Multi-dimensional linear stability analysis of S-ROCK methods for Itô stochastic differential equations

Yoshio Komori* and Kevin Burrage†,**

*Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka, Japan
†Department of Computer Science, University of Oxford, Wolfson Building, Parks Road, Oxford, UK
**Discipline of Mathematics, Queensland University of Technology, Brisbane, Australia

Abstract. In this paper, stochastic orthogonal Runge-Kutta Chebyshev methods are dealt with for strong approximations to solutions of Itô stochastic differential equations (SDEs). Recently, strong first order methods for non-commutative Itô SDEs have been proposed by the present authors. It is known that when the number of stages is large, the methods have very large stability domains in mean square (MS) for a scalar linear test equation. On the other hand, Buckwar and Sickenberger (2012) have recently proposed MS stability analyses for systems of Itô SDEs. Our aim is to investigate MS stability properties of these methods and other existing numerical schemes by their approach.

Keywords: stochastic orthogonal Runge-Kutta Chebyshev method, mean square stability, strong approximation, explicit scheme
PACS: 2010 MSC: 60H10, 65L20, 65L06

INTRODUCTION

We are concerned with stochastic Runge-Kutta (SRK) methods for strong approximations to solutions of non-commutative Itô stochastic differential equations (SDEs). Among them, explicit and derivative-free SRK methods are very important due to low computational costs especially when nonlinear SDEs are considered and the dimension of them is not small. We can typically give the following disadvantages about other methods:

- implicit or drift-implicit SRK methods [1, 2, 3, 4] can lead to solving a large nonlinear system of equations if the drift coefficient of SDEs is nonlinear,
- numerical methods which need derivatives [5] have to spend much effort to calculate them as the number of the diffusion coefficients becomes large when they are nonlinear.

In general, however, explicit methods have another problem caused by numerical stability. For stiff SDEs, explicit methods have usually to spend much computational efforts due to a very small step size required for stability. This leads to unnecessary computational costs if you do not need such high accuracy that the step size offers. In order to overcome the problem, Abdulle and Li [6] have developed a family of explicit stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. The methods are of strong order a half for non-commutative Itô SDEs, whereas they reduce to the first order Chebyshev methods when they are applied to ordinary differential equations (ODEs). The approaches developed by them are very important because they make it possible for us to stabilize explicit SRK methods. In fact, Komori and Burrage [7] have successfully derived SROCK methods by stabilizing SRK methods in an SRK family proposed by Rößler [8]. In [7], MS stability regions have been investigated for the methods and other existing methods, based on a scalar homogeneous autonomous test stochastic differential equation (SDE) [9, p. 138].

Differently from the case of ODEs, however, the scalar test SDE is not always enough to investigate stability for a multi-dimensional homogeneous autonomous equation [10]. Thus, very recently some researchers have proposed or started to use multi-dimensional test SDEs [10, 11, 12]. These facts motivate us. In order to investigate MS stability properties of the methods, we will adopt a two-dimensional test SDE with non-commutative noise terms proposed by Buckwar and Sickenberger [11].

SROCK METHODS

Consider the autonomous d-dimensional SDE

$$\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{f}(\boldsymbol{y}(t))\mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t))\mathrm{d}W_j(t), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1}$$

where the $W_j(t)$ $(1 \le j \le m)$ are independent Wiener processes and $\boldsymbol{y}_0$ is independent of $W_j(t) - W_j(0)$ for $t > 0$. If a global Lipschitz condition is satisfied, the SDE has exactly one continuous global solution on the entire interval $[0, \infty)$ [9, p. 113]. For a given time T_{end}, let t_n be an equidistant grid point nh $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and let $\boldsymbol{y}_n$ be a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (1).

