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Multi-Colored Rooted Tree Analysis of the Weak Order Conditions of a Stochastic Runge-Kutta Family under a Commutativity Condition

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Abstract

The present paper is aimed at multi-colored rooted tree analysis to give a transparent expression of weak order conditions of a stochastic Runge-Kutta family for stochastic differential equations satisfying a commutativity condition. As a result, a new explicit stochastic Runge-Kutta scheme is given, which is of weak order 2 and order 4 in the deterministic mean.

1 Introduction

Stochastic differential equations (SDEs) appear to make mathematical models in many fields [7, 13]. Since SDEs are analytically unsolvable in most cases, however, numerical methods for SDEs have been studied by many researchers.

The methods are categorized into two types based on the meaning of approximation. One type provides an approximate solution in the mean square sense [3]. The other type provides an approximation in the weak sense. The present paper deals with the latter.

Many numerical methods in the weak sense (weak schemes) have been proposed for multi-dimensional SDEs with multiplicative noise in the multi-dimensional Wiener process case. Let us introduce results concerning the schemes that attains weak order 2 for such SDEs. Klauder and Petersen [6] have proposed a weak scheme in the sense defined by the separation of an approximate solution into its deterministic and stochastic parts. Milstein [10] and Tocino and Ardanuy [15] have proposed weak schemes with derivatives of the drift or the diffusion coefficients. Kloeden and Platen [7, 11] have proposed a derivative-free scheme by replacing necessary derivatives of higher order by finite differences of higher order. On the other hand, Rößler [12] has proposed other derivative-free schemes by assuming a commutativity condition [1, 14].

The paper [12] has disclosed the important fact that schemes with only the random variables corresponding to the increment of Wiener process can attain weak order 2 under the commutativity condition. This will lead to our scheme given in Section 4 of the present paper.

As we can see in the papers mentioned above, the calculation to derive weak order conditions is a troublesome task in general. The rooted tree analysis for ordinary differential equations (ODEs) by Butcher [4, 5] can, however, become a help to relieve the task. In fact, Komori [8] has extended it to obtain weak order conditions of a stochastic Runge-Kutta family for SDEs with a multi-dimensional Wiener process. By utilizing the results, we will transparently get weak conditions under the commutativity condition in the later section.

The aim of the present paper is to show the analysis of weak order conditions under the commutativity condition, and give an explicit stochastic Runge-Kutta scheme of weak order 2.

Next, let us introduce the definition of weak (global) order. For the d-dimensional stochastic integral equation

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_0^t \boldsymbol{g}_0(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_0^t \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s), \quad 0 \le t \le T_{end},$$

where $W_j(s)$ is a scalar Wiener process and $\circ$ means the Stratonovich formulation, we give equidistant grid points $\tau_n \stackrel{\text{def}}{=} nh$ $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and consider discrete approximations $\boldsymbol{y}_n$ to $\boldsymbol{y}(\tau_n)$. Let $C_P^l(\boldsymbol{R}^d, \boldsymbol{R})$ denote the totality of l times continuously differentiable $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^d$, all of whose partial derivatives of order less than or equal to l have polynomial growth. Then, the definition is given as follows [2].

Definition 1.1 *Suppose that discrete approximations $\boldsymbol{y}_n$ are given by a scheme. Then, we say that the scheme is of weak (global) order q if for each $G \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$, $\exists C > 0$ (independent of h) and $\delta > 0$ such that*

$$|E[G(\boldsymbol{y}(\tau_M)] - E[G(\boldsymbol{y}_M)]| \le Ch^q, \quad h \in (0, \delta).$$

The organization of the present paper is as follows. In Section 2 we will give a general expression of weak order conditions for a stochastic Runge-Kutta family by the extended rooted tree analysis. In Section 3 we will give a similar expression under the commutativity condition. In Section 4 we will find a solution of the order conditions after giving simplifying assumptions, and give some numerical experiments under the commutativity conditions. In Section 5 we will give the summary. In the appendix, we will show the expectations of elementary weights and elementary numerical weights for weak order 2.

2 Weak order conditions by multi-colored rooted trees

The aim in this section is to express weak order conditions by multi-colored rooted trees. Because the theoretical content like proof has been already given in [8], we only give here the brief introduction to the expression of weak order conditions.

2.1 The Stratonovich-Taylor expansion for the stochastic differential equation solution

The purpose of this subsection is to represent the truncated Stratonovich-Taylor expansion of

$$\boldsymbol{y}(\tau_{n+1}) = \boldsymbol{y}_n + \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_0(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s) \qquad (2.\ 1)$$

by functions on the set of multi-colored rooted trees (MRTs).

Let us suppose that any component of the d-vector valued function $\boldsymbol{g}_j$ belongs to $C_P^{2q}(\boldsymbol{R}^d, \boldsymbol{R})$ $(0 \leq j \leq m)$, and denote by $\boldsymbol{y}_{2q}(\tau_{n+1})$ the truncated expansion of $\boldsymbol{y}(\tau_{n+1})$ satisfying $\lambda(x) + \sigma(x) \leq 2q$, where $\lambda(x)$ means the multiplicity of integrals with respect to a time variable or Wiener processes, and $\sigma(x)$ means the multiplicity of integrals with respect to a time variable for a multiple stochastic integral x appearing in the expansion.

First, we are to introduce MRT.

Definition 2.1 (Multi-colored rooted tree (MRT)) *A multi-colored rooted tree with a root* ⓙ *(colored with a label j from 0 to m) is a tree recursively defined in the following way.*

1) *$\tau^{(j)}$ is the primitive tree having only a vertex* ⓙ*.*

2) *If $t_1, \ldots, t_k$ are multi-colored trees, then $[t_1, \ldots, t_k]^{(j)}$ is also a multi-colored rooted tree with the root* ⓙ*.*

The totality of MRTs is denoted by T.

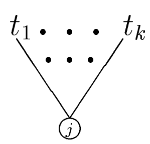


Figure 1: Generation of the tree $[t_1, \ldots, t_k]^{(j)}$

Next, let $\rho(t)$ be the number of vertices of $t \in T$, $r(t)$ the number of vertices of t with the color 0, and $\nu(t)$ the number of different ways of numbering on t in the way that along each outwardly directed arc the numbers increase and vertices of a subtree are consecutively numbered. (See Fig. 2.)

Figure 2: Examples of numbering on t

Furthermore, we introduce an integral operator and three functions on T. For any integrable function H of $\boldsymbol{y}$ and $s > \tau_n$,

$$J_0[H](s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} H(\boldsymbol{y}(s_1))\mathrm{d}s_1, \qquad J_j[H](s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} H(\boldsymbol{y}(s_1)) \circ \mathrm{d}W_j(s_1)$$

$(1 \le j \le m)$. In the sequel, it is clear that the upper bound of integral interval is τ_{n+1} only with respect to the last integral variable in the multiple integrals. Thus, we will omit this symbol for ease of notation.

Definition 2.2 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows.*

$$\Phi(\tau^{(j)}) = J_j[1], \qquad \Phi(t) = J_j\left[\prod_{i=1}^{k} \Phi(t_i)\right] \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

Definition 2.3 (Elementary differential $\boldsymbol{F}(t)$ on T) *An elementary differential is a possibly multilinear operator recursively given as follows.*

$$\begin{aligned} &\boldsymbol{F}(\tau^{(j)}) = \boldsymbol{g}_j(\boldsymbol{y}_n), \\ &\boldsymbol{F}(t) = \boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)] \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}. \end{aligned}$$

Definition 2.4 (Elementary coefficient $\beta(t)$ on T) *The index $\beta(t)$ $(t \in T)$ is defined recursively.*

$$\beta(\tau^{(j)}) = 1, \qquad \beta(t) = \frac{1}{k!}\prod_{i=1}^{k} \beta(t_i) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

Note that $0 \le j \le m$ in these definitions.

Then, we obtain one of the important results.

Theorem 2.1 *The finitely truncated expansion has the following expression.*

$$\boldsymbol{y}_{2q}(\tau_{n+1}) = \boldsymbol{y}_n + \sum_{i=1}^{2q} \sum_{\substack{\rho(t)+r(t)=i \\ t \in T}} \nu(t)\beta(t)\boldsymbol{F}(t)\Phi(t).$$

2.2 The Taylor expansion for a stochastic Runge-Kutta family

In order to obtain an approximate solution $\boldsymbol{y}_{n+1}$ of the solution $\boldsymbol{y}(t_{n+1})$ of (2. 1), we consider the stochastic Runge-Kutta family given by

$$\begin{aligned}
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s}\sum_{j=0}^{m} c_i^{(j)} \boldsymbol{Y}_i^{(j)}, \\
\boldsymbol{Y}_{i_a}^{(j_a)} &= \sum_{j_b=0}^{m} \eta_{i_a}^{(j_a,j_b)} \Bigg\{ b_{i_a}^{(j_a,j_b)} \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s}\sum_{j_c=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c)} \boldsymbol{Y}_{i_b}^{(j_c)}) \\
&\qquad + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s}\sum_{j_c=0}^{m} \gamma_{i_a i_b}^{(j_a,j_b,j_c)} \boldsymbol{Y}_{i_b}^{(j_c)} \Bigg\}
\end{aligned} \tag{2. 2}$$

$(1 \le i_a \le s,\ 0 \le j_a \le m)$, where each $\eta_{i_a}^{(j_a,j_b)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\eta_{i_a}^{(j_a,j_b)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j_b = 0), \\ K_2 h^k & (j_b \ne 0) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. If $b_{i_a}^{(j_a,j_b)} \ne 0$, by setting $\tilde{\eta}_{i_a}^{(j_a,j_b)} \stackrel{\text{def}}{=} \eta_{i_a}^{(j_a,j_b)} b_{i_a}^{(j_a,j_b)}$ and $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c)} \stackrel{\text{def}}{=} \gamma_{i_a i_b}^{(j_a,j_b,j_c)} / b_{i_a}^{(j_a,j_b)}$ we can rewrite this in the following simpler form:

$$\begin{aligned}
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s}\sum_{j=0}^{m} c_i^{(j)} \boldsymbol{Y}_i^{(j)}, \\
\boldsymbol{Y}_{i_a}^{(j_a)} &= \sum_{j_b=0}^{m} \tilde{\eta}_{i_a}^{(j_a,j_b)} \Bigg\{ \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s}\sum_{j_c=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c)} \boldsymbol{Y}_{i_b}^{(j_c)}) \\
&\qquad + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s}\sum_{j_c=0}^{m} \tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c)} \boldsymbol{Y}_{i_b}^{(j_c)} \Bigg\}
\end{aligned} \tag{2. 3}$$

$(1 \le i_a \le s,\ 0 \le j_a \le m)$. Note that these formulations include stochastic Rosenbrock-Wanner methods [9].

