_n) - h^*(\mu_n) - d\mu_n \\ \text{subject to} & \text{(i)}\quad \mu\in R^n. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_n) \quad \begin{array}{lll} f'(c-x_1)=\mu_1, & g'(x_1)=\mu_1-\mu_2 & \\ f'(x_{k-1}-x_k)=\mu_k, & g'(x_k)=\mu_k-\mu_{k+1} & 2\le k\le n-1 \\ f'(x_{n-1}-x_n)=\mu_n, & h'(x_n-d)=\mu_n. & \end{array}$$

## 7. Dynamic programming

We show that dynamic programming solves both primal and dual.

### 7.1. Three-variable pair

We consider the minimization problem ($\mathrm{CP}_3$) and the maximization problem ($\mathrm{CD}_3$).

#### 7.1.1. Primal ($\mathrm{CP}_3$)

Let $U$ be the minimum value of ($\mathrm{CP}_3$). Let $u_1(x)$ be the minimum value of two-variable subproblem:

$$(\mathrm{SP}_2) \quad \begin{array}{ll} \text{minimize} & f(x-y)+g(y)+f(y-z)+h(z-d) \\ \text{subject to} & \text{(i)} \quad (y,z)\in R^2. \end{array}$$

Let $u_2(y)$ be the minimum value of one-variable subproblem:

$$(\mathrm{SP}_1) \quad \begin{array}{ll} \text{minimize} & f(y-z)+h(z-d) \\ \text{subject to} & \text{(i)} \quad z\in R^3. \end{array}$$

Finally let $u_3(z) := h(z-d)$. Then we have a recursive formula

$$\begin{aligned} u_3(z) &= h(z-d) \\ u_2(y) &= \min_{z\in R^1}[f(y-z)+u_3(z)] \\ u_1(x) &= \min_{y\in R^1}[f(x-y)+g(y)+u_2(y)] \\ U &= \min_{x\in R^1}[f(c-x)+g(x)+u_1(x)]. \end{aligned}$$

Now let us solve the forementioned problem:

$$(\mathrm{P}_3) \quad \begin{array}{ll} \text{minimize} & (c-x)^2+x^2+(x-y)^2+y^2+(y-z)^2+(z-d)^2 \\ \text{subject to} & \text{(i)} \quad (x,y,z)\in R^3. \end{array}$$

Then the recursive formula

$$u_3(z) = (z - d)^2$$

$$u_2(y) = \min_{z \in R^1} [(y - z)^2 + u_3(z)]$$

$$u_1(x) = \min_{y \in R^1} [(x - y)^2 + y^2 + u_2(y)]$$

$$U = \min_{x \in R^1} [(c - x)^2 + x^2 + u_1(x)]$$

has a solution

$$u_3(z) = (z^2 - 2zd + d^2)$$

$$u_2(y) = \frac{1}{2}(y^2 - 2yd + d^2), \qquad \hat{z}(y) = \frac{1}{2}(y + d)$$

$$u_1(x) = \frac{1}{5}(3x^2 - 2xd + 2d^2), \qquad \hat{y}(x) = \frac{1}{5}(2x + d)$$

$$U = \frac{1}{13}(8c^2 - 2cd + 5d^2), \qquad \hat{x}(c) = \frac{1}{13}(5c + d).$$

Thus we get a minimum point $(\hat{x}, \hat{y}, \hat{z})$, where

$$\hat{x} = \hat{x}(c) = \frac{1}{13}(5c + d)$$

$$\hat{y} = \hat{y}(\hat{x}) = \frac{1}{5}(2\hat{x} + d) = \frac{1}{13}(2c + 3d)$$

$$\hat{z} = \hat{z}(\hat{y}) = \frac{1}{2}(\hat{y} + d) = \frac{1}{13}(c + 8d).$$

Hence $(P_3)$ attains a minimum $U = \frac{1}{13}(8c^2 - 2cd + 5d^2)$ at a point

$$(\hat{x}, \hat{y}, \hat{z}) = \frac{1}{13}(5c + d, 2c + 3d, c + 8d).$$

See (6).

