Weak second order S-ROCK methods for Stratonovich stochastic differential equations☆

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Abstract

It is well known that the numerical solution of stiff stochastic ordinary differential equations leads to a step size reduction when explicit methods are used. This has led to a plethora of implicit or semi-implicit methods with a wide variety of stability properties. However, for stiff stochastic problems in which the eigenvalues of a drift term lie near the negative real axis, such as those arising from stochastic partial differential equations, explicit methods with extended stability regions can be very effective. In the present paper our aim is to derive explicit Runge-Kutta schemes for non-commutative Stratonovich stochastic differential equations, which are of weak order two and which have large stability regions. This will be achieved by the use of a technique in Chebyshev methods for ordinary differential equations.

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1. Introduction

While it has been customary to treat the numerical solution of stiff ordinary differential equations (ODEs) by implicit methods, there is a class of explicit methods with extended stability regions that are well suited to solving stiff problems whose eigenvalues lie near the negative real axis. Such problems include parabolic partial differential equations when solved by the method of lines.

An original contribution is by van der Houwen and Sommeijer [1] who have constructed explicit s-stage Runge-Kutta (RK) methods whose stability functions are shifted Chebyshev polynomials $T_s(1 + z/s^2)$. These have stability intervals along the negative real axis $[-2s^2, 0]$. The corresponding RK methods satisfy a three term recurrence relation which make them efficient to implement, but their drawback is that they are of order one. Lebedev [2, 3] and Medovikov [4] have constructed high order methods by computing the zeros of the optimal stability polynomials for maximal stability. But, the method is sensitive to the ordering of these zeros and there is no recurrence relationship.

Abdulle and Medovikov [5] have developed a new strategy to construct second order Chebyshev methods with nearly optimal stability domain. These methods are based on a weighted orthogonal polynomial and so the numerical methods satisfy a three-term recurrence relationship. In this case the stability interval is $[-l_s, 0]$ where $l_s \approx 0.82s^2$. These ideas have been extended by Abdulle [6] who constructed a family of s-stage damped Chebyshev methods of order four that possess nearly optimal stability along the negative real axis and a three-term recurrence relationship. For these methods, $l_s \approx 0.35s^2$.

One of the drawbacks with Chebyshev methods is that the stability region can collapse to $s-1$ single points on the negative real axis due to the mini-max property of Chebyshev polynomials. Accordingly, we require the modulus of the stability polynomial to be bounded by a damping factor $\eta < 1$. The stability interval shrinks slightly but a strip around the negative real axis is included in stability region. With $\eta = 0.95$, $l_s \approx 0.81s^2$ for the second order Chebyshev methods.

In the case of stochastic differential equations (SDEs) the issues are much more complex. Nevertheless, Abdulle and Cirilli [7] have developed a family of explicit stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. These methods are of weak order one for non-commutative Stratonovich SDEs. They reduce to the first order Chebyshev methods when there is no noise. Such an approach is important because there are very few good numerical methods for solving stiff SDEs.

We are concerned with weak second order stochastic Runge-Kutta (SRK) methods, especially derivative-free ones, for non-commutative SDEs. Kloeden and Platen [8, pp. 486–487] have proposed a derivative-free scheme of weak order two for non-commutative Itô SDEs. Tocino and Vigo-Aguiar [9] have also proposed it as an example in their Runge-Kutta family. Komori [10] has proposed a different scheme for non-commutative Stratonovich SDEs, which has an advantage that it can reduce the random variables that need to be simulated. This scheme, however, still has a drawback in that its computational costs linearly depend on the dimension of the Wiener process for each diffusion coefficient. Rößler [11] and Debrabant and Rößler [12] have proposed new schemes which overcome the drawback while keeping the advantage for Stratonovich or Itô SDEs. Komori and Burrage [13] have also proposed an efficient SRK scheme which overcomes the drawback by improving the scheme in [10].

Abdulle and Cirilli's approach is important because it is difficult to construct implicit or drift-implicit methods of weak order two for stiff SDEs [8, 14, 15]. In the present paper we shall put all these ideas together. We will construct a family of s-stage SRK methods of weak order two for non-commutative Stratonovich SDEs and with extended mean square stability regions. The method will reduce to the second order Chebyshev methods of Abdulle and Medovikov [5] when the noise terms are set to zero. In Section 2 we will give some background material on Chebyshev methods for ODEs. In Section 3 we will give background material on SDEs. In Section 4 we will give a framework of SRK methods, while in Section 5 we will derive our new class of methods based on the stability analysis. Section 6 will present numerical results and Section 7 our conclusions.

2. Chebyshev methods for ODEs

Consider the autonomous N-dimensional ODEs given by

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{1}$$

The class of s-stage RK methods for solving (1) is

$$\boldsymbol{Y}_i = \boldsymbol{y}_n + h\sum_{j=1}^{s} a_{ij}\boldsymbol{f}(\boldsymbol{Y}_j) \quad (1 \le i \le s), \quad \boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\sum_{j=1}^{s} b_j\boldsymbol{f}(\boldsymbol{Y}_j). \tag{2}$$

For an equidistant grid point $t_n \stackrel{\text{def}}{=} nh$ $(n = 1, 2, \ldots, M)$ with step size h (M is a natural number), $\boldsymbol{y}_n$ denotes a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (1). A RK method is explicit if $a_{ij} = 0$ $(i \leq j)$.