In this section we deal with the simple formulation and represent the truncated Taylor expansion of $\boldsymbol{y}_{n+1}$ centered at $\boldsymbol{y}_n$ by functions on the set of multi-colored rooted tree with labels (MRTL).

Let us denote by $\boldsymbol{y}_{n+1,2q}$ the truncated expansion of $\boldsymbol{y}_{n+1}$ satisfying $\bar{\lambda}(x) + \bar{\sigma}(x) \le 2q$, where $\bar{\lambda}(x)$ means the multiplicity of products with respect to $\tilde{\eta}_{i_a}^{(\cdot,\cdot)}$, and $\bar{\sigma}(x)$ means the multiplicity of products with respect to $\tilde{\eta}_{i_a}^{(\cdot,\cdot)}$ except $\tilde{\eta}_{i_a}^{(\cdot,0)}$ for a monomial x of $\tilde{\eta}_{i_a}^{(\cdot,\cdot)}$ appearing in the expansion.

First, we introduce several matrices related to the formula parameters of (2. 3). Let us define

$$A^{(j)} \stackrel{\text{def}}{=} \begin{bmatrix}
\alpha_{11}^{(0,j,0)} & \cdots & \alpha_{11}^{(m,j,0)} & \cdots & \alpha_{s+1,1}^{(0,j,0)} & \cdots & \alpha_{s+1,1}^{(m,j,0)} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha_{11}^{(0,j,m)} & \cdots & \alpha_{11}^{(m,j,m)} & \cdots & \alpha_{s+1,1}^{(0,j,m)} & \cdots & \alpha_{s+1,1}^{(m,j,m)} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha_{1s}^{(0,j,0)} & \cdots & \alpha_{1s}^{(m,j,0)} & \cdots & \alpha_{s+1,s}^{(0,j,0)} & \cdots & \alpha_{s+1,s}^{(m,j,0)} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha_{1s}^{(0,j,m)} & \cdots & \alpha_{1s}^{(m,j,m)} & \cdots & \alpha_{s+1,s}^{(0,j,m)} & \cdots & \alpha_{s+1,s}^{(m,j,m)} \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}$$

for $\alpha_{i_a i_b}^{(j_a,j_b,j_c)}$. Similarly, define the matrix $\tilde{\Gamma}^{(j)}$ for $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c)}$, and set $\tilde{A}^{(j)} \stackrel{\text{def}}{=} A^{(j)} + \tilde{\Gamma}^{(j)}$. In addition, define the $(m+1)(s+1) \times (m+1)(s+1)$ diagonal matrix $D^{(j)}$ by

$$D^{(j)} \stackrel{\text{def}}{=} \text{diag}(\tilde{\eta}_1^{(0,j)}, \cdots, \tilde{\eta}_1^{(m,j)}, \cdots, \tilde{\eta}_{s+1}^{(0,j)}, \cdots, \tilde{\eta}_{s+1}^{(m,j)}).$$

Next, we introduce MRTL and a function on its set.

Definition 2.5 (Multi-colored rooted tree with labels (MRTL)) *A multi-colored rooted tree with labels, denoted by t_X, is one attached by labels according to the following rule.*

1) *The label of the root is X.*

2) *The label of the other vertices is decided by the number of branches and the color of the parent vertex:*

   - *the label is $\tilde{A}^{(j)}$ if the parent vertex has a single branch and is colored with j.*
   - *the label is $A^{(j)}$ if the parent vertex has more than one branch and is colored with j.*

The totality of MRTL's whose roots are labeled with X, is denoted by $\mathcal{T}_X$. For example, some MRTL's are listed in Fig. 3.

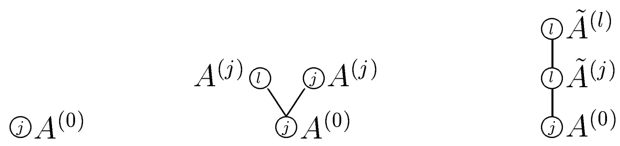


Figure 3: Examples of trees in $\mathcal{T}_{A^{(0)}}$

Definition 2.6 (Elementary numerical weight $\bar{\Phi}(t)$ on $\mathcal{T}_X$) *An elementary numerical weight of $t \in \mathcal{T}_X$ is given recursively as follows.*

$$\bar{\Phi}(\tau_X^{(j)}) = \mathbf{1} D^{(j)} X, \quad \bar{\Phi}(t) = (\prod_{i=1}^{k} \bar{\Phi}(t_i)) D^{(j)} X \quad \text{for } t = [t_1, \ldots, t_k]_X^{(j)}$$

$(0 \le j \le m)$, where $\mathbf{1}$ stands for an $(m+1)(s+1)$-dimensional row vector of 1's, and $\prod_{i=1}^{k} \bar{\Phi}(t_i)$ means the elementwise product of row vectors $\bar{\Phi}(t_i)$.

Then, we obtain another important result.

Theorem 2.2 *The finitely truncated expansion of the numerical solution by the stochastic Runge-Kutta family has the following expression.*

$$\boldsymbol{y}_{n+1,2q} = \boldsymbol{y}_n + \sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_{A^{(0)}}}} \nu(\hat{t}) \beta(\hat{t}) \boldsymbol{F}(\hat{t}) \bar{\Phi}_{(m+1)s+1}(t).$$

where $\bar{\Phi}_{(m+1)s+1}(t)$ denotes the $((m+1)s+1)$-st element of $\bar{\Phi}(t)$ and $\hat{t}$ means an MRT obtained by removing all labels from $t \in \mathcal{T}_{A^{(0)}}$.

2.3 Order conditions of the stochastic Runge-Kutta family

We give the transparent way of seeking the order conditions by utilizing the multi-colored rooted tree analysis.

Let us suppose the sufficient smoothness of g_j's and the regularity of the time discrete approximation. Then, the condition concerning convergence order in the important theorem presented by Platen [7, 11] can be rewritten as follows [8]: there exist constants $K < \infty$ and $r \in \{1, 2, \ldots\}$ independent of h such that for all $n = 0, \ldots, M-1$ and $(p_1, \ldots, p_L) \in \{1, \ldots, d\}^L$ $(1 \le L \le 2q+1)$,

$$\left| E\left[\prod_{j=1}^{L} (\boldsymbol{y}_{n+1} - \boldsymbol{y}_n)_{p_j} - \prod_{j=1}^{L} (\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n)_{p_j} \middle| \mathcal{F}_n \right] \right| \le K(1 + \max_{0 \le k \le n} |\boldsymbol{y}_k|^{2r}) h^{q+1} \qquad \text{(w.p.1).} \tag{2. 4}$$

Here, $(\boldsymbol{z})_{p_j}$ and $\mathcal{F}_n$ denote, respectively, the p_j-th component of $\boldsymbol{z}$ and a non-anticipating sub-σ-algebra generated by the discretized Wiener processes $W_j(\tau_i)$'s $(0 \le i \le n, 1 \le j \le m)$. If (2. 4) is satisfied, the time discrete approximation $\boldsymbol{y}_M$ converges to the $\boldsymbol{y}(\tau_M)$ with weak (global) order q as $h \to 0$.

Furthermore, let us rewrite (2. 4) by utilizing the multi-colored tree expression. We can replace $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ in (2. 4) with $\boldsymbol{y}_{n+1,2q} - \boldsymbol{y}_n$ by noting that any term in the expansion of $\boldsymbol{y}_{n+1} - \boldsymbol{y}_{n+1,2q}$ centered at $\boldsymbol{y}_n$ does not prevent the inequality from holding. After the replacement, the substitution of the results in Theorems 2.1 and 2.2 into the expression in the left-hand side of (2. 4) yields

$$E\left[\prod_{j=1}^{L} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_{A(0)}}} \nu(\hat{t})\beta(\hat{t})\boldsymbol{F}(\hat{t})\bar{\Phi}_{(m+1)s+1}(t) \Big)_{p_j} - \prod_{j=1}^{L} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(t)+r(t)=i \\ t \in T}} \nu(t)\beta(t)\boldsymbol{F}(t)\Phi(t) \Big)_{p_j} \middle| \mathcal{F}_n \right]$$

$$= \sum_{i_1=1}^{2q} \sum_{\substack{\rho(\hat{t}_1)+r(\hat{t}_1)=i_1 \\ t_1 \in \mathcal{T}_{A(0)}}} \cdots \sum_{i_L=1}^{2q} \sum_{\substack{\rho(\hat{t}_L)+r(\hat{t}_L)=i_L \\ t_L \in \mathcal{T}_{A(0)}}} \prod_{j=1}^{L} (\nu(\hat{t}_j)\beta(\hat{t}_j)(\boldsymbol{F}(\hat{t}_j))_{p_j}) \times E\left[\prod_{j=1}^{L} \bar{\Phi}_{(m+1)s+1}(t_j) - \prod_{j=1}^{L} \Phi(\hat{t}_j) \middle| \mathcal{F}_n \right] \qquad \text{(w.p.1).}$$

Eventually, since $\tilde{\eta}_{i_a}^{(\cdot, j_b)}$ is independent of $\boldsymbol{y}_n$, (2. 4) holds if the condition

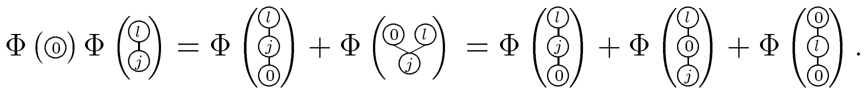
$$E\left[\prod_{j=1}^{L} \bar{\Phi}_{(m+1)s+1}(t_j) \right] = E\left[\prod_{j=1}^{L} \Phi(\hat{t}_j) \right] \tag{2. 5}$$

follows for any $t_1, \ldots, t_L \in \mathcal{T}_{A(0)}$ satisfying $\sum_{j=1}^{L}(\rho(\hat{t}_j) + r(\hat{t}_j)) \le 2q$.