### 7.1.2. **Dual** $(CD_3)$

Let $V$ be the maximum value of $(CD_3)$. Let $v_1(\lambda)$ be the minimum value of two-variable subproblem:

$$(SD_2) \quad \begin{array}{ll} \text{minimize} & [f^*(\lambda) + g^*(\lambda - \mu) + f^*(\mu) + g^*(\mu - \nu) + f^*(\nu) + h^*(\nu)] + d\nu \\ \text{subject to} & \text{(i)} \quad (\mu, \nu) \in R^2. \end{array}$$

Let $v_2(\mu)$ be the minimum value of one-variable subproblem:

$$(\mathrm{SD}_1)\quad \begin{array}{ll} \text{minimize} & [f^*(\mu) + g^*(\mu - \nu) + f^*(\nu) + h^*(\nu)] + d\nu \\ \text{subject to} & \text{(i)} \quad \nu \in R^1. \end{array}$$

Finally let $v_3(\nu) := f^*(\nu) + h^*(\nu) + d\nu$. Then we have a recursive formula

$$\begin{aligned} v_3(\nu) &= f^*(\nu) + h^*(\nu) + d\nu \\ v_2(\mu) &= \min_{\nu \in R^1}[f^*(\mu) + g^*(\mu - \nu) + v_3(\nu)] \\ v_1(\lambda) &= \min_{\mu \in R^1}[f^*(\lambda) + g^*(\lambda - \mu) + v_2(\mu)] \\ V &= \operatorname*{Max}_{\lambda \in R^1}[c\lambda - v_1(\lambda)]. \end{aligned}$$

Now let us solve the forementioned problem:

$$(\mathrm{D}_3)\quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2 + \nu^2] - 2d\nu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu, \nu) \in R^3. \end{array}$$

Then the recursive formula

$$\begin{aligned} v_3(\nu) &= 2\nu^2 + 2d\nu \\ v_2(\mu) &= \min_{\nu \in R^1}[\mu^2 + (\mu - \nu)^2 + v_3(\nu)] \\ v_1(\lambda) &= \min_{\mu \in R^1}[\lambda^2 + (\lambda - \mu)^2 + v_2(\mu)] \\ V &= \operatorname*{Max}_{\lambda \in R^1}[2c\lambda - v_1(\lambda)] \end{aligned}$$

has a solution

$$\begin{aligned} v_3(\nu) &= 2\nu^2 + 2d\nu \\ v_2(\mu) &= \frac{1}{3}(5\mu^2 + 2\mu d - d^2), \qquad \nu^*(\mu) = \frac{1}{3}(\mu - d) \\ v_1(\lambda) &= \frac{1}{8}(13\lambda^2 + 2\lambda - 3d^2), \qquad \mu^*(\lambda) = \frac{1}{8}(3\lambda - d) \\ V &= \frac{1}{13}(8c^2 - 2cd + 5d^2), \qquad \lambda^*(c) = \frac{1}{13}(8c - d). \end{aligned}$$

Thus we get a maximum point $(\lambda^*, \mu^*, \nu^*)$, where

$$\lambda^* = \lambda^*(c) = \frac{1}{13}(8c - d)$$

$$\mu^* = \mu^*(\lambda^*) = \frac{1}{8}(3\lambda^* - d) = \frac{1}{13}(3c - 2d)$$

$$\nu^* = \nu^*(\mu^*) = \frac{1}{3}(\mu^* - d) = \frac{1}{13}(c - 5d).$$

Hence $(D_3)$ attains a maximum $V = \frac{1}{13}(8c^2 - 2cd + 5d^2)$ at a point

$$(\lambda^*, \mu^*, \nu^*) = \frac{1}{13}(8c - d, 3c - 2d, c - 5d).$$

See also (7).

### 7.2. *n*-variable pair

We consider the minimization problem $(CP_n)$ and the maximization problem $(CD_n)$.

