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SEVERAL COMPLETENESSES ON ν-GENERALIZED METRIC SPACES

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Abstract

We can consider several completenesses on ν-generalized metric spaces. In this paper, we study the relationship between the completenesses. In particular, we study κ-completeness.

1. Introduction

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers. Let X be a set. Then we denote by $\#X$ the cardinal number of X. We define a subset $X^{(k)}$ of X^k as follows: $(x_1, x_2, \ldots, x_k) \in X^{(k)}$ iff $(x_1, x_2, \ldots, x_k) \in X^k$ and $x_1, x_2, \ldots, x_k$ are all different. For $k, \ell \in \mathbf{N}$, we denote by $k \,\%\, \ell$ the remainder when k is divided by ℓ.

In 2000, Branciari introduced the following interesting concept, named ν-generalized metric space.

Definition 1 (Branciari [2]). Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbf{N}$. Then (X, d) is said to be a *ν-generalized metric space* if the following hold:

(N1) $d(x, y) = 0 \Leftrightarrow x = y$.

(N2) $d(x, y) = d(y, x)$.

(N3) $d(x, y) \le D(x, u_1, u_2, \ldots, u_\nu, y)$ for any $x, u_1, u_2, \ldots, u_\nu, y \in X$ such that $x, u_1, u_2, \ldots, u_\nu, y$ are all different, where

$$D(x, u_1, u_2, \ldots, u_\nu, y) = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y).$$

We have studied the topological structure on this concept. See [4, 5, 6, 8, 9, 10, 11, 12] and references therein. For example, we know the following:

- 1- and 3-generalized metric spaces have the compatible topology.
- For any $\nu \in \mathbf{N} \setminus \{1, 3\}$, there exists a ν-generalized metric space which does not have the compatible topology.
- Every ν-generalized metric space has the strongly compatible topology.
- Every ν-generalized metric space has a sequentially compatible topology.

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We can consider several completenesses on ν-generalized metric spaces. See Section 2. In this paper, we study the relationship between the completenesses. In particular, we study κ-completeness.

2. Preliminaries

In this section, we give some preliminaries.

DEFINITION 2 ([1, 2, 7, 11, 13]). Let (X, d) be a ν-generalized metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

- $\{x_n\}$ is said to be *Cauchy* if $\lim_n \sup_{m>n} d(x_m, x_n) = 0$ holds.
- $\{x_n\}$ is said to be *κ-Cauchy* if
$$\lim_{n\to\infty} \sup\{d(x_n, x_{n+1+j\kappa}) : j = 0, 1, 2, \ldots\} = 0$$
holds, where $\kappa \in \mathbf{N}$.
- $\{x_n\}$ is said to be *$\sum$-Cauchy* if
$$\sum_{j=1}^{\infty} d(x_j, x_{j+1}) < \infty$$
holds.
- $\{x_n\}$ is said to be *$(\sum, \neq)$-Cauchy* if x_n $(n \in \mathbf{N})$ are all different and $\{x_n\}$ is $\sum$-Cauchy.
- $\{x_n\}$ is said to *converge* to x if $\lim_n d(x_n, x) = 0$ holds.
- $\{x_n\}$ is said to *converge exclusively* to x if
$$\lim_{n\to\infty} d(x_n, x) = 0 \qquad \text{and} \qquad \liminf_{n\to\infty} d(x_n, y) > 0$$
hold for any $y \in X \setminus \{x\}$.

DEFINITION 3 ([1, 2, 7, 11]). Let (X, d) be a ν-generalized metric space. Let $\kappa \in \mathbf{N}$.

- X is said to be *complete* if every Cauchy sequence converges.
- X is *κ-complete* if every κ-Cauchy sequence converges.
- X is said to be *$\sum$-complete* if every $\sum$-Cauchy sequence converges.
- X is *$(\sum, \neq)$-complete* if every $(\sum, \neq)$-Cauchy sequence converges.

REMARK. X is complete iff X is 1-complete.

DEFINITION 4 (see Example 1.1 in [3]). Let (X, d) be a ν-generalized metric space. X is said to be *Hausdorff* if $\lim_n d(x_n, x) = \lim_n d(x_n, y) = 0$ implies $x = y$.