Next, we show a way of seeking the expectation in the right-hand side of (2. 5) with the help of MRTs. In the multiple Stratonovich integrals, the usual chain rule holds as in the deterministic case. Hence, we can rewrite the product of elementary weights or the composition of subtrees in a elementary weight by the following:

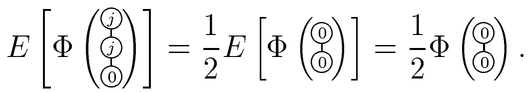
- The product of elementary weights of two MRTs t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to the root of t_2 and an MRT generated by grafting t_2 to the root of t_1.
- The elementary weight of an MRT having subtrees t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to t_2's own root and an MRT generated by grafting t_2 to t_1's own root.

For example, we have



In addition, by utilizing the relationship between multiple Stratonovich integrals and multiple Itô integrals ([11], p. 173), we can rewrite the expectations of MRTs whose each vertex has no more than one branch as follows:

- The expectation of an elementary weight vanishes unless the even number of vertices are of colors different from 0 and each of these vertices has a parent or child vertex with the same color.
- The expectation of an elementary weight of an MRT in which a vertex has a parent or child vertex with the same color is equal to a half of that of another MRT given by replacing the two vertices with one vertex with the color 0. For example,



  Note that there is no longer need of the expectation in the right-hand side.

In this way we obtain the expectations of elementary weights or the products of them in Appendix A.

Next, we give a way of seeking the expectations in the left-hand side of (2. 5) with the help of MRTL's. From the observation of calculations for some elementary numerical weights according to Definition 2.6, we can see that the $((m+1)s+1)$-st element of an elementary numerical weight can be obtained directly from a diagram for an MRTL by the following procedure.

- Trace vertices in the direction from the root to upper vertices.
- For the root vertex, prepare indices i_1 and j'_1 and write down $c_{i_1}^{(j'_1)}$. Then, write down $\tilde{\eta}_{i_1}^{(j'_1,j)}$ if the color is j.
- For each vertex except the root, prepare new indices i_{k+1} and j'_{k+1} and write down $\alpha_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1})}$ if the label is $A^{(j)}$, or $\tilde{\alpha}_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1})} \stackrel{\text{def}}{=} \alpha_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1})} + \tilde{\gamma}_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1})}$ if it is $\tilde{A}^{(j)}$, where i_k and j'_k mean the indices of the parent vertex. Then, write down $\tilde{\eta}_{i_{k+1}}^{(j'_{k+1},l)}$ if the color is l.
- Finally, sum over all values $(1,\dots,s)$ and $(0,\dots,m)$ of all indices in relation to i and j', respectively.

For example,

$$\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} \textcircled{0}_{\tilde{A}^{(j)}} \\ \textcircled{j}_{A^{(0)}} \end{pmatrix} = \sum_{i_1,i_2=1}^{s}\sum_{j_1',j_2'=0}^{m} c_{i_1}^{(j_1')}\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')}\tilde{\eta}_{i_2}^{(j_2',0)}.$$

Now, let us assume the following.

Assumption 2.1 *The expectation of the* $((m+1)s+1)$*-st element of an elementary numerical weight or the product of those is equal to* 0 *if the odd number of vertices are of the same color* $j(\neq 0)$.

Then, we can obtain the expectations of the $((m+1)s+1)$-st elements of elementary numerical weights or the products of them for weak order 2 as in Appendix B.

3 Weak order conditions in a commutative case

In the previous section we have shown the expression of weak order conditions by MRTL's. In this section we will obtain a similar expression under a commutativity condition.

3.1 Special case

In general, if elements t_1, t_2 in T differ in the structure or in the coloring, $\boldsymbol{F}(t_1) \neq \boldsymbol{F}(t_2)$ holds. Before the commutative case, we consider the expression of order conditions in a special case that $\boldsymbol{F}(t)$ is equivalent for some t's whose structures are the same but whose colorings differ.

Suppose that $\boldsymbol{F}(t)$ is equivalent for the t's under an assumption O. As preliminaries, we define a subset of T, say T_O, such that

- for each $t \in T$, an element u in T_O exists and satisfies $\boldsymbol{F}(t) = \boldsymbol{F}(u)$,
- for elements u_1, u_2 in T_O, when they differ in the coloring even if they are the same in the structure, $\boldsymbol{F}(u_1) \neq \boldsymbol{F}(u_2)$ holds.

In addition, we set

$$V_u \stackrel{\text{def}}{=} \{t \mid \boldsymbol{F}(t) = \boldsymbol{F}(u),\ t \in T\}$$

for $u \in T_O$. Then, from Theorem 2.1 we obtain

$$\boldsymbol{y}_{2q}(\tau_{n+1}) = \boldsymbol{y}_n + \sum_{i=1}^{2q} \sum_{\substack{\rho(u)+r(u)=i \\ u \in T_O}} \nu(u)\beta(u)\boldsymbol{F}(u)\left\{\sum_{t\in V_u}\Phi(t)\right\} \qquad (3.\ 1)$$

by noting that $\nu(t)$ and $\beta(t)$ do not depend on the coloring of $t \in V_u$.

Similarly, we define a subset of $\mathcal{T}_X$, say $\mathcal{T}_{X,O}$, such that

- for each $t \in \mathcal{T}_X$, an element u in $\mathcal{T}_{X,O}$ exists and satisfies $\boldsymbol{F}(\hat{t}) = \boldsymbol{F}(\hat{u})$,
- for elements u_1, u_2 in $\mathcal{T}_{X,O}$, when they differ in the coloring even if they are the same in the structure, $\boldsymbol{F}(\hat{u}_1) \neq \boldsymbol{F}(\hat{u}_2)$ holds,

and set

$$\mathcal{V}_{X,u} \stackrel{\text{def}}{=} \{t \mid \boldsymbol{F}(\hat{t}) = \boldsymbol{F}(\hat{u}),\ t \in \mathcal{T}_X\}$$

for $u \in \mathcal{T}_{X,O}$. Then, from Theorem 2.2 we obtain

$$\boldsymbol{y}_{n+1,2q} = \boldsymbol{y}_n + \sum_{i=1}^{2q} \sum_{\substack{\rho(u)+r(u)=i \\ u\in\mathcal{T}_{A^{(0)},O}}} \nu(\hat{u})\beta(\hat{u})\boldsymbol{F}(\hat{u}) \left\{ \sum_{t\in\mathcal{V}_{A^{(0)},u}} \bar{\Phi}_{(m+1)s+1}(t) \right\}. \tag{3. 2}$$

As in Subsection 2.3, from (2. 4), (3. 1) and (3. 2) we can see that (2. 4) holds if the condition

$$E\left[\prod_{j=1}^{L} \left\{ \sum_{t\in\mathcal{V}_{A^{(0)},u_j}} \bar{\Phi}_{(m+1)s+1}(t) \right\}\right] = E\left[\prod_{j=1}^{L} \left\{ \sum_{t\in\mathcal{V}_{A^{(0)},u_j}} \Phi(\hat{t}) \right\}\right] \tag{3. 3}$$

holds for any $u_1, \ldots, u_L \in \mathcal{T}_{A^{(0)},O}$ $(1 \le L \le 2q)$ satisfying $\sum_{j=1}^{L}(\rho(\hat{u}_j) + r(\hat{u}_j)) \le 2q$.

3.2 Commutative case

Let us consider the commutative case that for any $\boldsymbol{y}$ and $1 \le j, l \le m$ $(j \ne l)$

$$\boldsymbol{g}_j^{(1)}(\boldsymbol{y})\boldsymbol{g}_l(\boldsymbol{y}) = \boldsymbol{g}_l^{(1)}(\boldsymbol{y})\boldsymbol{g}_j(\boldsymbol{y}) \tag{3. 4}$$

holds. Fig. 4 shows the general form of the trees whose elementary differentials are equal under this commutativity condition. The big triangles stand for a common MRT.

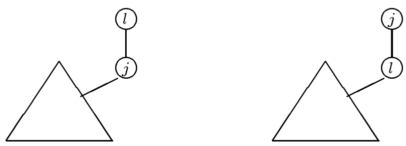
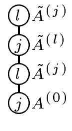
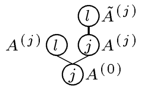
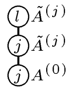
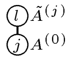


Figure 4: Generation of equivalent trees under the commutativity condition

Let us adopt the commutativity condition as O in the previous subsection. Among the MRTL's shown in Appendix B, we can choose those in Table 1 as $v \in \mathcal{T}_{A^{(0)},O}$ for which the number of elements in $\mathcal{V}_{A^{(0)},v}$ is greater than 1.

Table 1: Elements in $\mathcal{V}_{A^{(0)},v}$

i	1	2	3	4
v_i				

We are interested in combinations of MRTL's for which both sides of (3. 3) do not vanish. For some of such combinations under Assumption 2.1, let us demonstrate how to calculate the expression in the left-hand side of (3. 3). By setting that $L=1$ and $u_1=v_1$ in the expression, we obtain

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}+\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]$$

$$\begin{aligned}=&\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',l,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',l)}\right]\\ &+\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',l,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',l,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',j)}\right].\end{aligned}$$

Similarly, we can calculate it for v_2. Next, by setting that $L=2$, $u_1=v_3$ and $u_2=\tau_{A^{(0)}}^{(l)}$, we obtain

$$E\left[\left\{\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}+\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right\}\bar{\Phi}_{(m+1)s+1}\left(\textcircled{l}A^{(0)}\right)\right]$$

$$\begin{aligned}=&\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',j,j_3')}c_{i_4}^{(j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',l)}\right]\\ &+\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',l,j_3')}c_{i_4}^{(j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',l)}\right].\end{aligned}$$

For v_4, it needs to be calculated on two settings. One of them is that $L=2$ and $u_1=u_2=v_4$. The other is that $L=3$, $u_1=v_3$, $u_2=\tau_{A^{(0)}}^{(j)}$ and $u_3=\tau_{A^{(0)}}^{(l)}$.