#### 7.2.1. Primal $(CP_n)$

Let $U$ be the minimum value of $(CP_n)$. Let $u_k(x_k)$ be the minimum value of $(n-k)$-variable subproblem:

$$(SP_{n-k}) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{l=k+1}^{n-1} [f(x_{l-1} - x_l) + g(x_l)] + f(x_{n-1} - x_n) + h(x_n - x_{n+1}) \\ \text{subject to} & \text{(i)} \quad (x_{k+1}, x_{k+2}, \dots, x_n) \in R^{n-k}, \qquad \text{(ii)} \quad x_{n+1} = d \end{array}$$

where $1 \le k \le n-1$. Finally let $u_n(x_n) := h(x_n - d)$. Then we have a recursive formula

$$u_n(x_n) = h(x_n - d)$$

$$u_k(x_k) = \min_{x_{k+1} \in R^1} [f(x_k - x_{k+1}) + g(x_{k+1}) + u_{k+1}(x_{k+1})] \qquad 1 \le k \le n-1$$

$$U = \min_{x_1 \in R^1} [f(c - x_1) + g(x_1) + u_1(x_1)].$$

Let $\hat{x}_{k+1}(x_k)$ be a minimizer. Then a sequence $\{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n\}$ is called a *policy*.

Now let us solve the forementioned problem:

$$(\mathrm{P}_n) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}[(x_{k-1}-x_k)^2+x_k^2]+(x_{n-1}-x_n)^2+(x_n-x_{n+1})^2 \\ \text{subject to} & \text{(i)} \quad x \in R^n, \qquad \text{(ii)} \quad x_0=c,\ x_{n+1}=d. \end{array}$$

Then the recursive formula

$$\begin{aligned} u_n(x_n) &= (x_n-d)^2 \\ u_k(x_k) &= \min_{x_{k+1}\in R^1}[(x_k-x_{k+1})^2+x_{k+1}^2+u_{k+1}(x_{k+1})] \qquad 1 \le k \le n-1 \\ U &= \min_{x\in R^1}[(c-x_1)^2+x_1^2+u_1(x_1)]. \end{aligned}$$

has a solution

$$\begin{aligned} u_n(x_n) &= x_n^2-2x_nd+d^2 \\ u_k(x_k) &= \frac{1}{F_{2n-2k+1}}(F_{2n-2k}x_k^2-2x_kd+F_{2n-2k-1}d^2), \\ & \qquad \hat{x}_{k+1}(x_k) = \frac{1}{F_{2n-2k+1}}(F_{2n-2k-1}x_k+d) \qquad 1 \le k \le n-1, \\ U &= \frac{1}{F_{2n+1}}(F_{2n}c^2-2cd+F_{2n-1}d^2), \qquad \hat{x}_1(c) = \frac{1}{F_{2n+1}}(F_{2n-1}c+d). \end{aligned}$$

Thus we get a minimum point $\hat{x}=(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_k,\ldots,\hat{x}_n)$ as follows:

$$\begin{aligned} \hat{x}_1 &= \hat{x}_1(c) = \frac{1}{F_{2n+1}}(F_{2n-1}c+F_2d) \\ \hat{x}_2 &= \hat{x}_2(\hat{x}_1) = \frac{1}{F_{2n-1}}(F_{2n-3}\hat{x}_1+d) = \frac{1}{F_{2n+1}}(F_{2n-3}c+F_4d) \\ &\vdots \\ \hat{x}_{k+1} &= \hat{x}_{k+1}(\hat{x}_k) = \frac{1}{F_{2n-2k+1}}(F_{2n-2k-1}\hat{x}_k+d) = \frac{1}{F_{2n+1}}(F_{2n-2k-1}c+F_{2k+2}d) \\ &\vdots \\ \hat{x}_n &= \hat{x}_n(\hat{x}_{n-1}) = \frac{1}{F_3}(F_1\hat{x}_{n-1}+d) = \frac{1}{F_{2n+1}}(F_1c+F_{2n}d). \end{aligned}$$

Hence $(\mathrm{P}_n)$ attains a minimum $U=\dfrac{1}{F_{2n+1}}(F_{2n}c^2-2cd+F_{2n-1}d^2)$ at a point

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_{n-2} \\ \hat{x}_{n-1} \\ \hat{x}_n \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n-1}c + F_2 d \\ F_{2n-3}c + F_4 d \\ F_{2n-5}c + F_6 d \\ \vdots \\ F_5 c + F_{2n-4} d \\ F_3 c + F_{2n-2} d \\ F_1 c + F_{2n} d \end{pmatrix}.$$

See (8).