For (1), let us consider a simpler version [7] of the SRK methods proposed by Rößler [8], that is,

$$\begin{aligned}
&\boldsymbol{H}_{i_a}^{(0)} = \boldsymbol{y}_n + \sum_{i_b=1}^{i_a-1} A_{i_a i_b}^{(0)} h\boldsymbol{f}\left(\boldsymbol{H}_{i_b}^{(0)}\right), \qquad \boldsymbol{H}_{s-2}^{(j)} = \boldsymbol{y}_n + \sum_{i_b=1}^{s} A_{s-2,i_b}^{(1)} h\boldsymbol{f}\left(\boldsymbol{H}_{i_b}^{(0)}\right),\\
&\boldsymbol{H}_{s-1}^{(j)} = \boldsymbol{y}_n + \sum_{i_b=1}^{s} A_{s-1,i_b}^{(1)} h\boldsymbol{f}\left(\boldsymbol{H}_{i_b}^{(0)}\right) + \sum_{l=1}^{m} B_{s-1,s-2}^{(1)} \tilde{\zeta}^{(l,j)} \boldsymbol{g}_l\left(\boldsymbol{H}_{s-2}^{(l)}\right),\\
&\boldsymbol{H}_{s}^{(j)} = \boldsymbol{y}_n + \sum_{i_b=1}^{s} A_{s,i_b}^{(1)} h\boldsymbol{f}\left(\boldsymbol{H}_{i_b}^{(0)}\right) + \sum_{l=1}^{m} B_{s,s-2}^{(1)} \tilde{\zeta}^{(l,j)} \boldsymbol{g}_l\left(\boldsymbol{H}_{s-2}^{(l)}\right),\\
&\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \sum_{i=1}^{s} \alpha_i h\boldsymbol{f}\left(\boldsymbol{H}_{i}^{(0)}\right) + \sum_{i=s-2}^{s} \sum_{j=1}^{m} \left(\beta_i^{(1)} \Delta W_j + \beta_i^{(2)} \sqrt{h}\right) \boldsymbol{g}_j\left(\boldsymbol{H}_{i}^{(j)}\right),
\end{aligned} \tag{2}$$

where

$$\Delta W_j \stackrel{\text{def}}{=} W_j(t_n + h) - W_j(t_n), \qquad \tilde{\zeta}^{(j,j)} \stackrel{\text{def}}{=} \frac{1}{2\sqrt{h}}\left((\Delta W_j)^2 - h\right), \qquad \tilde{\zeta}^{(l,j)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{h}} \int_{t_n}^{t_{n+1}} \left(W_l(u) - W_l(t_n)\right) \mathrm{d}W_j(u)$$

for $j \ne l$. If we want to achieve strong order one by (2), parameters $\beta_{s-1}^{(1)}$, $\beta_s^{(1)}$, $\beta_{s-2}^{(2)}$, $\beta_{s-1}^{(2)}$ and $\beta_s^{(2)}$ are automatically determined by free parameters $\beta_{s-2}^{(1)}$, $B_{s-1,s-2}^{(1)}$ and $B_{s,s-2}^{(1)}$ because of order conditions [7].

When we construct SROCK methods, parameters $A_{i_a i_b}^{(0)}$ and α_i are determined by the Chebyshev formulation. If we embed the first order Chebyshev method [6] with a damping factor η_1 into (2), these parameters are given as follows:

$$\begin{aligned}
&A_{i_a,i_a-1}^{(0)} \stackrel{\text{def}}{=} \mu_{i_a-1} \quad (2 \le i_a \le s), \qquad A_{i_a,i_a-2}^{(0)} \stackrel{\text{def}}{=} (1+\kappa_{i_a-1}) A_{i_a-1,i_a-2}^{(0)} \quad (3 \le i_a \le s),\\
&A_{i_a i_b}^{(0)} \stackrel{\text{def}}{=} (1+\kappa_{i_a-1}) A_{i_a-1,i_b}^{(0)} - \kappa_{i_a-1} A_{i_a-2,i_b}^{(0)} \quad (1 \le i_b \le i_a - 3, \quad 4 \le i_a \le s),\\
&\alpha_s \stackrel{\text{def}}{=} \mu_s, \qquad \alpha_{s-1} \stackrel{\text{def}}{=} (1+\kappa_s) A_{s,s-1}^{(0)}, \qquad \alpha_{i_b} \stackrel{\text{def}}{=} (1+\kappa_s) A_{s,i_b}^{(0)} - \kappa_s A_{s-1,i_b}^{(0)} \quad (1 \le i_b \le s-2),
\end{aligned} \tag{3}$$