On the other hand, let us calculate the expression in the right-hand side of (3. 3). It clearly vanishes when $L=1$ and $u_1=v_1$. For v_2, by setting that $L=1$ and $u_1=v_2$, we obtain

$$\begin{aligned}&E\left[\left\{\Phi\begin{pmatrix}&\textcircled{l}\\ \textcircled{l}&\textcircled{j}\\ \textcircled{j}&\end{pmatrix}+\Phi\begin{pmatrix}&\textcircled{j}\\ \textcircled{l}&\textcircled{l}\\ \textcircled{j}&\end{pmatrix}\right\}\right]\\ =\ &E\left[\Phi\begin{pmatrix}\textcircled{l}\\ \textcircled{j}\\ \textcircled{l}\\ \textcircled{j}\end{pmatrix}+\Phi\begin{pmatrix}\textcircled{l}\ \textcircled{l}\\ \textcircled{j}\\ \textcircled{j}\end{pmatrix}+\Phi\begin{pmatrix}\textcircled{j}\\ \textcircled{l}\\ \textcircled{l}\\ \textcircled{j}\end{pmatrix}+\Phi\begin{pmatrix}\textcircled{j}\ \textcircled{l}\\ \textcircled{l}\\ \textcircled{j}\end{pmatrix}\right]\\ =\ &E\left[\Phi\begin{pmatrix}\textcircled{l}\ \textcircled{l}\\ \textcircled{j}\\ \textcircled{j}\end{pmatrix}\right]=2E\left[\Phi\begin{pmatrix}\textcircled{l}\\ \textcircled{l}\\ \textcircled{j}\\ \textcircled{j}\end{pmatrix}\right]=E\left[\Phi\begin{pmatrix}\textcircled{0}\\ \textcircled{j}\\ \textcircled{j}\end{pmatrix}\right]=\frac{1}{2}E\left[\Phi\begin{pmatrix}\textcircled{0}\\ \textcircled{0}\end{pmatrix}\right]=\frac{1}{2}\Phi\begin{pmatrix}\textcircled{0}\\ \textcircled{0}\end{pmatrix}=\frac{1}{4}h^2.\end{aligned}$$

For v_3 or v_4, we can calculate similarly.

4 Solution of order conditions

In the previous section we have shown the weak order conditions by MRTL's, and demonstrated the calculation of the expectations of elementary weights and elementary numerical weights appearing in the conditions. In this section we will find a solution of the conditions for weak order 2 under the commutativity condition.

4.1 Simplifying assumption

As seen in (3. 3), the conditions for weak order are generally given in the form of expectations. By replacing expectations with monomials for trees which has only a few vertices, however, we can reduce the number of the order conditions. In relation to $\tau_{A^{(0)}}^{(0)}$, $\tau_{A^{(0)}}^{(j)}$,

$[\tau^{(j)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, $[\tau^{(j)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$, $[\tau^{(0)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$ and $[\tau^{(l)}_{\tilde{A}^{(j)}}]^{(j)}_{A^{(0)}}$, let us assume that the following equations hold (simplifying assumptions):

$$\begin{aligned}
&\sum c^{(j_1')}_{i_1}\tilde{\eta}^{(j_1',0)}_{i_1} = h, \quad \sum c^{(j_1')}_{i_1}\tilde{\eta}^{(j_1',j)}_{i_1} = \triangle W_j,\\
&\sum c^{(j_1')}_{i_1}\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}\tilde{\eta}^{(j_2',j)}_{i_2} = \frac{(\triangle W_j)^2}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\eta}^{(j_1',0)}_{i_1}\tilde{\alpha}^{(j_1',0j_2')}_{i_1 i_2}\tilde{\eta}^{(j_2',j)}_{i_2} = \frac{h\triangle W_j}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}\tilde{\eta}^{(j_2',0)}_{i_2} = \frac{h\triangle W_j}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}\tilde{\eta}^{(j_2',l)}_{i_2} = \frac{\triangle W_j\triangle W_l}{2} \qquad (j\neq l),
\end{aligned}$$

where $\triangle W_j$'s $(j=1,\ldots,m)$ are mutually independent random variables satisfying

$$E\left[(\triangle W_j)^k\right] = \begin{cases} 0 & (k=1,3,5),\\ (k-1)h^{k/2} & (k=2,4),\\ O(h^3) & (k\geq 6). \end{cases}$$

Then, the next order conditions are satisfied:

$$\begin{aligned}
&\sum c^{(j_1')}_{i_1}E\left[\tilde{\eta}^{(j_1',0)}_{i_1}\right] = h, \qquad \sum c^{(j_1')}_{i_1}c^{(j_2')}_{i_2}E\left[\tilde{\eta}^{(j_1',0)}_{i_1}\tilde{\eta}^{(j_2',0)}_{i_2}\right] = h^2,\\
&\sum c^{(j_1')}_{i_1}c^{(j_2')}_{i_2}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\right] = h,\\
&\sum c^{(j_1')}_{i_1}c^{(j_2')}_{i_2}c^{(j_3')}_{i_3}c^{(j_4')}_{i_4}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\tilde{\eta}^{(j_4',j)}_{i_4}\right] = 3h^2,\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\right] = \frac{h}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}\tilde{\alpha}^{(j_3',j,j_4')}_{i_3 i_4}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\tilde{\eta}^{(j_4',j)}_{i_4}\right] = \frac{3h^2}{4},\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}c^{(j_4')}_{i_4}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\tilde{\eta}^{(j_4',j)}_{i_4}\right] = \frac{3h^2}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',0,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}E\left[\tilde{\eta}^{(j_1',0)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\right] = \frac{h^2}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',0)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\right] = \frac{h^2}{2},\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}\tilde{\alpha}^{(j_3',j,j_4')}_{i_3 i_4}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',l)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\tilde{\eta}^{(j_4',l)}_{i_4}\right]\\
&+2\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}\tilde{\alpha}^{(j_3',l,j_4')}_{i_3 i_4}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',l)}_{i_2}\tilde{\eta}^{(j_3',l)}_{i_3}\tilde{\eta}^{(j_4',j)}_{i_4}\right]\\
&+\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',l,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}\tilde{\alpha}^{(j_3',l,j_4')}_{i_3 i_4}E\left[\tilde{\eta}^{(j_1',l)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\tilde{\eta}^{(j_3',l)}_{i_3}\tilde{\eta}^{(j_4',j)}_{i_4}\right] = h^2,\\
&\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',j,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}c^{(j_4')}_{i_4}E\left[\tilde{\eta}^{(j_1',j)}_{i_1}\tilde{\eta}^{(j_2',l)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\tilde{\eta}^{(j_4',l)}_{i_4}\right]\\
&+\sum c^{(j_1')}_{i_1}\tilde{\alpha}^{(j_1',l,j_2')}_{i_1 i_2}c^{(j_3')}_{i_3}c^{(j_4')}_{i_4}E\left[\tilde{\eta}^{(j_1',l)}_{i_1}\tilde{\eta}^{(j_2',j)}_{i_2}\tilde{\eta}^{(j_3',j)}_{i_3}\tilde{\eta}^{(j_4',l)}_{i_4}\right] = h^2.
\end{aligned}$$

4.2 Explicit stochastic Runge-Kutta methods

We consider the explicit stochastic Runge-Kutta methods under the commutativity condition, and show how to solve the order conditions.

First of all, we set $\eta_i^{(0,0)} = h$, $\eta_i^{(j,j)} = \triangle W_j$ $(j \geq 1)$, and $b_i^{(j',j)} = 1$ if $j' = j$, or 0 otherwise for any i, which mean

$$\tilde{\eta}_i^{(0,0)} = h, \qquad \tilde{\eta}_i^{(j,j)} = \triangle W_j \ (j \geq 1), \qquad \tilde{\eta}_i^{(j',j)} = 0 \ (j' \neq j). \tag{4. 1}$$

Next, let us set $\alpha_{i_a i_b}^{(j'_a,j,j'_b)} = 0$ if $j'_a \neq j$ and introduce

$$\alpha_{i_a i_b}^{(j,j'_b)} \stackrel{\text{def}}{=} \alpha_{i_a i_b}^{(j,j,j'_b)}, \qquad A_{i_a}^{(j,j'_b)} \stackrel{\text{def}}{=} \sum_{i_b=1}^{i_a-1} \alpha_{i_a i_b}^{(j,j'_b)}$$

for ease of notation.