**7.2.2. Dual** $(\mathrm{CD}_n)$

Let $V$ be the maximum value of $(\mathrm{CD}_n)$. Let $v_k(\mu_k)$ be the minimum value of $(n-k)$-variable subproblem:

$$(\mathrm{SD}_{n-k}) \quad \begin{array}{ll} \text{minimize} & \sum_{l=k}^{n-1}[f^*(\mu_l) + g^*(\mu_l - \mu_{l+1})] + f^*(\mu_n) + h^*(\mu_n) + d\mu_n \\ \text{subject to} & \text{(i)} \quad (\mu_{k+1}, \mu_{k+2}, \dots, \mu_n) \in R^{n-k} \end{array}$$

where $1 \le k \le n-1$. Finally let $v_n(\mu_n) := f^*(\mu_n) + h^*(\mu_n) + d\mu_n$. Then we have a recursive formula

$$v_n(\mu_n) = f^*(\mu_n) + h^*(\mu_n) + d\mu_n$$

$$v_k(\mu_k) = \min_{\mu_{k+1} \in R^1}[f^*(\mu_k) + g^*(\mu_k - \mu_{k+1}) + v_{k+1}(\mu_{k+1})] \qquad 1 \le k \le n-1$$

$$V = \operatorname*{Max}_{\mu_1 \in R^1}[c\mu_1 - v_1(\mu_1)].$$

Let $\mu_{k+1}^*(\mu_k)$ be a minimizer and $\mu_1^*(c)$ be a maximizer. Then a sequence $\{\mu_1^*, \mu_2^*, \dots, \mu_n^*\}$ is called a *policy.*

Now let us solve the forementioned problem:

$$(\mathrm{D}_n) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \mu_n^2 - 2d\mu_n \\ \text{subject to} & \text{(i)} \quad \mu \in R^n. \end{array}$$

Then the recursive formula

$$v_n(\mu_n) = 2\mu_n^2 + d\mu_n$$

$$v_k(\mu_k) = \min_{\mu_{k+1} \in R^1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2 + v_{k+1}(\mu_{k+1})] \qquad 1 \le k \le n-1$$

$$V = \operatorname*{Max}_{\mu_1 \in R^1}[c\mu_1 - v_1(\mu_1)]$$

has a solution

$$v_n(\mu_n) = 2\mu_n^2 + 2d\mu_n$$

$$v_k(\mu_k) = \frac{1}{F_{2n-2k+2}}(F_{2n-2k+3}\mu_k^2 + 2\mu_k d - F_{2n-2k}d^2),$$

$$\mu_{k+1}^*(\mu_k) = \frac{1}{F_{2n-2k+2}}(F_{2n-2k}\mu_k - d) \qquad 1 \le k \le n-1,$$

$$V = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2), \qquad \mu_1^*(c) = \frac{1}{F_{2n+1}}(F_{2n}c - F_1 d).$$

Thus we get a maximum point $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_k^*, \ldots, \mu_n^*)$, where

$$\mu_1^* = \mu_1^*(c) = \frac{1}{F_{2n+1}}(F_{2n}c - F_1 d)$$

$$\mu_2^* = \mu_2^*(\mu_1^*) = \frac{1}{F_{2n}}(F_{2n-2}\mu_1^* - d) = \frac{1}{F_{2n+1}}(F_{2n-2}c - F_3 d)$$

$$\vdots$$

$$\mu_{k+1}^* = \mu_{k+1}^*(\mu_k^*) = \frac{1}{F_{2n-2k+2}}(F_{2n-2k}\mu_k^* - d) = \frac{1}{F_{2n+1}}(F_{2n-2k}c - F_{2k+1}d)$$

$$\vdots$$

$$\mu_n^* = \mu_n^*(\mu_{n-1}^*) = \frac{1}{F_4}(\mu_{n-1}^* - d) = \frac{1}{F_{2n+1}}(F_2 c - F_{2n-1}d).$$

Hence $(\mathrm{D}_n)$ attains a maximum $V = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$ at a point

$$\mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \\ \vdots \\ \mu_{n-2}^* \\ \mu_{n-1}^* \\ \mu_n^* \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n}c - F_1 d \\ F_{2n-2}c - F_3 d \\ F_{2n-4}c - F_5 d \\ \vdots \\ F_6 c - F_{2n-5}d \\ F_4 c - F_{2n-3}d \\ F_2 c - F_{2n-1}d \end{pmatrix}.$$

See also (9).

