Stochastic Runge-Kutta methods with deterministic high order for ordinary differential equations

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Abstract. Our aim is to show that the embedding of deterministic Runge-Kutta methods with higher order than necessary order to achieve a weak order can enrich the properties of stochastic Runge-Kutta methods with respect to not only practical errors but also stability. This will be done through the comparisons between our new schemes and an efficient weak second order scheme with minimized error constant proposed by Debrabant and Rößler (2009).

Keywords: Itô stochastic differential equation, weak approximation, explicit scheme, mean square stability
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INTRODUCTION

We are concerned with weak second order explicit stochastic Runge-Kutta (SRK) methods for non-commutative stochastic differential equations (SDEs). Among such methods, derivative-free methods are especially important because they can numerically solve SDEs with less computational efforts, compared with other methods which need derivatives.

In fact, weak second order and derivative-free methods have been recently studied by many researchers. Kloeden and Platen [1, pp. 486–487] have proposed a derivative-free scheme of weak order two for non-commutative Itô SDEs. Tocino and Vigo-Aguiar [2] have also proposed it as an example in their Runge-Kutta family. Komori [3] has proposed a different scheme which is for non-commutative Stratonovich SDEs and which has an advantage that it can reduce the random variables that need to be simulated. This scheme, however, still has a drawback that its computational costs linearly depend on the dimension of the Wiener process for each diffusion coefficient. Rößler [4] or Debrabant and Rößler [5] have proposed new schemes which overcome the drawback while keeping the advantage for Stratonovich or Itô SDEs, respectively.

Komori and Burrage [6] have also proposed an efficient SRK scheme which overcomes the drawback by improving the scheme in [3]. In addition, they have indicated that, even in a 10-dimensional Wiener process case, not only the scheme in [6] but also the other one in [3] can perform much better than a scheme [4] in terms of computational costs. The classical Runge-Kutta method is embedded in both methods [3, 6]. This fact motivates us.

In the present paper we consider embedding deterministic high order Runge-Kutta methods into weak second order SRK methods proposed by Rößler [7] for non-commutative Itô SDEs. For these SRK methods, we will study their stability properties and investigate their effectiveness in computation by numerical experiments.

PRELIMINARIES

Consider the autonomous d-dimensional Itô stochastic differential equation (SDE)

$$\mathrm{d}y(t) = g_0(y(t))\mathrm{d}t + \sum_{j=1}^{m} g_j(y(t))\mathrm{d}W_j(t), \quad t > 0, \quad y(0) = x_0, \tag{1}$$

where $W_j(t)$ is a scalar Wiener process and x_0 is independent of $W_j(t) - W_j(0)$ for $t > 0$. We assume a global Lipschitz condition is satisfied such that the SDE has exactly one continuous global solution on the entire interval $[0, \infty)$ [8, p. 113]. For a given time T_{end}, let t_n be an equidistant grid point nh $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M

is a natural number) and y_n a discrete approximation to the solution $y(t_n)$ of (1). In addition, suppose that all moments of the initial value x_0 exist and any component of g_j is sufficiently smooth, and define weak order in a usual way [1, p. 327].

SRK METHOD

On the base of the SRK framework proposed by Rößler [7], we consider the following SRK method for (1):

$$
\begin{aligned}
H_i^{(0)} &= y_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h g_0\left(H_k^{(0)}\right) + \sum_{k=s-2}^{i-1}\sum_{l=1}^{m} B_{ik}^{(0)} \Delta\hat{W}_l g_l\left(H_k^{(l)}\right) \quad (1 \le i \le s),\\
H_i^{(j)} &= y_n + \sum_{k=1}^{i} A_{ik}^{(1)} h g_0\left(H_k^{(0)}\right) + \sum_{k=s-2}^{i-1}\sum_{l=1}^{m} B_{ik}^{(1)} \sqrt{h} g_l\left(H_k^{(l)}\right) \quad (s-2 \le i \le s \text{ and } 1 \le j \le m),\\
\hat{H}_i^{(j)} &= y_n + \sum_{k=1}^{s} A_{ik}^{(2)} h g_0\left(H_k^{(0)}\right) + \sum_{k=s-2}^{s}\sum_{\substack{l=1\\ l\ne j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} g_l\left(H_k^{(l)}\right) \quad (s-2 \le i \le s \text{ and } 1 \le j \le m),\\
y_{n+1} &= y_n + \sum_{i=1}^{s} \alpha_i h g_0\left(H_i^{(0)}\right) + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \Delta\hat{W}_j g_j\left(H_i^{(j)}\right) + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} g_j\left(H_i^{(j)}\right)\\
&\quad + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(3)} \Delta\hat{W}_j g_j\left(\hat{H}_i^{(j)}\right) + \sum_{i=s-2}^{s}\sum_{j=1}^{m} \beta_i^{(4)} \sqrt{h} g_j\left(\hat{H}_i^{(j)}\right),
\end{aligned}
\tag{2}
$$