Because (4. 1) lets Assumption 2.1 hold, from Appendices A and B we can see that the order conditions including the simplifying assumptions are as follows. In the sequel, we suppose $j, l \neq 0$ and $j \neq l$.

$$\sum c_{i_1}^{(0)} = 1, \tag{4. 2}$$
$$\sum c_{i_1}^{(j)} = 1, \tag{4. 3}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} = \frac{1}{2}, \tag{4. 4}$$
$$\sum c_{i_1}^{(0)} A_{i_1}^{(0,j)} = \frac{1}{2}, \tag{4. 5}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,0)} = \frac{1}{2}, \tag{4. 6}$$
$$\sum c_{i_1}^{(0)} A_{i_1}^{(0,0)} = \frac{1}{2}, \tag{4. 7}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,0)} = \frac{1}{4}, \tag{4. 8}$$
$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,j)} A_{i_2}^{(j,j)} = \frac{1}{4}, \tag{4. 9}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,0)} A_{i_2}^{(0,j)} = 0, \tag{4. 10}$$
$$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,j)}\right)^2 = \frac{1}{2}, \tag{4. 11}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,0)} A_{i_1}^{(j,j)} = \frac{1}{4}, \tag{4. 12}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \alpha_{i_2 i_3}^{(j,j)} A_{i_3}^{(j,j)} = \frac{1}{24}, \tag{4. 13}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \left(A_{i_2}^{(j,j)}\right)^2 = \frac{1}{12}, \tag{4. 14}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,j)} = \frac{1}{8}, \tag{4. 15}$$
$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,j)}\right)^3 = \frac{1}{4}, \tag{4. 16}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,j)} = \frac{1}{6}, \tag{4. 17}$$

$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,j)}\right)^2 = \frac{1}{3}, \tag{4. 18}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} = \frac{1}{2}, \tag{4. 19}$$
$$\sum c_{i_1}^{(j)} \left(A_{i_1}^{(j,l)}\right)^2 = \frac{1}{2}, \tag{4. 20}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} A_{i_1}^{(j,l)} = \frac{1}{4}, \tag{4. 21}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,l)} + \sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,j)} = \frac{1}{4}, \tag{4. 22}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,l)} = \frac{1}{4}, \tag{4. 23}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} \alpha_{i_1 i_2}^{(j,j)} A_{i_2}^{(j,l)} + \sum c_{i_1}^{(j)} A_{i_1}^{(j,l)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,j)} = \frac{1}{4}, \tag{4. 24}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,l)} = \frac{1}{8}, \tag{4. 25}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} A_{i_2}^{(l,l)} A_{i_2}^{(l,j)} = 0, \tag{4. 26}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \left(A_{i_2}^{(j,l)}\right)^2 = \frac{1}{4}, \tag{4. 27}$$
$$\sum c_{i_1}^{(j)} A_{i_1}^{(j,j)} \left(A_{i_1}^{(j,l)}\right)^2 = \frac{1}{4}, \tag{4. 28}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} \alpha_{i_2 i_3}^{(l,j)} A_{i_3}^{(j,l)} + \sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,l)} \alpha_{i_2 i_3}^{(l,l)} A_{i_3}^{(l,j)} = 0, \tag{4. 29}$$
$$\sum c_{i_1}^{(j)} \alpha_{i_1 i_2}^{(j,j)} \alpha_{i_2 i_3}^{(j,l)} A_{i_3}^{(l,l)} = \frac{1}{8}. \tag{4. 30}$$

Note that $\alpha_{i_a i_b}^{(j,j_b')} = 0$ $(i_a \geq i_b,\ \forall j, j_b')$ and $\gamma_{i_a i_b}^{(j_a',j,j_b')} = 0$ $(\forall i_a, i_b, j_a', j, j_b')$ because we consider explicit stochastic Runge-Kutta methods.

The system of the conditions (4. 3), (4. 4), (4. 13), (4. 14), (4. 15), (4. 16), (4. 17) and (4. 18) has the same algebraic structure as that of the order conditions for ordinary Runge-Kutta methods to attain order 4 in the deterministic mean (Butcher:2003, pp. 90-91). Hence, the stage number s has to be at least 4. In the sequel, let us deal with the case where $s = 4$.

Now that $s = 4$, (2. 3) can attain order 4 in the deterministic mean. For this, we add the following six conditions:

$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} \alpha_{i_2 i_3}^{(0,0)} A_{i_3}^{(0,0)} = \frac{1}{24}, \tag{4. 31}$$
$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} \left(A_{i_2}^{(0,0)}\right)^2 = \frac{1}{12}, \tag{4. 32}$$
$$\sum c_{i_1}^{(0)} A_{i_1}^{(0,0)} \alpha_{i_1 i_2}^{(0,0)} A_{i_2}^{(0,0)} = \frac{1}{8}, \tag{4. 33}$$
$$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,0)}\right)^3 = \frac{1}{4}, \tag{4. 34}$$
$$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,0)} A_{i_2}^{(0,0)} = \frac{1}{6}, \tag{4. 35}$$
$$\sum c_{i_1}^{(0)} \left(A_{i_1}^{(0,0)}\right)^2 = \frac{1}{3}, \tag{4. 36}$$

which come from $[[[\tau_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[[\tau_{A^{(0)}}^{(0)}, \tau_{A^{(0)}}^{(0)}]_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$, $[\tau_{A^{(0)}}^{(0)}, [\tau_{\tilde{A}^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}]_{A^{(0)}}^{(0)}$,

$[\tau^{(0)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$, $[[\tau^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{\tilde{A}^{(0)}}]^{(0)}_{A^{(0)}}$ and $[\tau^{(0)}_{A^{(0)}}, \tau^{(0)}_{A^{(0)}}]^{(0)}_{A^{(0)}}$, and are the counterparts of (4. 13), (4. 14), (4. 15), (4. 16), (4. 17) and (4. 18).

To find a solution, we first simplify the equations from (4. 19) to (4. 30). By noting that we may suppose $A_i^{(j,j)} = A_i^{(l,l)}$ and $A_i^{(j,l)} = A_i^{(l,j)}$ for any i, we have $\alpha_{43}^{(j,l)} A_2^{(j,l)} = 0$ from (4. 13), (4. 29) and (4. 30).

1) If $A_2^{(j,l)} = 0$, $\alpha_{43}^{(j,l)} A_3^{(l,l)} = 0$ holds from (4. 26) and (4. 27).

   - If $\alpha_{43}^{(j,l)} = 0$, $A_3^{(j,l)} = A_4^{(j,l)} = 1$ holds from (4. 22), (4. 24) and (4. 27).
   - If $A_3^{(l,l)} = 0$, $A_4^{(j,l)} = 1$ holds from (4. 21) and (4. 28). The substitution of $A_2^{(j,l)} = 0$ and $A_4^{(j,l)} = 1$ into (4. 19) and (4. 20) yields $A_3^{(j,l)} = 1$ (also note (4. 27)). Then, $\alpha_{43}^{(j,l)} = 0$ holds from (4. 22) and (4. 27).

2) If $\alpha_{43}^{(j,l)} = 0$, $A_2^{(l,j)} = 0$ holds from (4. 23) and (4. 26). Then, $A_3^{(j,l)} = A_4^{(j,l)} = 1$ holds from 1).

Hence, we have

$$\alpha_{43}^{(j,l)} = A_2^{(j,l)} = 0, \qquad A_3^{(j,l)} = A_4^{(j,l)} = 1.$$

By substituting these into the equations from (4. 19) to (4. 30) and rewriting them, we obtain

$$c_3^{(j)} + c_4^{(j)} = \frac{1}{2}, \tag{4. 37}$$

$$c_3^{(j)} A_3^{(j,j)} + c_4^{(j)} A_4^{(j,j)} = \frac{1}{4}, \tag{4. 38}$$

$$c_4^{(j)} \alpha_{43}^{(j,j)} = \frac{1}{4}, \tag{4. 39}$$

$$\alpha_{42}^{(j,l)} A_2^{(l,l)} = \frac{1}{2}, \tag{4. 40}$$

$$\alpha_{32}^{(j,l)} = \alpha_{42}^{(j,l)}. \tag{4. 41}$$

As we have mentioned, the system of the conditions (4. 3), (4. 4), (4. 13), (4. 14), (4. 15), (4. 16), (4. 17) and (4. 18) has the same algebraic structure as that of the order conditions for ordinary Runge-Kutta methods of order 4. Hence, we can utilize the results known in the deterministic case to solve the system of the order conditions. The following five special cases where a solution of it surely exists are known ([4], pp. 164–165):

Case I	$A_2^{(j,j)} \notin \{0, \frac{1}{2}, \frac{1}{2} \pm \frac{\sqrt{3}}{6}, 1\}, \quad A_3^{(j,j)} = 1 - A_2^{(j,j)}$,
Case II	$c_2^{(j)} = 0, \quad A_2^{(j,j)} \neq 0, \quad A_3^{(j,j)} = \frac{1}{2}$,
Case III	$c_3^{(j)} \neq 0, \quad A_2^{(j,j)} = \frac{1}{2}, \quad A_3^{(j,j)} = 0$,
Case IV	$c_4^{(j)} \neq 0, \quad A_2^{(j,j)} = 1, \quad A_3^{(j,j)} = \frac{1}{2}$,
Case V	$c_3^{(j)} \neq 0, \quad A_2^{(j,j)} = A_3^{(j,j)} = \frac{1}{2}$.

In Cases I and V, for example, the solutions are given by the following Butcher tableaux

$$\begin{array}{c|c} \boldsymbol{A}^{(j,j)} & \left[\alpha_{i_a i_b}^{(j,j)}\right] \\ \hline & \left(\boldsymbol{c}^{(j)}\right)^{\mathrm{T}} \end{array},$$

respectively:

Case I

$$
\begin{array}{c|cccc}
0 & & & & \\
\delta_0 & \delta_0 & & & \\
A_3^{(j,j)} & \dfrac{A_3^{(j,j)}\delta_1}{2\delta_0} & \dfrac{A_3^{(j,j)}}{2\delta_0} & & \\
1 & \dfrac{12\left(A_3^{(j,j)}\right)^3 - 24\left(A_3^{(j,j)}\right)^2 + 17A_3^{(j,j)} - 4}{2\delta_0\delta_2} & \dfrac{A_3^{(j,j)}\delta_1}{2\delta_0\delta_2} & \dfrac{\delta_0}{\delta_2} & \\
\hline
 & \dfrac{\delta_2}{12A_3^{(j,j)}\delta_0} & \dfrac{1}{12A_3^{(j,j)}\delta_0} & \dfrac{1}{12A_3^{(j,j)}\delta_0} & \dfrac{\delta_2}{12A_3^{(j,j)}\delta_0}
\end{array},
$$

where $\delta_0 \stackrel{\text{def}}{=} 1 - A_3^{(j,j)}$, $\delta_1 \stackrel{\text{def}}{=} 1 - 2A_3^{(j,j)}$ and $\delta_2 \stackrel{\text{def}}{=} 6A_3^{(j,j)} - 1 - 6\left(A_3^{(j,j)}\right)^2$, and

Case V

$$
\begin{array}{c|cccc}
0 & & & & \\
\dfrac{1}{2} & \dfrac{1}{2} & & & \\
\dfrac{1}{2} & \dfrac{1}{2} - \dfrac{1}{6A_3^{(j,j)}} & \dfrac{1}{6A_3^{(j,j)}} & & \\
1 & 0 & 1 - 3A_3^{(j,j)} & 3A_3^{(j,j)} & \\
\hline
 & \dfrac{1}{6} & \dfrac{2}{3} - A_3^{(j,j)} & A_3^{(j,j)} & \dfrac{1}{6}
\end{array}.
$$

The solutions in Cases II, III, IV and V, however, do not satisfy (4. 37) and (4. 39), simultaneously. Hence, the steps that are taken to find a solution of all the conditions are as follows:

Step 1) Select the solution in Case I as that of the system (4. 3), (4. 4), (4. 13), (4. 14), (4. 15), (4. 16), (4. 17) and (4. 18), substitute it into (4. 37), (4. 38), (4. 39), (4. 40) and (4. 41), and solve them.