## 8. Identical dual

We have analyzed a pair of primal and dual from a complementary duality. Now we consider the pair from an identical duality.

**8.1.** (P$^*$) **vs** (D$^*$)

Let $x = \{x_k\}_0^{n+1}$, $\mu = \{\mu_k\}_1^n$ be a pair of sequences of real number with $x_0 = c$, $x_{n+1} = d$. Then a complementary identity

$$(\mathrm{CI}_n) \quad (c - x_1)\mu_1 + x_1(\mu_1 - \mu_2) + \sum_{k=2}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})]$$

$$+ (x_{n-1} - x_n)\mu_n + (x_n - d)\mu_n = c\mu_1 - d\mu_n$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n}$, $v = \{v_k\}_1^{2n}$ from $x = \{x_k\}_0^{n+1}$, $\mu = \{\mu_k\}_1^n$ through

$$(13) \quad \begin{aligned} &y_1 = c - x_1, \quad y_2 = x_1, \quad y_3 = x_1 - x_2, \quad y_4 = x_2, \quad y_5 = x_2 - x_3 \ldots, \\ &\qquad y_{2n-2} = x_{n-1}, \quad y_{2n-1} = x_{n-1} - x_n, \quad y_{2n} = x_n - d \\ &v_1 = \mu_1, \quad v_2 = \mu_1 - \mu_2, \quad v_3 = \mu_2, \quad v_4 = \mu_2 - \mu_3, \quad v_5 = \mu_3 \ldots, \\ &\qquad v_{2n-2} = \mu_{n-1} - \mu_n, \quad v_{2n-1} = \mu_n, \quad v_{2n} = \mu_n, \end{aligned}$$

respectively. Then an identity

$$(\mathrm{C}^*) \quad cv_1 - dv_{2n} = \sum_{k=1}^{2n} y_k v_k$$

holds *under a constraint*—a linear system of $2n$-equation on $4n$-variable $(y, v)$—:

$$(\mathrm{C}) \qquad \begin{array}{ll} c = y_1 + y_2 & v_1 = v_2 + v_3 \\ y_2 = y_3 + y_4 & v_3 = v_4 + v_5 \\ \vdots & \vdots \\ y_{2n-4} = y_{2n-3} + y_{2n-2} & v_{2n-3} = v_{2n-2} + v_{2n-1} \\ y_{2n-2} = y_{2n-1} + y_{2n} + d & v_{2n-1} = v_{2n}. \end{array}$$

An equality (C$^*$) with constraint (C) is called a $4n$-variable *conditional complementarity.* This is simply written as (C$^*$) under (C).

Now let $y = \{y_k\}_1^{2n}$, $v = \{v_k\}_1^{2n}$ satisfy (C). Then an elementary inequality with equality

$$2xy \leq x^2 + y^2 \qquad \text{on } R^2; \; x = y$$

yields

$$2cv_1 - 2dv_{2n} \le \sum_{k=1}^{2n}(y_k^2 + v_k^2).$$

Thus we have an inequality

$$2cv_1 - \sum_{k=1}^{2n} v_k^2 - 2dv_{2n} \le \sum_{k=1}^{2n} y_k^2.$$

The sign of equality holds iff

$$(\mathrm{EC}^*) \quad y_k = v_k \qquad 1 \le k \le 2n.$$

Hence we have a pair of conditional minimization problem:

$$(\mathrm{P}^*) \quad \begin{array}{lll} \text{minimize} & & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ \text{subject to} & (1) & y_1 + y_2 = c \\ & (2) & y_3 + y_4 = y_2 \\ & \vdots & \\ & (n-1) & y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & (n) & y_{2n-1} + y_{2n} + d = y_{2n-2} \\ & (n+1) & y \in R^{2n} \end{array}$$

and conditional maximization problem:

$$(\mathrm{D}^*) \quad \begin{array}{lll} \text{Maximize} & & 2cv_1 - (v_1^2 + v_2^2 + \cdots + v_{2n}^2) - 2dv_{2n} \\ \text{subject to} & [1] & v_2 + v_3 = v_1 \\ & [2] & v_4 + v_5 = v_3 \\ & \vdots & \\ & [n-1] & v_{2n-2} + v_{2n-1} = v_{2n-3} \\ & [n] & v_{2n} = v_{2n-1} \\ & [n+1] & v \in R^{2n}. \end{array}$$