where the α_i, $\beta_i^{(r_a)}$, $A_{ik}^{(r_b)}$, and $B_{ik}^{(r_b)}$ $(1 \le r_a \le 4$ and $0 \le r_b \le 2)$ denote the parameters of the method and where

$$
\tilde{\eta}_i^{(j,l)} = \begin{cases} (\Delta\hat{W}_j \Delta\hat{W}_l - \sqrt{h}\Delta\tilde{W}_j)/(2\sqrt{h}) & (j < l),\\ (\Delta\hat{W}_j \Delta\hat{W}_l + \sqrt{h}\Delta\tilde{W}_j)/(2\sqrt{h}) & (j > l),\\ ((\Delta\hat{W}_j)^2 - h)/(2\sqrt{h}) & (j = l), \end{cases}
$$

the $\Delta\tilde{W}_l$ $(1 \le l \le m-1)$ are independent two-point distributed random variables with $P(\Delta\tilde{W}_j = \pm\sqrt{h}) = 1/2$ and the $\Delta\hat{W}_j$ $(1 \le j \le m)$ are independent three-point distributed random variables with $P(\Delta\hat{W}_j = \pm\sqrt{3h}) = 1/6$ and $P(\Delta\hat{W}_j = 0) = 2/3$ [1, p. 225].

Concerning (2), the following are remarkable:

- the stage number s has to be at least 3 in order to achieve weak order two [7],
- the $H_i^{(0)}$ $(1 \le i \le s-2)$ do not need the values of g_l for $l \ge 1$,
- only $B_{s-1,s-2}^{(r)}$ and $B_{s,s-2}^{(r)}$ in the $B_{ik}^{(r)}$ $(1 \le i \le s$ and $1 \le k \le m)$ can be nonzero for $r = 0, 1$ [5],
- without loss of generality, we can assume $\sum_{k=s-2}^{s} B_{s-2,k}^{(2)} = 0$, which leads to $B_{s-2,k}^{(2)} = 0$ for $k = s-2, s-1, s$ due to order conditions [5],
- from order conditions, we obtain
$$B_{s-1,s-2}^{(0)} = \frac{\alpha_{s-1}/\delta_1 \pm \sqrt{\gamma_1}}{2\alpha_{s-1}(\alpha_{s-1} + \alpha_s)}$$
if
$$\gamma_1 \stackrel{\text{def}}{=} \alpha_{s-1}\alpha_s\big(-1 + 2(\alpha_{s-1} + \alpha_s)\big) \ge 0, \tag{3}$$
where $\delta_1 = \pm 1$.

MEAN SQUARE STABILITY

In order to study stability properties, let us deal with the scalar test SDE

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sum_{j=1}^{m} \sigma_j y(t)\mathrm{d}w_j(t), \qquad t > 0, \qquad y(0) = x_0, \tag{4}$$