Step 2) Select one among the five cases above and adopt its solution as that of the system (4. 2), (4. 7), (4. 31), (4. 32), (4. 33), (4. 34), (4. 35) and (4. 36) since those are exactly the conditions for ordinary Runge-Kutta methods of order 4.

Step 3) Substitute the solution in Step 1) into (4. 6), (4. 8) and (4. 12), and seek $A_2^{(j,0)}$, $A_3^{(j,0)}$ or $A_4^{(j,0)}$. Here, note that the equations are not linearly independent with respect to the parameters.

Step 4) Substitute the solution in Step 2) into (4. 5) and (4. 11), and seek $A_2^{(0,j)}$, $A_3^{(0,j)}$ or $A_4^{(0,j)}$. Here, it is remarkable that the equations are equivalent when the parameters are 0 or 1 except for $A_2^{(0,j)} = A_3^{(0,j)} = A_4^{(0,j)} = 0$.

Step 5) Substitute the results in Steps 1) and 4) into (4. 10), and seek $\alpha_{32}^{(j,0)}$, $\alpha_{42}^{(j,0)}$ or $\alpha_{43}^{(j,0)}$. Here, it is remarkable that the equation has the trivial solution $\alpha_{32}^{(j,0)} = \alpha_{42}^{(j,0)} = \alpha_{43}^{(j,0)} = 0$.

Step 6) Substitute the results in Steps 2) and 4) into (4. 9), and seek $\alpha_{32}^{(0,j)}$, $\alpha_{42}^{(0,j)}$ or $\alpha_{43}^{(0,j)}$.

By following the steps, let us find a solution of all the conditions. In Step 1), from the solution in Case I and (4. 39) we have $A_3^{(j,j)} = \frac{1}{3}$. Then, (4. 37) and (4. 38) hold since $c_3^{(j)} = \frac{3}{8}$ and $c_4^{(j)} = \frac{1}{8}$. From (4. 40) and (4. 41), $\alpha_{32}^{(j,l)} = \alpha_{42}^{(j,l)} = \frac{3}{4}$. We chose the solution of Case V in Step 2). We obtain

$$A_2^{(j,0)} = -2A_3^{(j,0)} + 2, \quad A_4^{(j,0)} = 3A_3^{(j,0)} - 2$$

in Step 3). Let us set $A_2^{(0,j)} = A_4^{(0,j)} = 1$ and $A_3^{(0,j)} = 0$ in Step 4). This makes (4. 5) and (4. 11) equivalent and means $c_2^{(0)} + c_4^{(0)} = \frac{1}{2}$. Hence, $c_3^{(0)} = \frac{1}{3}$ in the present case. In Step 5) let us set $\alpha_{32}^{(j,0)} = \alpha_{42}^{(j,0)} = \alpha_{43}^{(j,0)} = 0$. In Step 6) we set $\alpha_{42}^{(0,j)} = \alpha_{43}^{(0,j)} = 0$ and obtain $\alpha_{32}^{(0,j)} = \frac{9}{8}$.

We finally obtain as a solution of the conditions

$$\begin{array}{c|cc}
\left[\alpha_{i_a i_b}^{(0,0)}\right] & \left[\alpha_{i_a i_b}^{(j,0)}\right] & \\
\left[\alpha_{i_a i_b}^{(0,j)}\right] & \left[\alpha_{i_a i_b}^{(j,j)}\right] & \left[\alpha_{i_a i_b}^{(j,l)}\right] \\
\hline
\left(\boldsymbol{c}^{(0)}\right)^{\mathrm{T}} & \left(\boldsymbol{c}^{(j)}\right)^{\mathrm{T}} &
\end{array}
=
\begin{array}{cccc|ccccccc}
 & & & & & & & & & & \\
\frac{1}{2} & & & & 2-2\alpha_{31}^{(j,0)} & & & & & & \\
0 & \frac{1}{2} & & & \alpha_{31}^{(j,0)} & 0 & & & & & \\
0 & 0 & 1 & & 3\alpha_{31}^{(j,0)}-2 & 0 & 0 & & & & \\
 & & & & & & & & & & \\
1 & & & & \frac{2}{3} & & & & 0 & & \\
-\frac{9}{8} & \frac{9}{8} & & & \frac{1}{12} & \frac{1}{4} & & & \frac{1}{4} & \frac{3}{4} & \\
1 & 0 & 0 & & -\frac{5}{4} & \frac{1}{4} & 2 & & \frac{1}{4} & \frac{3}{4} & 0 \\
\hline
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & & &
\end{array} .$$

4.3 Numerical experiments

We show the results of numerical experiments to confirm that the explicit scheme in the previous subsection attains weak order 2 when $\alpha_{31}^{(j,0)} = 0$. Here, the following two SDEs are considered. The first one is

$$\begin{aligned}
\mathrm{d}\boldsymbol{y}(t) &= \left(R - \frac{1}{2}\sum_{j=1}^{m} B_j^2\right)\boldsymbol{y}(t)\mathrm{d}t + \sum_{j=1}^{m} B_j \boldsymbol{y}(t) \circ \mathrm{d}W_j(t), \quad 0 \le t \le 1, \\
\boldsymbol{y}(0) &= \boldsymbol{y}_0.
\end{aligned} \tag{4. 42}$$

The commutativity condition is given by $B_j B_l = B_l B_j$ $(j \ne l)$. This is, for example, satisfied when the matrices are diagonal. The second one is

$$\begin{aligned}
\mathrm{d}\boldsymbol{y}(t) &= \left(R\boldsymbol{y}(t) - \frac{1}{4}\sum_{j=1}^{2}\begin{bmatrix} b_{j1} \\ b_{j2} \end{bmatrix} \frac{\partial Q}{\partial \boldsymbol{y}}(\boldsymbol{y}(t)) \begin{bmatrix} b_{j1} \\ b_{j2} \end{bmatrix}\right)\mathrm{d}t \\
&\quad + \sqrt{Q(\boldsymbol{y}(t))}\sum_{j=1}^{2}\begin{bmatrix} b_{j1} \\ b_{j2} \end{bmatrix} \circ \mathrm{d}W_j(t), \quad 0 \le t \le 1, \\
\boldsymbol{y}(0) &= \boldsymbol{y}_0,
\end{aligned} \tag{4. 43}$$

where $Q(\boldsymbol{y})$ is a non-negative function. The commutativity condition is given by $b_{11}b_{22} = b_{12}b_{21}$.

In (4. 42), we set $m = 2$,

$$R = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix}, \quad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, we sought $\boldsymbol{y}_M$ by means of the scheme, and calculated the arithmetic variances $\langle y_{M,i}^2 \rangle - \langle y_{M,i} \rangle^2$ of the ith element of $\boldsymbol{y}_M$ and $\langle y_{M,1} y_{M,2} \rangle$ as approximate values of variances $V[y_i(1)]$ $(i = 1, 2)$ and $E[y_1(1) y_2(1)]$, respectively. The notation $\langle \cdot \rangle$ stands for an arithmetic mean. On the other hand, the their exact values were sought by means of $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = RE[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t) y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 2 & \frac{1}{16} \\ -3 & -\frac{27}{16} & 1 \\ \frac{1}{16} & -6 & -\frac{15}{4} \end{bmatrix} \begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t) y_2(t)] \\ E[y_2^2(t)] \end{bmatrix}.$$

In (4. 43), we set $b_{11} = b_{12} = \frac{1}{2}$, $b_{21} = b_{22} = \frac{1}{4}$,

$$R = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \quad Q(\boldsymbol{y}) \stackrel{\mathrm{def}}{=} y_1^2 - y_1 y_2 + y_2^2 + 1, \quad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution satisfies $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = RE[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t) y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{27}{16} & \frac{5}{16} \\ -\frac{43}{16} & -\frac{37}{16} & \frac{21}{16} \\ \frac{5}{16} & -\frac{101}{16} & -\frac{59}{16} \end{bmatrix} \begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t) y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} + \begin{bmatrix} \frac{5}{16} \\ \frac{5}{16} \\ \frac{5}{16} \end{bmatrix}.$$

In both experiments, 1×10^6 sets of independent trajectories were simulated for each step. The results are indicated in Figures 5 and 6. These illustrate that the scheme is of weak order 2.