where

$$\omega_0 \stackrel{\text{def}}{=} 1 + \frac{\eta_1}{s^2}, \qquad \mu_1 \stackrel{\text{def}}{=} \frac{T_s(\omega_0)}{\omega_0 T_s'(\omega_0)}, \qquad \mu_i \stackrel{\text{def}}{=} \frac{2T_s(\omega_0) T_{i-1}(\omega_0)}{T_s'(\omega_0) T_i(\omega_0)}, \qquad \kappa_i \stackrel{\text{def}}{=} \frac{T_{i-2}(\omega_0)}{T_i(\omega_0)}$$

for $i = 2, 3, \ldots, s$ and $T_k(x)$ is the Chebyshev polynomial of degree k. On the other hand, if we embed the second order Chebyshev method [13] with a damping factor η_2 into (2), the parameters are given as follows:

$$\begin{aligned}
&A_{i_a,i_a-1}^{(0)} \stackrel{\text{def}}{=} \tilde{\mu}_{i_a-1} \quad (2 \le i_a \le s-1), \qquad A_{i_a,i_a-2}^{(0)} \stackrel{\text{def}}{=} (1+\tilde{\kappa}_{i_a-1}) A_{i_a-1,i_a-2}^{(0)} \quad (3 \le i_a \le s-1),\\
&A_{i_a i_b}^{(0)} \stackrel{\text{def}}{=} (1+\tilde{\kappa}_{i_a-1}) A_{i_a-1,i_b}^{(0)} - \tilde{\kappa}_{i_a-1} A_{i_a-2,i_b}^{(0)} \quad (1 \le i_b \le i_a - 3, \quad 4 \le i_a \le s-1), \qquad A_{s,s-1}^{(0)} \stackrel{\text{def}}{=} \theta_s,\\
&A_{s,i_b}^{(0)} \stackrel{\text{def}}{=} A_{s-1,i_b}^{(0)} \quad (1 \le i_b \le s-2), \qquad \alpha_s \stackrel{\text{def}}{=} \frac{\tau_s}{\theta_s}, \qquad \alpha_{s-1} \stackrel{\text{def}}{=} 2\theta_s - \frac{\tau_s}{\theta_s}, \qquad \alpha_i \stackrel{\text{def}}{=} A_{s-1,i}^{(0)} \quad (1 \le i \le s-2),
\end{aligned} \tag{4}$$

where τ_s, θ_s, $\tilde{\mu}_1$, $\tilde{\mu}_i$ and $\tilde{\kappa}_i$ $(2 \le i \le s)$ depend on η_2 and are sought by some numerical algorithm [13]. For a given s we will call (2) an SROCKD1 method or SROCKD2 method corresponding to (3) or (4), and for all s we will assume

$$A_{s-2,i_b}^{(1)} = A_{s-1,i_b}^{(1)} = A_{s,i_b}^{(1)} = \alpha_{i_b} \quad (1 \le i_b \le s)$$

in the SROKCD1 methods and

$$A_{s-2,i_b}^{(1)} = A_{s-1,i_b}^{(1)} = A_{s,i_b}^{(1)} = A_{s-1,i_b}^{(0)} \quad (1 \le i_b \le s-2), \qquad A_{s-2,i_b}^{(1)} = A_{s-1,i_b}^{(1)} = A_{s,i_b}^{(1)} = 0 \quad (i_b = s-1, s)$$

in the SROCKD2 methods.