PROPOSITION 5 (Proposition 6 in [1]). *Let (X, d) be a ν-generalized metric space and let $\kappa, \lambda \in \mathbf{N}$ such that λ is divisible by κ. Then the following hold:*

(i) *Every κ-Cauchy sequence is λ-Cauchy.*
(ii) *If X is λ-complete, then X is κ-complete.*

Lemma 6 (Lemma 15 in [7]). *Let (X,d) be a 2-complete, ν-generalized metric space. Then X is Hausdorff.*

Lemma 7 (Proposition 5.4 in [11]). *Let (X,d) be a $\sum$-complete, ν-generalized metric space. Then X is Hausdorff.*

Lemma 8 (Lemma 11 in [7]). *Let (X,d) be a ν-generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X converging to some $z \in X$. Let $\{y_n\}$ be a sequence in X satisfying $\lim_n d(x_n, y_n) = 0$. Then $\{y_n\}$ also converges to z.*

Lemma 9 (Lemma 13 in [7]). *Let (X,d) be a $(\sum, \neq)$-complete, ν-generalized metric space. Then X is complete.*

Theorem 10 (Proposition 17 in [7]). *Let (X,d) be a ν-generalized metric space where ν is odd. Then the following are equivalent:*
- *X is complete.*
- *X is $(\sum, \neq)$-complete.*

Theorem 11 (Proposition 18 in [7]). *Let (X,d) be a ν-generalized metric space. Then the following are equivalent:*
- *X is 2-complete.*
- *X is $(\sum, \neq)$-complete and Hausdorff.*

3. Lemmas

In this section, we prove some lemmas.

Lemma 12. *Let (X,d) be a ν-generalized metric space. Let $\{a_n\}$ and $\{b_n\}$ be sequences in X satisfying*

$$\lim_{n\to\infty} \sup\{d(a_n, b_m) : m \geq n\} = 0. \tag{1}$$

Define two subsets of X by

$$A = \{x \in X : \#\{n \in \mathbf{N} : a_n = x\} = \infty\} \tag{2}$$

and

$$B = \{x \in X : \#\{n \in \mathbf{N} : b_n = x\} = \infty\}. \tag{3}$$

Then the following hold:
(i) $\lim_n d(a, b_n) = 0$ *holds for all $a \in A$.*
(ii) $\lim_n d(a_n, b) = 0$ *holds for all $b \in B$.*

(iii) *If* $A \neq \emptyset$ *and* $B \neq \emptyset$ *hold, then there exists* $z \in X$ *satisfying* $A = B = \{z\}$.

(iv) *If* $\#\{a_n : n \in \mathbf{N}\} < \infty$ *and* $\#\{b_n : n \in \mathbf{N}\} < \infty$ *hold, then there exist* $z \in X$ *and* $\mu \in \mathbf{N}$ *satisfying* $a_n = b_n = z$ *for all* $n \in \mathbf{N}$ *with* $n \geq \mu$.

Proof. We first show (i). Fix $a \in A$. Let $\varepsilon > 0$ be fixed. Then there exists $\lambda \in \mathbf{N}$ satisfying

$$\sup\{d(a_n, b_m) : m \geq n\} < \varepsilon,$$

for $n \in \mathbf{N}$ with $n \geq \lambda$. We can choose $\mu \geq \lambda$ satisfying $a_\mu = a$. We have

$$\sup\{d(a, b_m) : m \geq \mu\} = \sup\{d(a_\mu, b_m) : m \geq \mu\} < \varepsilon.$$

So we obtain (i).

We next show (ii). Fix $b \in B$. Let $\varepsilon > 0$ be fixed. Then there exists $\lambda \in \mathbf{N}$ satisfying

$$\sup\{d(a_n, b_m) : m \geq n \geq \lambda\} < \varepsilon.$$

Fix $\ell \in \mathbf{N}$ with $\ell \geq \lambda$. Then we can choose $\mu \geq \ell$ satisfying $b_\mu = b$. We have

$$d(a_\ell, b) = d(a_\ell, b_\mu) \leq \sup\{d(a_n, b_m) : m \geq n \geq \lambda\} < \varepsilon.$$

So we obtain (ii).

Let us prove (iii). By (i) and (ii), for all $(a, b) \in A \times B$, we have

$$\lim_{n\to\infty} d(a, b_n) = \lim_{n\to\infty} d(a_n, b) = 0,$$

which implies $a = b$. Thus, we obtain (iii).

(iv) follows from (iii). □

Lemma 13. *Let* (X, d) *be a* ν*-generalized metric space. Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences in* X. *Assume that there exists* $\mu \in \mathbf{N}$ *satisfying*

$$\lim_{n\to\infty} \sup\{d(a_n, b_m) : m \geq n + \mu\} = 0.$$

Then

$$\lim_{n\to\infty} \sup\{\max\{d(a_n, b_m), d(a_m, b_n) : m \geq n\} = 0$$

holds.