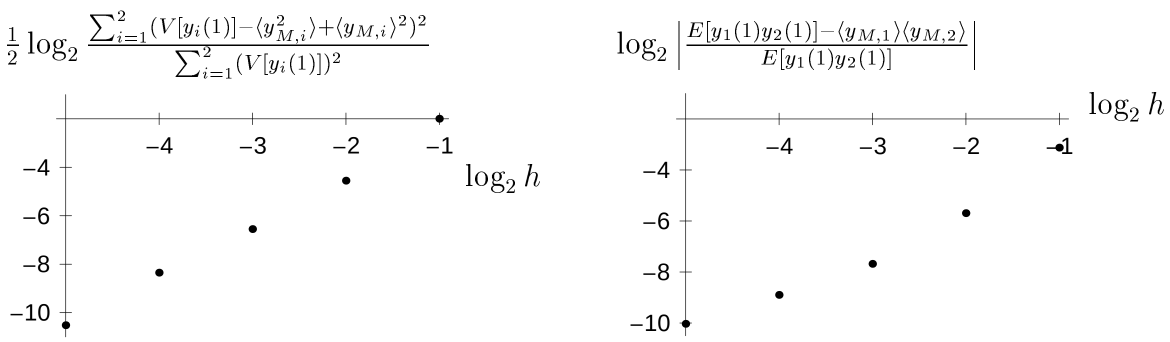


Figure 5: Relative errors in (4. 42)

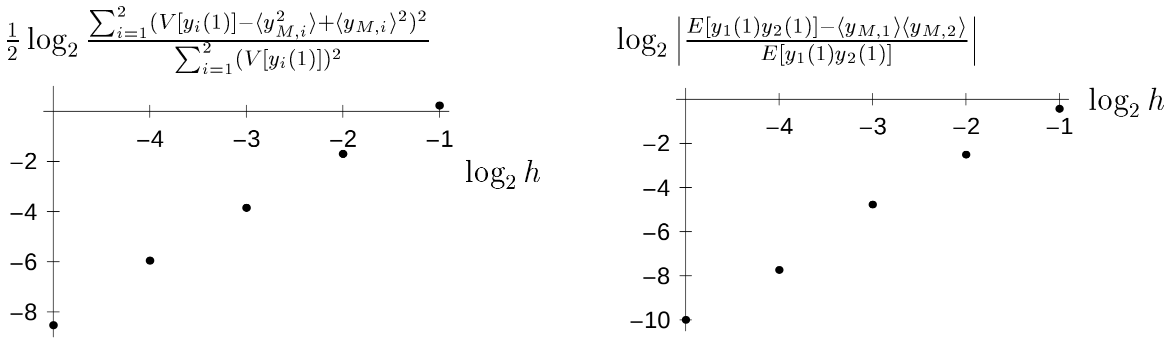


Figure 6: Relative errors in (4. 43)

5 Summary

First, we have generally derived the expression of the weak order conditions of the stochastic Runge-Kutta family in a situation that some derivatives of a drift or diffusion coefficients are equivalent. On the basis of that, we have obtained the order conditions in the commutative case. Second, after adding the six conditions for order 4 in the deterministic mean, we have found a solution of all the conditions with the help of the simplifying conditions and the results in ODEs. Third, we have performed the numerical experiments and shown the explicit stochastic Runge-Kutta method with four stages is of weak order 2.

Appendix

A Expectations of elementary weights

We show the expectations of elementary weights or the products of them for weak order 2. In what follows, we suppose $j, l \neq 0$ and $j \neq l$ for ease of notation. Only the expectations that does not vanish are shown here.

$$
\begin{aligned}
&E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ |\\ \textcircled{0}\end{smallmatrix}\right)\right]=\frac{1}{2}h^2,\\
&E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ |\\ \textcircled{j}\\ |\\ \textcircled{j}\end{smallmatrix}\right)\right]=\frac{1}{2}E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ |\\ \textcircled{0}\end{smallmatrix}\right)\right]=\frac{1}{4}h^2,\\
&E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ |\\ \textcircled{j}\\ |\\ \textcircled{0}\end{smallmatrix}\right)\right]=\frac{1}{2}E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ |\\ \textcircled{0}\end{smallmatrix}\right)\right]=\frac{1}{4}h^2,\\
&E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}&&\textcircled{j}\\ &\textcircled{0}&\end{smallmatrix}\right)\right]=2E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ |\\ \textcircled{j}\\ |\\ \textcircled{0}\end{smallmatrix}\right)\right]=\frac{1}{2}h^2,\\
&E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}&&\textcircled{j}\\ &\textcircled{j}&\end{smallmatrix}\right)\right]=E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ |\\ \textcircled{0}\\ |\\ \textcircled{j}\end{smallmatrix}\right)\right]+E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ |\\ \textcircled{j}\\ |\\ \textcircled{j}\end{smallmatrix}\right)\right]=E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ |\\ \textcircled{j}\\ |\\ \textcircled{j}\end{smallmatrix}\right)\right]=\frac{1}{4}h^2,
\end{aligned}
$$

$$E\left[\Phi\left(j(j(j(j)))\right)\right] = \frac{1}{2}E\left[\Phi\left(j(j(0))\right)\right] = \frac{1}{8}h^2,$$

$$E\left[\Phi\left(j(j(l(l)))\right)\right] = \frac{1}{2}E\left[\Phi\left(j(j(0))\right)\right] = \frac{1}{8}h^2,$$

$$E\left[\Phi\left(j(j(j,j))\right)\right] = 2E\left[\Phi\left(j(j(j(j(j))))\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(j(j(l,l))\right)\right] = 2E\left[\Phi\left(j(j(j(l(l))))\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(j(j,j(j))\right)\right] = E\left[\Phi\left(j(j(j(j)))\right)\right] + E\left[\Phi\left(j(j(j,j))\right)\right] = \frac{3}{8}h^2,$$

$$E\left[\Phi\left(j(j,l(l))\right)\right] = E\left[\Phi\left(j(j(l(l)))\right)\right] + E\left[\Phi\left(j(l(j,l))\right)\right] = E\left[\Phi\left(j(j(l(l)))\right)\right] = \frac{1}{8}h^2,$$

$$E\left[\Phi\left(j(l,j(l))\right)\right] = E\left[\Phi\left(j(l(j(l)))\right)\right] + E\left[\Phi\left(j(j(l,l))\right)\right] = E\left[\Phi\left(j(l,l)\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(j(j,j,j)\right)\right] = 2E\left[\Phi\left(j(j,j(j))\right)\right] = \frac{3}{4}h^2,$$

$$E\left[\Phi\left(j(j,l,l)\right)\right] = 2E\left[\Phi\left(j(j,l(l))\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(j(0)\right)\Phi\left(j\right)\right] = E\left[\Phi\left(j(0,j)\right)\right] + E\left[\Phi\left(j(j(0))\right)\right] = \frac{1}{2}h^2,$$

$$E\left[\Phi\left(0(j)\right)\Phi\left(j\right)\right] = E\left[\Phi\left(0(j,j)\right)\right] + E\left[\Phi\left(j(0(j))\right)\right] = E\left[\Phi\left(0(j,j)\right)\right] = \frac{1}{2}h^2,$$

$$E\left[\Phi\left(j(j(j))\right)\Phi\left(j\right)\right] = E\left[\Phi\left(j(j,j(j))\right)\right] + E\left[\Phi\left(j(j(j(j)))\right)\right] = \frac{1}{2}h^2,$$

$$E\left[\Phi\left(j(j(l))\right)\Phi\left(l\right)\right] = E\left[\Phi\left(j(l,j(l))\right)\right] + E\left[\Phi\left(l(j(j(l)))\right)\right] = E\left[\Phi\left(j(l,j(l))\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(j(l(l))\right)\Phi\left(j\right)\right] = E\left[\Phi\left(j(j,l(l))\right)\right] + E\left[\Phi\left(j(j(l(l)))\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(j(j,j)\right)\Phi\left(j\right)\right] = E\left[\Phi\left(j(j,j,j)\right)\right] + E\left[\Phi\left(j(j(j,j))\right)\right] = h^2,$$

$$E\left[\Phi\left(j(j,l)\right)\Phi\left(l\right)\right] = E\left[\Phi\left(j(j,l,l)\right)\right] + E\left[\Phi\left(l(j(j,l))\right)\right] = E\left[\Phi\left(j(j,l,l)\right)\right] = \frac{1}{4}h^2,$$

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{l}\ \textcircled{l}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\textcircled{j}\right)\right]=E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\,\textcircled{l}\,\textcircled{l}\\ \textcircled{j}\end{smallmatrix}\right)\right]+E\left[\Phi\left(\begin{smallmatrix}\textcircled{l}\ \textcircled{l}\\ \textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\right]=\frac{1}{2}h^2,$$

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\right]=\frac{1}{2}E\left[\Phi\left(\textcircled{0}\right)\right]=\frac{1}{2}h,$$

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\right]=2E\left[\Phi\left(\begin{smallmatrix} & \textcircled{j}\\ \textcircled{j} & \textcircled{j}\\ \textcircled{j} & \end{smallmatrix}\right)\right]=\frac{3}{4}h^2,$$

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\right]=2E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\right]=\frac{3}{2}h^2,$$

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{l}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\begin{smallmatrix}\textcircled{l}\\ \textcircled{j}\end{smallmatrix}\right)\right]=2E\left[\Phi\left(\begin{smallmatrix} & \textcircled{l}\\ \textcircled{l} & \textcircled{j}\\ \textcircled{j} & \end{smallmatrix}\right)\right]=\frac{1}{2}h^2,$$

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{l}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{l}\right)\right]=E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\ \textcircled{l}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\textcircled{l}\right)\right]+E\left[\Phi\left(\begin{smallmatrix}\textcircled{l}\\ \textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\textcircled{l}\right)\right]=\frac{1}{2}h^2,$$

$$E\left[\Phi\left(\textcircled{0}\right)\right]=h,$$

$$E\left[\Phi\left(\textcircled{0}\right)\Phi\left(\textcircled{0}\right)\right]=2E\left[\Phi\left(\begin{smallmatrix}\textcircled{0}\\ \textcircled{0}\end{smallmatrix}\right)\right]=h^2,$$

$$E\left[\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\right]=2E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\right]=h,$$

$$E\left[\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\right]=4E\left[\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\Phi\left(\begin{smallmatrix}\textcircled{j}\\ \textcircled{j}\end{smallmatrix}\right)\right]=3h^2$$

B Expectations of elementary numerical weights

We show only the expectations that does not vanish, of elementary numerical weights or the products of them for weak order 2. For ease of notation, we omit all indices and the range of values of all indices in all summations.