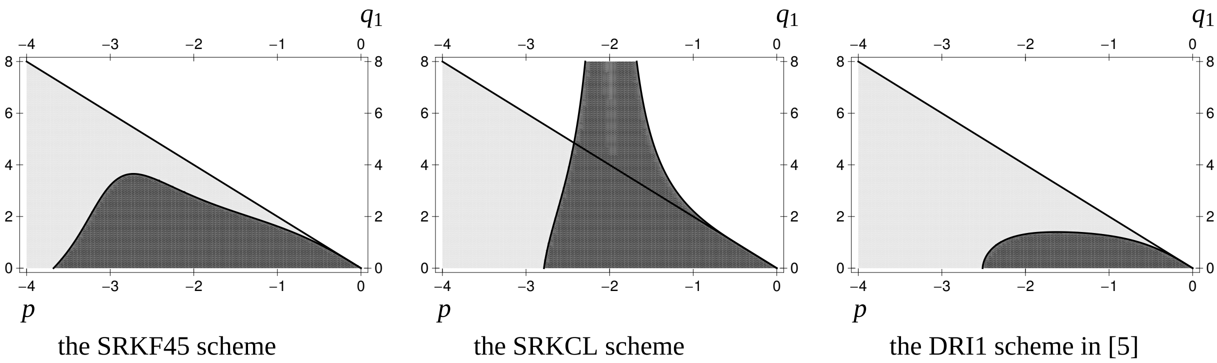

FIGURE 1. MS stability regions of SRK schemes for $m = 1$

where λ and σ_j $(1 \leq j \leq m)$ are real values and where $x_0 \neq 0$ with probability 1. By applying (2) to (4), we obtain $y_{n+1} = Ry_n$ and the amplification factor R contains random variables. To study stability properties for weak schemes, thus, we need to investigate the MS-stability region $\{(p, q_1, q_2, \ldots, q_m) | \hat{R} \leq 1\}$, where $\hat{R}(p, q_1, q_2, \ldots, q_m) \stackrel{\text{def}}{=} E[R^2]$, $p \stackrel{\text{def}}{=} h\lambda$, and $q_j \stackrel{\text{def}}{=} h\sigma_j^2$ [9, 10]. Using order conditions and assumptions to simplify $\hat{R}$, we obtain

$$
\begin{aligned}
&\hat{R}(p, q_1, q_2, \ldots, q_m) \\
= \;& \left(1 + p \sum_{i=1}^{s} \alpha_i Q_{i-1}(p)\right)^2 \\
&+ \sum_{j=1}^{m} q_j \left\{ \frac{1}{2(B^{(1)}_{s-1,s-2})^2} \left(\hat{Q}_{s-1}(p) - \hat{Q}_{s-2}(p)\right) + \left(1 + \frac{1}{2}p + \frac{\alpha_s A^{(0)}_{s,s-1}(\alpha_{s-1} \pm \delta_1 \sqrt{\gamma_0})}{2\alpha_{s-1}(\alpha_{s-1} + \alpha_s)} p^2\right) \hat{Q}_{s-2}(p) \right\} \\
&+ \frac{1}{2} \sum_{j=1}^{m} q_j^2 \left(\hat{Q}_{s-2}(p)\right)^2 + \frac{1}{2} \sum_{j=1}^{m} \sum_{\substack{l=1 \\ l \neq j}}^{m} q_j q_l \left(\frac{B^{(2)}_{s-1,s-2}\hat{Q}_{s-2}(p) + 2B^{(2)}_{s-1,s-1}\hat{Q}_{s-1}(p)}{B^{(2)}_{s-1,s-2} + 2B^{(2)}_{s-1,s-1}} \right)^2
\end{aligned}
$$

for (2). Here,

$$
Q_0(p) \stackrel{\text{def}}{=} 1, \qquad Q_i(p) \stackrel{\text{def}}{=} 1 + p \sum_{k=1}^{i} A^{(0)}_{i+1,k} Q_{k-1}(p) \quad (i \geq 1), \qquad \hat{Q}_i(p) \stackrel{\text{def}}{=} 1 + p \sum_{k=1}^{i} A^{(1)}_{ik} Q_{k-1}(p).
$$

MS-stability regions of three schemes are given with dark-colored parts in Fig. 1. In the figure SRKF45 or SRKCL denotes the SRK scheme in which the Fehlberg 4(5) [11, p. 177] or the classical Runge-Kutta scheme is embedded, respectively. Note that all the three schemes satisfy (3), which is a critical restriction. The parts enclosed by the two straight lines $q_1 = -2p$ and $q_1 = 0$ indicate the region in which $\lim_{t\to\infty} E[|y(t)|^2] = 0$ holds concerning (4) [10]. Thus, light-colored parts indicate the region in which the test SDE is stable, but the SRK schemes are not. We can see that the SRKF45 and SRKCL schemes are better than the DRI1 scheme in terms of MS-stability. Because we have chosen parameter values such that $\hat{Q}_{s-1}(p) = \hat{Q}_{s-2}(p) = 1 + p/2$ in the SRKCL scheme, $\hat{R}$ does not depend on any q_j when $p = -2$.