Proof. We first show

$$\lim_{n\to\infty} \sup\{d(a_m, b_n) : m \geq n\} = 0. \tag{4}$$

We consider the following three cases:

(a) $\#\{a_n : n \in \mathbf{N}\} < \infty$.
(b) $\#\{a_n : n \in \mathbf{N}\} = \infty$ and $\#\{b_n : n \in \mathbf{N}\} < \infty$.
(c) $\#\{a_n : n \in \mathbf{N}\} = \infty$ and $\#\{b_n : n \in \mathbf{N}\} = \infty$.

In the case of (a), define a subset A of X by (2). It is obvious that $0 < \#A < \infty$ and $a_n \in A$ for sufficiently large $n \in \mathbf{N}$. By Lemma 12 (i), we have $\lim_n d(a, b_n) = \lim_n d(a, b_{n+\mu}) = 0$ for $a \in A$. So we have

$$\lim_{n\to\infty} \sup\{d(a_m, b_n) : m \geq n\} = \lim_{n\to\infty} \max\{d(a, b_n) : a \in A\} = 0.$$

In the case of (b), define a subset B of X by (3). It is obvious that $0 < \#B < \infty$ and $b_n \in B$ for sufficiently large $n \in \mathbf{N}$. By Lemma 12 (ii), we have $\lim_n d(a_n, b) = \lim_n d(a_{n-\mu}, b) = 0$ for $b \in B$. So we have

$$\lim_{n\to\infty} \sup\{d(a_m, b_n) : m \geq n\} = \lim_{n\to\infty} \sup\{d(a_m, b) : m \geq n, b \in B\} = 0.$$

In the case of (c), we fix $\varepsilon > 0$. Then there exists $\lambda_0 \in \mathbf{N}$ satisfying

$$\sup\{d(a_n, b_m) : m \geq n + \mu\} < \varepsilon$$

for $n \geq \lambda_0$. We can choose $\lambda, \lambda_1 \in \mathbf{N}$ satisfying $\lambda_0 < \lambda_1 < \lambda_1 + \mu < \lambda$ and

$$\#\{a_n : \lambda_0 < n < \lambda_1\} \geq 2\nu + 3 \qquad \text{and} \qquad \#\{b_n : \lambda_1 + \mu < n < \lambda\} \geq 2\nu + 3.$$

Fix $m, n \in \mathbf{N}$ with $m \geq n \geq \lambda$. In the case where $a_m = b_n$ holds, we have $d(a_m, b_n) = 0 < \varepsilon$. In the other case, where $a_m \neq b_n$ holds, we can choose $f(1), \ldots, f(\nu), g(0), \ldots, g(\nu) \in \mathbf{N}$ satisfying

$$\begin{aligned} \lambda_0 < f(1) < \cdots < f(\nu) < \lambda_1 \\ < \lambda_1 + \mu < g(1) < \cdots < g(\nu) < \lambda < m \\ < m + \mu < g(0) \end{aligned}$$

and $(a_m, a_{f(1)}, \ldots, a_{f(\nu)}, b_n, b_{g(0)}, \ldots, b_{g(\nu)}) \in X^{(2\nu+3)}$. We consider the following three cases:

(c-1) ν is even.
(c-2) $\nu = 1$.
(c-3) ν is odd with $\nu \geq 3$.

In the case of (c-1), we have

$$\begin{aligned} d(a_m, b_n) &\leq D(a_m, b_{g(0)}, a_{f(1)}, b_{g(1)}, \ldots, a_{f(\nu/2-1)}, b_{g(\nu/2-1)}, a_{f(\nu/2)}, b_n) \\ &< (\nu + 1)\varepsilon. \end{aligned}$$

In the case of (c-2), we have

$$d(a_m, b_n) \leq D(a_m, b_{g(0)}, a_{f(1)}, b_n) < 3\varepsilon.$$

In the case of (c-3), we have

$$\begin{aligned} d(a_m, a_{f(1)}) &\le D(a_m, b_{g(0)}, a_{f(2)}, b_{g(2)}, \dots, a_{f(\nu/2+1/2)}, b_{g(\nu/2+1/2)}, a_{f(1)}) \\ &< (\nu+1)\varepsilon \end{aligned}$$

and hence

$$\begin{aligned} d(a_m, b_n) &\le D(a_m, a_{f(1)}, b_{g(1)}, \dots, a_{f(\nu/2-1/2)}, b_{g(\nu/2-1/2)}, a_{f(\nu/2+1/2)}, b_n) \\ &< (2\nu+1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have shown (4).