Let (AC) be an *augmentation* of the system (C) with the additional equality condition ($\mathrm{EC}^*$):

$$
(\mathrm{AC})\quad
\begin{array}{rlrl}
c &= y_1 + y_2 & v_1 &= v_2 + v_3 \\
y_2 &= y_3 + y_4 & v_3 &= v_4 + v_5 \\
&\vdots & &\vdots \\
y_{2n-4} &= y_{2n-3} + y_{2n-2} & v_{2n-3} &= v_{2n-2} + v_{2n-1} \\
y_{2n-2} &= y_{2n-1} + y_{2n} + d & v_{2n-1} &= v_{2n} \\
\multicolumn{4}{c}{y_k = v_k \qquad 1 \le k \le 2n.}
\end{array}
$$

The linear system (AC) is of $4n$-equation on $4n$-variable.

Let $(y, v)$ satisfy (AC). Then both sides become a common value with five expressions:

$$
\begin{aligned}
& y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\
& = cy_1 - dy_{2n} \\
(5\mathrm{V}^*)\quad & = 2cv_1 - (v_1^2 + v_2^2 + \cdots + v_{2n}^2) - 2dv_{2n} \\
& = v_1^2 + v_2^2 + \cdots + v_{2n}^2 \\
& = cv_1 - dv_{2n}.
\end{aligned}
$$

The system (AC) has indeed a unique common solution $(\hat{y}, v^*)$:

$$
\hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \vdots \\ \hat{y}_{2k-1} \\ \hat{y}_{2k} \\ \vdots \\ \hat{y}_{2n-3} \\ \hat{y}_{2n-2} \\ \hat{y}_{2n-1} \\ \hat{y}_{2n} \end{pmatrix}
= v^* = \begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \\ v_4^* \\ \vdots \\ v_{2k-1}^* \\ v_{2k}^* \\ \vdots \\ v_{2n-3}^* \\ v_{2n-2}^* \\ v_{2n-1}^* \\ v_{2n}^* \end{pmatrix}
= \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n}c - F_1 d \\ F_{2n-1}c + F_2 d \\ F_{2n-2}c - F_3 d \\ F_{2n-3}c + F_4 d \\ \vdots \\ F_{2n-2k+2}c - F_{2k-1}d \\ F_{2n-2k+1}c + F_{2k}d \\ \vdots \\ F_4 c - F_{2n-3}d \\ F_3 c + F_{2n-2}d \\ F_2 c - F_{2n-1}d \\ F_1 c - F_{2n-1}d \end{pmatrix}.
$$

The primal $(\mathrm{P}^*)$ has a minimum value

$$
m = c\hat{y}_1 - d\hat{y}_{2n} = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)
$$

at a path $\hat{y}$, while the dual (D$^*$) has a maximum value

$$M = cv_1^* - dv_{2n}^* = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$$

at a path $v^*$.

Both optimal solutions (point and value) are identical:

$$\hat{x} = \mu^*, \qquad m = M.$$

Further both are Fibonacci:

$$\left.\begin{aligned} \hat{x}_{2k-1} = \mu^*_{2k-1} &= \frac{1}{F_{2n+1}}(F_{2n-2k+2}c - F_{2k-1}d) \\ \hat{x}_{2k} = \mu^*_{2k} &= \frac{1}{F_{2n+1}}(F_{2n-2k+1}c + F_{2k}d) \end{aligned}\right. \qquad 1 \le k \le n-1,$$

$$\hat{x}_{2n} = \mu^*_{2n} = \frac{1}{F_{2n+1}}(F_1 c - F_{2n-1}d),$$

$$m = M = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2).$$

Thus *Fibonacci Identical Duality* (FID) holds between (P$^*$) and (D$^*$).

We remark that the $2n$*-variable* pair is a transliteration from $n$*-variable* one of (P$_n$) and (D$_n$).

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