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{0}\tilde{A}^{(0)}\\ \textcircled{0}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',0,j_2')}E\left[\tilde{\eta}_{i_1}^{(j_1',0)}\tilde{\eta}_{i_2}^{(j_2',0)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{0}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',j,j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',0)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(0)}\\ \textcircled{0}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',0,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',j,j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',0)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{j}\tilde{A}^{(0)}\\ \textcircled{0}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',0,j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',0)}\tilde{\eta}_{i_3}^{(j_3',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}A^{(0)}\textcircled{j}\ \textcircled{j}A^{(0)}\\ \textcircled{0}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',0,j_2')}\alpha_{i_1i_3}^{(j_1',0,j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',0)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}A^{(j)}\textcircled{0}\ \textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',0)}\tilde{\eta}_{i_3}^{(j_3',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',j,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{l}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{smallmatrix}\right)\right]=\sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',j,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',l,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{l}\,\tilde{A}^{(j)}\\ \vert\\ \textcircled{j}\,\tilde{A}^{(l)}\\ \vert\\ \textcircled{l}\,\tilde{A}^{(j)}\\ \vert\\ \textcircled{j}\,A^{(0)}\end{pmatrix}\right] = \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',l,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{j}\,\tilde{A}^{(l)}\\ \vert\\ \textcircled{l}\,\tilde{A}^{(l)}\\ \vert\\ \textcircled{l}\,\tilde{A}^{(j)}\\ \vert\\ \textcircled{j}\,A^{(0)}\end{pmatrix}\right] = \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\tilde{\alpha}_{i_2i_3}^{(j_2',l,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',l,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}A^{(j)}\,\textcircled{j} & & \textcircled{j}\,A^{(j)}\\ & \textcircled{j}\,\tilde{A}^{(j)} & \\ & \vert & \\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_2i_3}^{(j_2',j,j_3')}\alpha_{i_2i_4}^{(j_2',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}A^{(j)}\,\textcircled{l} & & \textcircled{l}\,A^{(j)}\\ & \textcircled{j}\,\tilde{A}^{(j)} & \\ & \vert & \\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_2i_3}^{(j_2',j,j_3')}\alpha_{i_2i_4}^{(j_2',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}A^{(l)}\,\textcircled{l} & & \textcircled{j}\,A^{(l)}\\ & \textcircled{l}\,\tilde{A}^{(j)} & \\ & \vert & \\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_2i_3}^{(j_2',l,j_3')}\alpha_{i_2i_4}^{(j_2',l,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} & & \textcircled{j}\,\tilde{A}^{(j)}\\ & & \vert\\ A^{(j)}\,\textcircled{j} & & \textcircled{j}\,A^{(j)}\\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} & & \textcircled{l}\,\tilde{A}^{(l)}\\ & & \vert\\ A^{(j)}\,\textcircled{j} & & \textcircled{l}\,A^{(j)}\\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',l,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} & & \textcircled{l}\,\tilde{A}^{(j)}\\ & & \vert\\ A^{(j)}\,\textcircled{l} & & \textcircled{j}\,A^{(j)}\\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix} & & \textcircled{j}\,\tilde{A}^{(l)}\\ & & \vert\\ A^{(j)}\,\textcircled{l} & & \textcircled{l}\,A^{(j)}\\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}\tilde{\alpha}_{i_3i_4}^{(j_3',l,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',l)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}A^{(j)}\,\textcircled{j} & \overset{A^{(j)}}{\textcircled{j}} & \textcircled{j}\,A^{(j)}\\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}\alpha_{i_1i_4}^{(j_1',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}A^{(j)}\,\textcircled{j} & \overset{A^{(j)}}{\textcircled{l}} & \textcircled{l}\,A^{(j)}\\ & \textcircled{j}\,A^{(0)} & \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')}\alpha_{i_1i_2}^{(j_1',j,j_2')}\alpha_{i_1i_3}^{(j_1',j,j_3')}\alpha_{i_1i_4}^{(j_1',j,j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',l)}\tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{0}\,\tilde{A}^{(j)}\\ \vert\\ \textcircled{j}\,A^{(0)}\end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(\textcircled{j}\,A^{(0)}\right)\right] = \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',j,j_2')}c_{i_3}^{(j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',0)}\tilde{\eta}_{i_3}^{(j_3',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{j}\,\tilde{A}^{(0)}\\ \vert\\ \textcircled{0}\,A^{(0)}\end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(\textcircled{j}\,A^{(0)}\right)\right] = \sum c_{i_1}^{(j_1')}\tilde{\alpha}_{i_1i_2}^{(j_1',0,j_2')}c_{i_3}^{(j_3')}E\left[\tilde{\eta}_{i_1}^{(j_1',0)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\begin{pmatrix}\textcircled{j}\,\tilde{A}^{(j)}\\ \vert\\ \textcircled{j}\,\tilde{A}^{(j)}\\ \vert\\ \textcircled{j}\,A^{(0)}\end{pmatrix}\bar{\Phi}_{(m+1)s+1}\left(\textcircled{j}\,A^{(0)}\right)\right]$$

$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} \tilde{\alpha}_{i_2 i_3}^{(j_2',j,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{l}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} \tilde{\alpha}_{i_2 i_3}^{(j_2',j,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)} \tilde{\eta}_{i_3}^{(j_3',l)} \tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{l}\tilde{A}^{(l)} \\ \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{j}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} \tilde{\alpha}_{i_2 i_3}^{(j_2',l,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',l)} \tilde{\eta}_{i_3}^{(j_3',l)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{j}\tilde{A}^{(l)} \\ \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{l}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} \tilde{\alpha}_{i_2 i_3}^{(j_2',l,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',l)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} A^{(j)}\textcircled{j} \quad \textcircled{j}A^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{j}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \alpha_{i_1 i_2}^{(j_1',j,j_2')} \alpha_{i_1 i_3}^{(j_1',j,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} A^{(j)}\textcircled{j} \quad \textcircled{l}A^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{l}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \alpha_{i_1 i_2}^{(j_1',j,j_2')} \alpha_{i_1 i_3}^{(j_1',j,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)} \tilde{\eta}_{i_3}^{(j_3',l)} \tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} A^{(j)}\textcircled{l} \quad \textcircled{l}A^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{j}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \alpha_{i_1 i_2}^{(j_1',j,j_2')} \alpha_{i_1 i_3}^{(j_1',j,j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',l)} \tilde{\eta}_{i_3}^{(j_3',l)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{j}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix}\right] = \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{j}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{j}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} c_{i_3}^{(j_3')} \tilde{\alpha}_{i_3 i_4}^{(j_3',j,j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{j}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{j}A^{(0)}\right) \bar{\Phi}_{(m+1)s+1} \left(\textcircled{j}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} c_{i_3}^{(j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',j)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} c_{i_3}^{(j_3')} \tilde{\alpha}_{i_3 i_4}^{(j_3',j,j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',l)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{j}\tilde{A}^{(l)} \\ \textcircled{l}A^{(0)} \end{pmatrix}\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} c_{i_3}^{(j_3')} \tilde{\alpha}_{i_3 i_4}^{(j_3',l,j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',l)} \tilde{\eta}_{i_3}^{(j_3',l)} \tilde{\eta}_{i_4}^{(j_4',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \begin{pmatrix} \textcircled{l}\tilde{A}^{(j)} \\ \textcircled{j}A^{(0)} \end{pmatrix} \bar{\Phi}_{(m+1)s+1} \left(\textcircled{j}A^{(0)}\right) \bar{\Phi}_{(m+1)s+1} \left(\textcircled{l}A^{(0)}\right)\right]$$
$$= \sum c_{i_1}^{(j_1')} \tilde{\alpha}_{i_1 i_2}^{(j_1',j,j_2')} c_{i_3}^{(j_3')} c_{i_4}^{(j_4')} E\left[\tilde{\eta}_{i_1}^{(j_1',j)} \tilde{\eta}_{i_2}^{(j_2',l)} \tilde{\eta}_{i_3}^{(j_3',j)} \tilde{\eta}_{i_4}^{(j_4',l)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \left(\textcircled{0}A^{(0)}\right)\right] = \sum c_{i_1}^{(j_1')} E\left[\tilde{\eta}_{i_1}^{(j_1',0)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1} \left(\textcircled{0}A^{(0)}\right) \bar{\Phi}_{(m+1)s+1} \left(\textcircled{0}A^{(0)}\right)\right] = \sum c_{i_1}^{(j_1')} c_{i_2}^{(j_2')} E\left[\tilde{\eta}_{i_1}^{(j_1',0)} \tilde{\eta}_{i_2}^{(j_2',0)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(ⓙ_{A^{(0)}}\right)\bar{\Phi}_{(m+1)s+1}\left(ⓙ_{A^{(0)}}\right)\right]=\sum c_{i_1}^{(j_1')}c_{i_2}^{(j_2')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\right],$$

$$E\left[\bar{\Phi}_{(m+1)s+1}\left(ⓙ_{A^{(0)}}\right)\bar{\Phi}_{(m+1)s+1}\left(ⓙ_{A^{(0)}}\right)\bar{\Phi}_{(m+1)s+1}\left(ⓙ_{A^{(0)}}\right)\bar{\Phi}_{(m+1)s+1}\left(ⓙ_{A^{(0)}}\right)\right]$$
$$=\sum c_{i_1}^{(j_1')}c_{i_2}^{(j_2')}c_{i_3}^{(j_3')}c_{i_4}^{(j_4')}E\left[\tilde{\eta}_{i_1}^{(j_1',j)}\tilde{\eta}_{i_2}^{(j_2',j)}\tilde{\eta}_{i_3}^{(j_3',j)}\tilde{\eta}_{i_4}^{(j_4',j)}\right].$$

The above expectations are expressed with the notation in (2. 3). By substituting $\tilde{\eta}_i^{(\cdot,j_a)}=\eta_i^{(\cdot,j_a)}b_i^{(\cdot,j_a)}$ and $\tilde{\gamma}_{il}^{(\cdot,j_a,*)}=\gamma_{il}^{(\cdot,j_a,*)}/b_i^{(\cdot,j_a)}$ for $j_a\in\{0,j,l\}$, however, we can readily obtain the expressions of the expectations for (2. 2).

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