NUMERICAL EXPERIMENTS

We apply numerical schemes to the following SDE [5]:

$$
\mathrm{d}y(t) = y(t)\mathrm{d}t + \sum_{j=1}^{10} \sigma_j \sqrt{y(t) + k_j}\,\mathrm{d}w_j(t), \qquad t > 0, \qquad y(0) = x_0,
$$

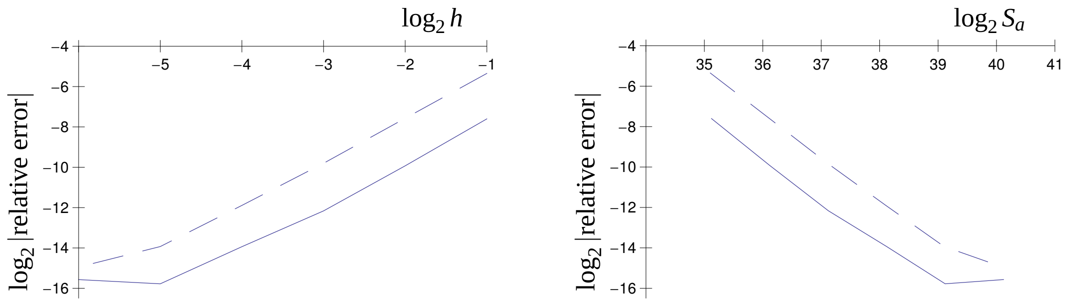

FIGURE 2. Relative errors about the fourth moment at $t = 1$.

where

$$\sigma_1 = \frac{1}{10},\quad \sigma_2 = \frac{1}{15},\quad \sigma_3 = \frac{1}{20},\quad \sigma_4 = \frac{1}{25},\quad \sigma_5 = \frac{1}{40},\quad \sigma_6 = \frac{1}{25},\quad \sigma_7 = \frac{1}{20},$$
$$\sigma_8 = \frac{1}{15},\quad \sigma_9 = \frac{1}{20},\quad \sigma_{10} = \frac{1}{25},\quad k_1 = \frac{1}{2},\quad k_2 = \frac{1}{4},\quad k_3 = \frac{1}{5},\quad k_4 = \frac{1}{10},$$
$$k_5 = \frac{1}{20},\quad k_6 = \frac{1}{2},\quad k_7 = \frac{1}{4},\quad k_8 = \frac{1}{5},\quad k_9 = \frac{1}{10},\quad k_{10} = \frac{1}{20}.$$

The fourth moment of its solution is given by

$$E\left[(X(t))^4\right] = \big(74342479604283 + 1749302625065840\mathrm{e}^{t} - 24798885546415218\mathrm{e}^{2t} \\ -263952793100784216\mathrm{e}^{3t} + 1531088033542529311\mathrm{e}^{4t}\big) \,/\, \big(124416 \times 10^{13}\big)$$

when $x_0 = 1$ [6].

Using the Mersenne twister [12], we simulate 256×10^6 independent trajectories for a given h. In this simulation, let us compare the SRKCL and DRI1 schemes. Here, remember that the DRI1 scheme is a scheme with minimized error constant and minimal stage number for weak order two. On the other hand, the SRKCL scheme is a four-stage scheme. The results are indicated in Fig. 2. The solid or dash lines denote the SRKCL scheme or the DRI1 scheme, respectively. In addition, S_a stands for the sum of the number of evaluations on the drift or diffusion coefficients and the number of generated pseudo random numbers. In this experiment we can see that the SRKCL scheme is better than the DRI1 scheme in terms of computational costs.

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REFERENCES

1. P. Kloeden, and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, New York, 1999, Corrected Third Printing.
2. A. Tocino, and J. Vigo-Aguiar, *SIAM J. Sci. Comput.* **24**, 507–523 (2002).
3. Y. Komori, *J. Comput. Appl. Math.* **206**, 158–173 (2007).
4. A. Rößler, *BIT* **47**, 657–680 (2007).
5. K. Debrabant, and A. Rößler, *Appl. Numer. Math.* **59**, 582–594 (2009).
6. Y. Komori, and K. Burrage, *J. Comput. Appl. Math.* **235**, 5326–5329 (2011).
7. A. Rößler, *SIAM J. Numer. Anal.* **47**, 1713–1738 (2009).
8. L. Arnold, *Stochastic Differential Equations: Theory and Applications*, John Wiley & Sons, New York, 1974.
9. Y. Komori, and T. Mitsui, *Monte Carlo Methods and Appl.* **1**, 279–300 (1995).
10. D. Higham, *BIT* **40**, 404–409 (2000).
11. E. Hairer, S. Nø rsett, and G. Wanner, *Solving Ordinary Differential Equations I, Nonstiff Problems*, Springer-Verlag, Berlin, 2000, Corrected Second Printing.
12. M. Matsumoto, and T. Nishimura, *ACM Transactions on Modeling and Computer Simulation (TOMACS)* **8**, 3–30 (1998).