In particular, we have

$$\lim_{n\to\infty} \sup\{d(a_m, b_n) : m \ge n + \mu\} = 0.$$

So, the above argument yields (1). Thus, we obtain the desired result. □

LEMMA 14. *Let (X, d) be a ν-generalized metric space. Let $\{a_n\}$ and $\{b_n\}$ be sequences in X satisfying*

$$\lim_{n\to\infty} \sup\{d(a_n, b_m) : m \ge n\} = 0.$$

Then

$$\lim_{n\to\infty} \sup\{d(a_m, b_n) : m \ge n\} = 0$$

holds. In other words, the sequence $\{u_n\}$ defined by

$$a_1, b_1, a_2, b_2, \dots, a_n, b_n, a_{n+1}, b_{n+1}, \dots$$

is 2-Cauchy.

PROOF. By Lemma 13, we obtain the desired result. □

LEMMA 15. *Let (X, d) be a ν-generalized metric space. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{e_n\}$ be sequences in X satisfying*

$$\lim_{n\to\infty} \sup\{d(a_n, b_m) : m \ge n\} = 0,$$

$$\lim_{n\to\infty} \sup\{d(b_n, c_m) : m \ge n\} = 0,$$

$$\lim_{n\to\infty} \sup\{d(c_n, e_m) : m \ge n\} = 0.$$

Assume $\#\{b_n : n \in \mathbf{N}\} = \#\{c_n : n \in \mathbf{N}\} = \infty$. Then

$$\lim_{n\to\infty} \sup\{d(a_n, e_m) : m \ge n\} = 0$$

holds.

PROOF. By Lemma 14, we have

$$\lim_{n\to\infty} \sup\{\max\{d(a_m, b_n), d(b_m, c_n), d(c_m, e_n)\} : m \geq n\} = 0.$$

Let $\varepsilon > 0$ be fixed. Then there exists $\mu \in \mathbf{N}$ satisfying

$$\max\{d(a_n, b_m), d(b_n, c_m), d(c_n, e_m)\} < \varepsilon$$

for $m, n \in \mathbf{N}$ with $m \geq \mu$ and $n \geq \mu$. We can choose $\lambda, \lambda_1 \in \mathbf{N}$ satisfying $\mu < \lambda_1 < \lambda$ and

$$\#\{b_n : \mu < n < \lambda_1\} \geq 2\nu + 2 \qquad \text{and} \qquad \#\{c_n : \lambda_1 < n < \lambda\} \geq 2\nu + 2.$$

Fix $m, n \in \mathbf{N}$ with $m \geq n \geq \lambda$. In the case where $a_n = e_m$ holds, we have $d(a_n, e_m) = 0 < \varepsilon$. In the other case, where $a_n \neq e_m$ holds, we can choose $f(1), \dots, f(\nu), g(1), \dots, g(\nu) \in \mathbf{N}$ satisfying

$$\mu < f(1) < \cdots < f(\nu) < \lambda_1 < g(1) < \cdots < g(\nu) < \lambda$$

and $(a_n, b_{f(1)}, \dots, b_{f(\nu)}, c_{g(1)}, \dots, c_{g(\nu)}, e_m) \in X^{(2\nu+2)}$. We consider the following three cases:

(a) ν is even.
(b) $\nu = 1$.
(c) ν is odd with $\nu \geq 3$.

In the case of (a), we have

$$\begin{aligned} d(a_n, e_m) &\leq D(a_n, b_{f(1)}, c_{g(1)}, b_{f(2)}, c_{g(2)}, \dots, b_{f(\nu/2)}, c_{g(\nu/2)}, e_m) \\ &< (\nu + 1)\varepsilon. \end{aligned}$$

In the case of (b), we have

$$d(a_n, e_m) \leq D(a_n, b_{f(1)}, c_{g(1)}, e_m) < 3\varepsilon.$$

In the case of (c), we have

$$\begin{aligned} d(a_n, c_{g(1)}) &\leq D(a_n, b_{f(2)}, c_{g(2)}, b_{f(3)}, c_{g(3)}, \dots, b_{f(\nu/2+3/2)}, c_{g(1)}) \\ &< (\nu + 1)\varepsilon \end{aligned}$$

and hence

$$\begin{aligned} d(a_n, e_m) &\leq D(a_n, c_{g(1)}, b_{f(1)}, c_{g(2)}, b_{f(2)}, \dots, c_{g(\nu/2-1/2)}, b_{f(\nu/2-1/2)}, c_{g(\nu/2+1/2)}, e_m) \\ &< (2\nu + 1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result. □

LEMMA 16. *Let (X, d) be a Hausdorff, ν-generalized metric space. Let $\{x_n\}$ be a sequence in X converging to z. Then $\{x_n\}$ converges exclusively to z.*

Proof. Let $w \in X$ satisfy $\liminf_n d(x_n, w) = 0$. Then there exists a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbf{N}$ satisfying $\lim_n d(x_{f(n)}, w) = 0$. Since $\lim_n d(x_{f(n)}, z) = 0$ holds, $\{x_{f(n)}\}$ converges to w and z. Since X is Hausdorff, $w = z$ holds. □

4. 1-Completeness

In this section, we prove 1-completeness is equivalent to 3-completeness.