MS STABILITY ANALYSIS FOR MULTI-DIMENSIONAL LINEAR SDES

Let us consider

$$\mathrm{d}\boldsymbol{y}(t) = F\boldsymbol{y}(t)\mathrm{d}t + \sum_{j=1}^{m} G_j \boldsymbol{y}(t)\mathrm{d}W_j(t), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{5}$$

where F and G_j $(1 \leq j \leq m)$ are real-valued square matrices of size d. This has the zero solution $\boldsymbol{y}(t) \equiv \boldsymbol{0}$ when $\boldsymbol{y}_0 = \boldsymbol{0}$ with probability one (w. p. 1). We call it the equilibrium position. By applying (2) to (5), we have

$$\boldsymbol{y}_{n+1} = \boldsymbol{R}_s \left(h, F, \left\{ \Delta W_j \right\}_{j=1}^{m}, \left\{ \tilde{\zeta}^{(l,j)} \right\}_{j,l=1}^{m}, \left\{ G_j \right\}_{j=1}^{m} \right) \boldsymbol{y}_n, \tag{6}$$

where

$$\boldsymbol{R}_s \left(h, F, \left\{ \Delta W_j \right\}_{j=1}^{m}, \left\{ \tilde{\zeta}^{(l,j)} \right\}_{j,l=1}^{m}, \left\{ G_j \right\}_{j=1}^{m} \right) \stackrel{\mathrm{def}}{=} C + \sum_{j=1}^{m} \Delta W_j D_j + \sum_{j,l=1}^{m} \tilde{\zeta}^{(l,j)} \sqrt{h} V_{jl} \tag{7}$$

and C, D_j and V_{jl} stand for $d \times d$ matrices. Clearly, it has the equilibrium position $\boldsymbol{y}_n \equiv \boldsymbol{0}$ with $\boldsymbol{y}_0 = \boldsymbol{0}$ (w. p. 1). In order to investigate the asymptotic MS stability of the equilibrium position for numerical methods, Buckwar and Sickenberger [11] have introduced the MS stability matrix of numerical methods: $\hat{\boldsymbol{R}}_s \stackrel{\mathrm{def}}{=} E[\boldsymbol{R}_s \otimes \boldsymbol{R}_s]$. Here, $\otimes$ stands for the Kronecker product. From (7), we obtain

$$E\left[\boldsymbol{R}_s \otimes \boldsymbol{R}_s\right] = (C \otimes C) + \sum_{j=1}^{m} h\left(D_j \otimes D_j\right) + \frac{1}{2} \sum_{j,l=1}^{m} h^2 \left(V_{jl} \otimes V_{jl}\right). \tag{8}$$

As a test SDE, Buckwar and Sickenberger [11] have proposed a non-commutative SDE. In (5), it has

$$d = m = 2, \qquad F = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad G_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \tag{9}$$

for real values λ, σ_1 and σ_2. For the SDE, the eigenvalues of the MS stability matrix of the SROCKD1 methods are

$$v_{D1}^{(0)} + v_{D1}^{(1+)}, \quad v_{D1}^{(0)} + v_{D1}^{(1-)}, \quad v_{D1}^{(0)} + v_{D1}^{(2+)} \quad \text{and} \quad v_{D1}^{(0)} + v_{D1}^{(2-)},$$

where

$$v_{D1}^{(0)} \stackrel{\mathrm{def}}{=} \frac{1}{2} c_1^2, \qquad v_{D1}^{(1\pm)} \stackrel{\mathrm{def}}{=} \frac{1}{2} c_1^2 (q_1 + q_2 \pm 1)^2, \qquad v_{D1}^{(2\pm)} \stackrel{\mathrm{def}}{=} \frac{1}{2} c_1^2 (q_1 - q_2 \pm 1)^2, \qquad q_i \stackrel{\mathrm{def}}{=} h\sigma_i^2 \quad (i = 1, 2),$$

$$c_1 \stackrel{\mathrm{def}}{=} 1 + \sum_{k=1}^{s} \alpha_k p + \sum_{k_1=2}^{s} \sum_{k_2=1}^{k_1-1} \alpha_{k_1} A_{k_1 k_2}^{(0)} p^2 + \cdots + \alpha_s A_{s,s-1}^{(0)} A_{s-1,s-2}^{(0)} \cdots A_{21}^{(0)} p^s, \qquad p \stackrel{\mathrm{def}}{=} h\lambda$$

(double sign in same order). We have obtained this result with the help of a symbolic computing package, *Mathematica*. Thus, the spectral radius of $\hat{\boldsymbol{R}}_s$, say, $\rho(\hat{\boldsymbol{R}}_s)$ is $v_{D1}^{(0)} + v_{D1}^{(1+)}$. If and only if $\rho(\hat{\boldsymbol{R}}_s) < 1$, the equilibrium position of (6) is asymptotically MS stable, whereas if and only if $2p + q_1 + q_2 < 0$, the equilibrium position of the test SDE is asymptotically MS stable [11].