Theorem 17. *Let (X, d) be a v-generalized metric space. Then the following are equivalent:*

(i) *X is complete.*

(ii) *X is 3-complete.*

Proof. By Proposition 5 (ii), we obtain (ii) ⇒ (i). Let us prove (i) ⇒ (ii). Let $\{x_n\}$ be a 3-Cauchy sequence in X. Define $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ by

$$a_n = x_{3n-2}, \qquad b_n = x_{3n-1} \qquad \text{and} \qquad c_n = x_{3n}$$

for $n \in \mathbf{N}$. We only have to consider the following two cases:

(a) $\#\{a_n : n \in \mathbf{N}\} < \infty$ and $\#\{b_n : n \in \mathbf{N}\} < \infty$.

(b) $\#\{b_n : n \in \mathbf{N}\} = \#\{c_n : n \in \mathbf{N}\} = \infty$.

In the case of (a), by Lemma 12 (iv), there exist $z \in X$ and $\mu \in \mathbf{N}$ satisfying $a_n = b_n = z$ for all $n \in \mathbf{N}$ with $n \geq \mu$. It is obvious that $\{a_n\}$ and $\{b_n\}$ converge to z. By Lemma 12 (i), $\{c_n\}$ converges to z. Therefore $\{x_n\}$ converges to z. In the case of (b), since $\{x_n\}$ is 3-Cauchy, we have

$$\lim_{n\to\infty} \sup\{\max\{d(a_n, b_m), d(b_n, c_m), d(c_n, a_{m+1})\} : m \geq n\} = 0.$$

By Lemma 13, we have

$$\lim_{n\to\infty} \sup\{d(c_n, a_m) : m \geq n\} = 0.$$

By Lemma 15, we have

$$\lim_{n\to\infty} \sup\{d(a_n, a_m) : m \geq n\} = 0.$$

Therefore we obtain that $\{a_n\}$ is Cauchy. Since X is complete, $\{a_n\}$ converges to some $z \in X$. By Lemma 8, $\{b_n\}$ and $\{c_n\}$ also converge to z. Therefore $\{x_n\}$ converges to z. □

5. 5-Completeness

In this section, we study 5-completeness.

THEOREM 18. *Let (X, d) be a ν-generalized metric space. Let $\lambda \in \mathbf{N}$ with $\lambda \geq 4$. Assume that X is λ-complete. Then X is $(\lambda - 2)$-complete.*

PROOF. Put $\kappa = \lambda - 2$. Let $\{x_n\}$ be a κ-Cauchy sequence in X. We define sequences $\{u_n^{(j)}\}$ by

$$u_n^{(j)} = x_{(n-1)\kappa+j}$$

for $j \in \{1, \ldots, \kappa\}$ and $n \in \mathbf{N}$. By Lemma 14, the sequence defined by

$$u_1^{(1)}, u_1^{(2)}, u_2^{(1)}, u_2^{(2)}, \ldots, u_n^{(1)}, u_n^{(2)}, u_{n+1}^{(1)}, u_{n+1}^{(2)}, \ldots$$

is 2-cauchy. Define a sequence $\{y_n\}$ by

$$u_1^{(1)}, u_1^{(2)}, u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, \ldots, u_1^{(\kappa)}, u_2^{(1)}, u_2^{(2)}, u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, \ldots, u_2^{(\kappa)}, \ldots.$$

Then $\{y_n\}$ is λ-Cauchy. Since X is λ-complete, $\{y_n\}$ converges to some $z \in X$. Since $\{x_n\}$ is a subsequence of $\{y_n\}$, $\{x_n\}$ also converges to z. Therefore X is κ-complete. □

LEMMA 19. *Let (X, d) be a κ-complete, ν-generalized metric space where $\kappa \in \mathbf{N} \backslash \{1, 3\}$ holds. Then X is Hausdorff.*

PROOF. We have proved the conclusion in the case where $\kappa = 2$; see Lemma 6. So we assume $\kappa \geq 4$. Let $\{x_n\}$ be a sequence in X converging to z and w in X. Then the sequence $\{y_n\}$ defined by

$$z, x_2, \underbrace{w, \ldots, w}_{\kappa-3}, x_\kappa, z, x_{\kappa+2}, \underbrace{w, \ldots, w}_{\kappa-3}, x_{2\kappa}, z, x_{2\kappa+2}, \underbrace{w, \ldots, w}_{\kappa-3}, x_{3\kappa}, z, \ldots$$

is κ-Cauchy. Since X is κ-complete, $\{y_n\}$ converges to some $x \in X$. It is obvious that $z = x = w$ holds. □

THEOREM 20. *Let (X, d) be a ν-generalized metric space. Let $\kappa \in \mathbf{N}$ with $\kappa \geq 4$. Assume that X is κ-complete. Then X is $(\kappa + 2)$-complete.*

PROOF. By Lemma 19, we note that X is Hausdorff. Put $\lambda = \kappa + 2$. Let $\{x_n\}$ be a λ-Cauchy sequence in X. We define sequences $\{u_n^{(j)}\}$ by

$$u_n^{(j)} = x_{(n-1)\lambda+j}$$

for $j \in \{1, \ldots, \lambda\}$ and $n \in \mathbf{N}$. We consider the following two cases:

(a) There exists $j \in \{1, 2, \ldots, \lambda\}$ satisfying

$$\#\{u_n^{(j)} : n \in \mathbf{N}\} = \infty \quad \text{and} \quad \#\{u_n^{(j+1)} : n \in \mathbf{N}\} = \infty,$$

where $\lambda + 1 = 1$.

(b) For any $j \in \{1, 2, \ldots, \lambda\}$, either

$$\#\{u_n^{(j)} : n \in \mathbf{N}\} < \infty \qquad \text{or} \qquad \#\{u_n^{(j+1)} : n \in \mathbf{N}\} < \infty$$

holds, where $\lambda + 1 = 1$.

In the case of (a), by Lemma 15, the sequence $\{y_n\}$ defined by

$$u_1^{(1)}, u_1^{(4)}, u_1^{(5)}, u_1^{(6)}, \ldots, u_1^{(\lambda)}, u_2^{(1)}, u_2^{(4)}, u_2^{(5)}, u_2^{(6)}, \ldots, u_2^{(\lambda)}, \ldots$$

is κ-Cauchy. Since X is κ-complete, $\{y_n\}$ converges to some $z \in X$. In particular, $\{u_n^{(1)}\}$ converges to z. Define a sequence $\{z_n\}$ by

$$u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_1^{(4)}, \underbrace{z, \ldots, z}_{\kappa-4}, u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}, \underbrace{z, \ldots, z}_{\kappa-4}, \ldots$$

Then in the case where $\kappa \geq 5$, it is obvious that $\{z_n\}$ is κ-Cauchy. In the other case, where $\kappa = 4$, by Lemma 13, $\{z_n\}$ is κ-Cauchy. Since X is κ-complete, $\{z_n\}$ converges. Since $\{u_n^{(1)}\}$ converges to z, $\{z_n\}$ converges to z. Therefore $\{u_n^{(2)}\}$ and $\{u_n^{(3)}\}$ converge to z. So $\{x_n\}$ converges to z.

In the case of (b), without loss of generality, we may assume $\#\{u_n^{(1)} : n \in \mathbf{N}\} < \infty$. We can choose z satisfying $\#\{n \in \mathbf{N} : u_n^{(1)} = z\} = \infty$. We consider the following two cases:

(b-1) $\#\{u_n^{(2)} : n \in \mathbf{N}\} < \infty$.

(b-2) $\#\{u_n^{(2)} : n \in \mathbf{N}\} = \infty$.

In the case of (b-1), by Lemma 12 (iv), $u_n^{(1)} = u_n^{(2)} = z$ holds for sufficiently large $n \in \mathbf{N}$. In the case of (b-2), by Lemma 12 (i), $\lim_n d(z, u_n^{(2)}) = 0$ holds. Since X is Hausdorff, $u_n^{(1)} = z$ holds for sufficiently large $n \in \mathbf{N}$. Since $\#\{u_n^{(3)} : n \in \mathbf{N}\} < \infty$ holds, $u_n^{(3)} = z$ holds for sufficiently large $n \in \mathbf{N}$. Thus, we can prove that $\{u_n^{(j)}\}$ converges to z for any $j \in \{1, \ldots, \lambda\}$. Therefore $\{x_n\}$ converges to z. Therefore X is κ-complete. □

Theorem 21. *Let (X, d) be a ν-generalized metric space. Then the following are equivalent:*

(i) *X is 5-complete.*

(ii) *X is $(2\kappa + 3)$-complete for any $\kappa \in \mathbf{N}$.*

(iii) *X is $(2\kappa + 3)$-complete for some $\kappa \in \mathbf{N}$.*

Proof. By Theorem 20, we can prove (i) ⇒ (ii). (ii) ⇒ (iii) obviously holds. By Theorem 18, We can prove (iii) ⇒ (i). □

6. 2-Completeness

In this section, we study 2-completeness.