Similarly, the eigenvalues of the MS stability matrix of the SROCKD2 methods are

$$v_{D2}^{(0)} + v_{D2}^{(1+)}, \quad v_{D2}^{(0)} + v_{D2}^{(1-)}, \quad v_{D2}^{(0)} + v_{D2}^{(2+)} \quad \text{and} \quad v_{D2}^{(0)} + v_{D2}^{(2-)},$$

where

$$v_{D2}^{(0)} \stackrel{\mathrm{def}}{=} c_2^2 \left(1 + 2\theta_s p + \tau_s p^2\right)^2, \qquad v_{D2}^{(1\pm)} \stackrel{\mathrm{def}}{=} \frac{1}{2} c_2^2 (q_1 + q_2 \pm 2)(q_1 + q_2), \qquad v_{D2}^{(2\pm)} \stackrel{\mathrm{def}}{=} \frac{1}{2} c_2^2 (q_1 - q_2 \pm 2)(q_1 - q_2),$$

$$c_2 \stackrel{\mathrm{def}}{=} 1 + \sum_{k=1}^{s-2} A_{s-1,k}^{(0)} p + \sum_{k_1=2}^{s-2} \sum_{k_2=1}^{k_1-1} A_{s-1,k_1}^{(0)} A_{k_1 k_2}^{(0)} p^2 + \cdots + A_{s-1,s-2}^{(0)} A_{s-2,s-3}^{(0)} \cdots A_{21}^{(0)} p^{s-2}$$

(double sign in same order). In this case, thus, $\rho(\hat{\boldsymbol{R}}_s)$ is $v_{D2}^{(0)} + v_{D2}^{(1+)}$.

Let us plot the MS stability domains of the methods, that is, $\{(p, q_1, q_2) | \rho(\hat{\boldsymbol{R}}_s) < 1\}$. They are given with colored parts in Fig. 1. The parts enclosed by mesh indicate the domain in which $2p + q_1 + q_2 < 0$ is satisfied. We can see that all the methods have large stability domains, although near the origin the stability domains of the SROCKD1 methods become slightly smaller than the domain in which $2p + q_1 + q_2 < 0$ is satisfied. In the talk, we will also show other results for methods including other existing schemes and will give numerical experiments.