Theorem 22. *Let (X, d) be a 2-complete, ν-generalized metric space. Then X is κ-complete for any $\kappa \in \mathbf{N}$.*

Proof. By Proposition 5 (ii), X is 1-complete. So we assume $\kappa \geq 3$. By Lemma 6, we note that X is Hausdorff. Let $\{x_n\}$ be a κ-Cauchy sequence in X. We define sequences $\{u_n^{(j)}\}$ by

$$u_n^{(j)} = x_{(n-1)\kappa+j}$$

for $j \in \{1, \dots, \kappa\}$ and $n \in \mathbf{N}$. Let $\{v_n^{(1,2)}\}$ be the sequence defined by

$$u_1^{(1)}, u_1^{(2)}, u_2^{(1)}, u_2^{(2)}, \dots, u_n^{(1)}, u_n^{(2)}, u_{n+1}^{(1)}, u_{n+1}^{(2)}, \dots.$$

By Lemma 14, $\{v_n^{(1,2)}\}$ is 2-Cauchy. Since X is 2-complete, $\{v_n^{(1,2)}\}$ converges to some $z^{(1,2)} \in X$. Similarly, we can prove that the sequence $\{v_n^{(2,3)}\}$ defined by

$$u_1^{(2)}, u_1^{(3)}, u_2^{(2)}, u_2^{(3)}, \dots, u_n^{(2)}, u_n^{(3)}, u_{n+1}^{(2)}, u_{n+1}^{(3)}, \dots$$

converges to some $z^{(2,3)} \in X$. So $\{u_n^{(2)}\}$ converges to $z^{(1,2)}$ and $z^{(2,3)}$. Since X is Hausdorff, we obtain $z^{(1,2)} = z^{(2,3)}$. Therefore $\{u_n^{(3)}\}$ also converges to $z^{(1,2)}$. Thus we can prove $\{u_n^{(j)}\}$ converges to $z^{(1,2)}$ for $j \in \{1, 2, \dots, \kappa\}$. Therefore $\{x_n\}$ converges to $z^{(1,2)}$. We have shown that X is κ-complete. □

Theorem 23. *Let (X, d) be a ν-generalized metric space. Then the following are equivalent:*

(i) *X is 2-complete.*
(ii) *X is 2κ-complete for any $\kappa \in \mathbf{N}$.*
(iii) *X is 2κ-complete for some $\kappa \in \mathbf{N}$.*

Proof. By Theorem 22, we obtain (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii) obviously holds. By Proposition 5 (ii), we obtain (iii) $\Rightarrow$ (i). □

Theorem 24. *Let (X, d) be a ν-generalized metric space where ν is odd. Then the following are equivalent:*

(i) *X is complete and Hausdorff.*
(ii) *X is κ-complete for any $\kappa \in \mathbf{N} \backslash \{1, 3\}$.*
(iii) *X is κ-complete for some $\kappa \in \mathbf{N} \backslash \{1, 3\}$.*

Proof. We first show (i) $\Rightarrow$ (ii). We assume (i). By Theorem 10, X is $(\sum, \neq)$-complete. By Theorem 11, X is 2-complete. By Theorem 22, X is κ-complete for any $\kappa \in \mathbf{N}$. (ii) $\Rightarrow$ (iii) obviously holds. Let us prove (iii) $\Rightarrow$ (i). We assume (iii). By Proposition 5 (ii), X is 1-complete, thus, X is complete. By Lemma 19, X is Hausdorff. □

7. $\sum$-Completeness

In this section, we study $\sum$-completeness.

Lemma 25. *Let (X, d) be a Hausdorff, ν-generalized metric space. Let $\{x_n\}$ be a sequence in X. Assume that there exist $\kappa \in \mathbf{N}$ and a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbf{N}$ satisfying the following:*

- $f(n) \% \kappa = n \% \kappa$.
- *Every subsequence $\{x_{g(n)}\}$ of $\{x_n\}$ converges provided $g(n) \geq f(n)$ and $g(n) \% \kappa = n \% \kappa$ hold for any $n \in \mathbf{N}$.*

Then $\{x_n\}$ converges.