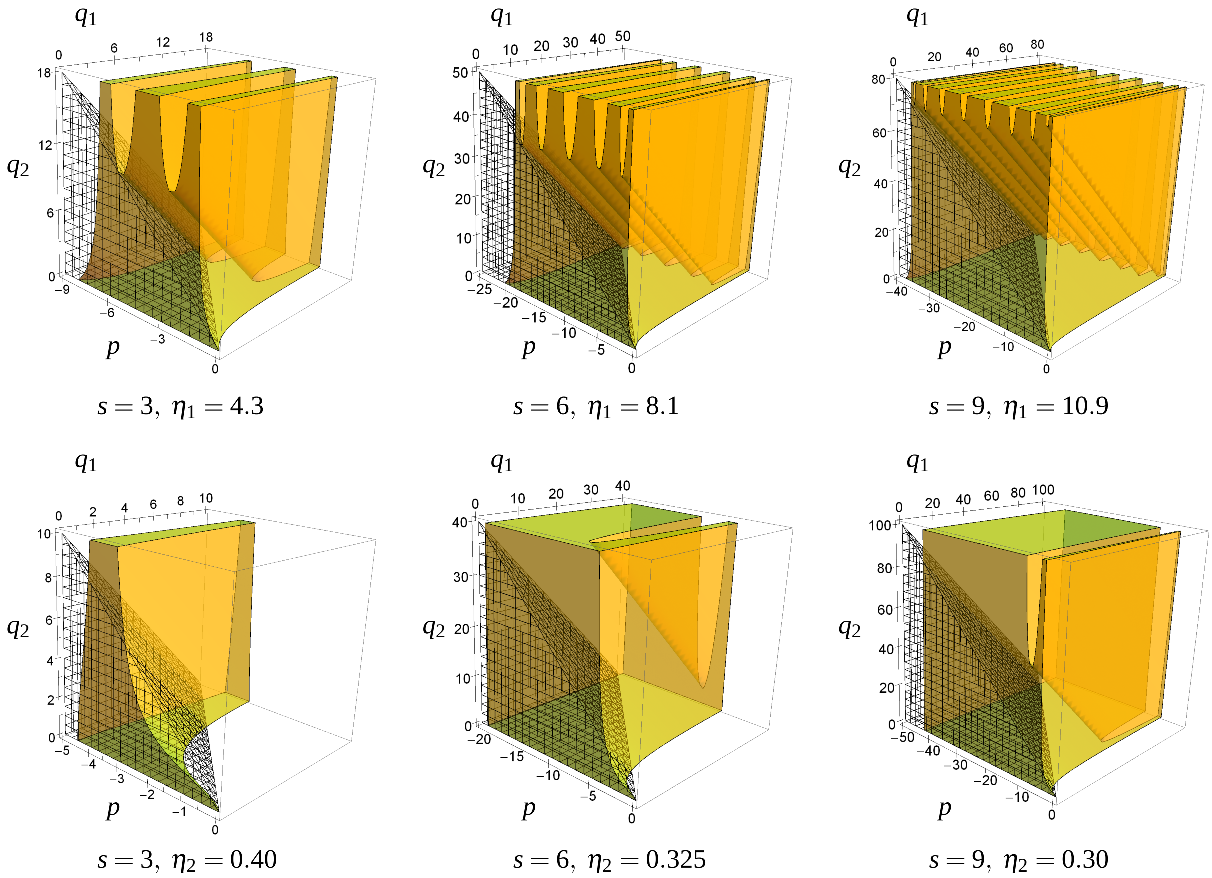

FIGURE 1. MS stability domain of the SROCKD1 methods (top) and the SROCKD2 methods (bottom)

ACKNOWLEDGMENTS

This work was partially supported by JSPS Grant-in-Aid for Scientific Research No. 23540143.

REFERENCES

1. J. Alcock, and K. Burrage, *BIT* **46**, 689–710 (2006).
2. K. Burrage, L. Lenane, and G. Lythe, *SIAM J. Sci. Comput.* **29**, 245–264 (2007).
3. G. Milstein, E. Platen, and H. Schurz, *SIAM J. Appl. Math.* **35**, 1010–1019 (1998).
4. H. Schurz, *Z. Angew. Math. Mech.* **79**, 375–382 (1999).
5. P. Kloeden, and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, New York, 1999, Corrected Third Printing.
6. A. Abdulle, and T. Li, *Commun. Math. Sci.* **6**, 845–868 (2008).
7. Y. Komori, and K. Burrage, *Kyushu Inst.Technol. Tech. Rep.* **CSSE-39** (2012), preprint of a paper submitted for publication.
8. A. Rößler, *SIAM J. Numer. Anal.* **48**, 922–952 (2010).
9. L. Arnold, *Stochastic Differential Equations: Theory and Applications*, John Wiley & Sons, New York, 1974.
10. E. Buckwar, and C. Kelly, *SIAM J. Numer. Anal.* **48**, 298–321 (2010).
11. E. Buckwar, and T. Sickenberger, *Appl. Numer. Math.* **62**, 842–859 (2012).
12. A. Tocino, and M. Senosiain, *J. Comput. Appl. Math.* **236**, 2660–2672 (2012).
13. A. Abdulle, and A. Medovikov, *Numer. Math.* **90**, 1–18 (2001).