Proof. From the assumption, $\{x_{f(n)}\}$ converges to some $z \in X$. Arguing by contradiction, we assume $\limsup_n d(x_n, z) > 0$. Then there exist $\varepsilon > 0$ and a subsequence $\{g(n)\}$ of $\{n\}$ in $\mathbf{N}$ satisfying $d(x_{g(n)}, z) \geq \varepsilon$ for any $n \in \mathbf{N}$. We can choose $\lambda \in \{0, 1, \dots, \kappa - 1\}$ satisfying

$$\#\{n \in \mathbf{N} : g(n) \% \kappa = \lambda\} = \infty.$$

Without loss of generality, we may assume $g(n) \% \kappa = \lambda$ for all $n \in \mathbf{N}$. We can choose a subsequence $\{h(n)\}$ of $\{n\}$ in $\mathbf{N}$ satisfying

$$\begin{aligned}
&h(n) \% \kappa = n \% \kappa, \\
&h(n) \geq f(n), \\
&h(2i\kappa + \lambda) \in g(\mathbf{N}), \\
&h(2i\kappa + \ell) \in f(\mathbf{N}), \\
&h(2i\kappa + \kappa + j) \in f(\mathbf{N})
\end{aligned}$$

for any $n, i \in \mathbf{N}$, $\ell \in \{0, 1, \dots, \kappa - 1\} \backslash \{\lambda\}$ and $j \in \{0, 1, \dots, \kappa - 1\}$. From the assumption, $\{x_{h(n)}\}$ converges to some $w \in X$. By Lemma 16, $\{x_{h(n)}\}$ converges exclusively to $w \in X$. So, since

$$\lim_{i \to \infty} d(x_{h(2i\kappa + \kappa + \lambda)}, z) = 0$$

holds, we have $z = w$. However, since

$$d(x_{h(2i\kappa + \lambda)}, z) \geq \varepsilon$$

holds for $i \in \mathbf{N}$, we have $z \neq w$, which implies a contradiction. Therefore we have shown $\lim_n d(x_n, z) = 0$. □

We give an alternative proof of the following:

Theorem 26 (Proposition 5.4 in [11]). *Let (X, d) be a $\sum$-complete, ν-generalized metric space. Then X is κ-complete for any $\kappa \in \mathbf{N}$.*

Proof. By Lemma 7, we first note that X is Hausdorff. Let $\{x_n\}$ be a κ-Cauchy sequence in X. Choose a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbf{N}$ satisfying the following:

- $f(n) \% \kappa = n \% \kappa$.
- $\sup\{d(x_\ell, x_{\ell+1+j\kappa}) : j = 0, 1, 2, \dots\} < 2^{-n}$ holds for any $\ell, n \in \mathbf{N}$ with $\ell \geq f(n)$.

Let $\{g(n)\}$ be a subsequence of $\{n\}$ in $\mathbf{N}$ satisfying the following:

- $g(n) \geq f(n)$.
- $g(n) \% \kappa = n \% \kappa$.

Then we have

$$\sum_{n=1}^{\infty} d(x_{g(n)}, x_{g(n+1)}) < \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

Since X is $\sum$-complete, $\{x_{g(n)}\}$ converges. We have shown the assumption of Lemma 25. So, by Lemma 25, $\{x_n\}$ converges. □

8. Conclusion

Throughout this section, we denote by $(\sum)$ that (X, d) is $\sum$-complete. Similarly for (2), $(\sum, \neq)$, (5) and (1).

THEOREM 27. *Let (X, d) be a ν-generalized metric space. Then*

$$(\textstyle\sum) \Rightarrow (2) \Rightarrow (\textstyle\sum, \neq) \Rightarrow (1)$$

and

$$(\textstyle\sum) \Rightarrow (2) \Rightarrow (5) \Rightarrow (1)$$

hold.

PROOF. By Theorem 26, we obtain $(\sum) \Rightarrow (2)$. By Theorem 11, we obtain $(2) \Rightarrow (\sum, \neq)$. By Lemma 9, we obtain $(\sum, \neq) \Rightarrow (1)$. By Theorem 22, we obtain $(2) \Rightarrow (5)$. By Proposition 5 (ii), we obtain $(5) \Rightarrow (1)$. □

THEOREM 28. *Let (X, d) be a ν-generalized metric space where ν is odd with $\nu \geq 5$. Then*

$$(\textstyle\sum) \Rightarrow (2) \Leftrightarrow (5) \Rightarrow (\textstyle\sum, \neq) \Leftrightarrow (1)$$

holds.

PROOF. By Theorem 24, we obtain $(2) \Leftrightarrow (5)$. By Lemma 9, we obtain $(\sum, \neq) \Leftrightarrow (1)$. □

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