Denote by A a $s \times s$ matrix $[a_{ij}]$ and define $\boldsymbol{b} \stackrel{\text{def}}{=} [b_1\ b_2\ \cdots\ b_s]^\top$ and $\boldsymbol{e} \stackrel{\text{def}}{=} [1\ 1\ \cdots\ 1]^\top$. When we apply (2) to the linear, scalar test problem

$$y'(t) = \lambda y(t), \quad t > 0, \quad \Re(\lambda) \leq 0, \quad y(0) = y_0, \tag{3}$$

we have $y_{n+1} = R(h\lambda)y_n$ where

$$R(z) \stackrel{\text{def}}{=} 1 + z\boldsymbol{b}^\top (I - Az)^{-1}\boldsymbol{e}. \tag{4}$$

Here R is called the stability function and for explicit methods $R(z)$ is a polynomial of degree s at most, namely

$$R(z) = 1 + \sum_{j=1}^{s} z^j \boldsymbol{b}^\top A^{j-1}\boldsymbol{e}. \tag{5}$$

The stability region of (2) is $S \stackrel{\text{def}}{=} \{z \mid |R(z)| \leq 1\}$. A method whose stability domain contains the whole left half of the complex plane is said to be A-stable, but such methods are by necessity implicit.

Van der Houwen and Sommeijer [1] constructed RK methods of order one that have maximal stability along the negative real axis, namely $[-2s^2, 0]$. These methods have stability polynomial given by

$$R(z) = T_s(1 + z/s^2), \tag{6}$$

where $T_n(x)$ is the Chebyshev polynomial of degree n defined by $T_n(\cos\theta) \stackrel{\text{def}}{=} \cos(n\theta)$ or by the three term recurrence relation

$$T_0(x) \stackrel{\text{def}}{=} 1, \quad T_1(x) \stackrel{\text{def}}{=} x, \quad T_j(x) \stackrel{\text{def}}{=} 2xT_{j-1}(x) - T_{j-2}(x), \quad j \geq 2.$$

The corresponding RK method whose stability function is given by (6) is

$$\begin{aligned} &K_0 \stackrel{\text{def}}{=} \boldsymbol{y}_n, \qquad K_1 \stackrel{\text{def}}{=} \boldsymbol{y}_n + \frac{h}{s^2}\boldsymbol{f}(K_0), \\ &K_j \stackrel{\text{def}}{=} 2\frac{h}{s^2}\boldsymbol{f}(K_{j-1}) + 2K_{j-1} - K_{j-2} \quad (2 \leq j \leq s), \qquad \boldsymbol{y}_{n+1} = K_s. \end{aligned} \tag{7}$$

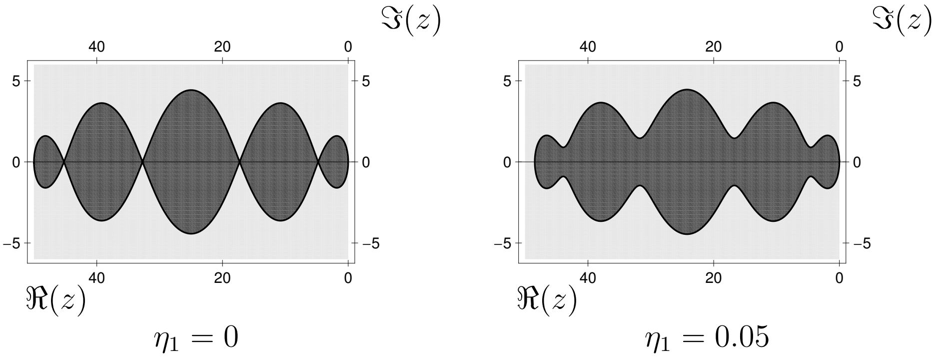

Figure 1: Stability region for $s = 5$ and $\eta_1 = 0, 0.05$

One of the drawbacks associated with this family of methods is that the stability region reduces to a single point at $s-1$ intermediate points in $[-2s^2, 0]$. This can be overcome by introducing a damping parameter η_1 that allows a strip around the negative real axis to be included in the stability domain at a cost of a slightly shortening of the stability interval. This can be achieved by setting

$$R_s(z) = \frac{T_s(\omega_0 + \omega_1 z)}{T_s(\omega_0)}, \qquad \omega_0 \overset{\text{def}}{=} 1 + \eta_1/s^2, \qquad \omega_1 \overset{\text{def}}{=} \frac{T_s(\omega_0)}{T'(s)(\omega_0)}. \tag{8}$$

See Figure 1.

The corresponding RK method can be written as a three term recurrence relation

$$\begin{aligned} &K_0 \overset{\text{def}}{=} \boldsymbol{y}_n, \qquad K_1 \overset{\text{def}}{=} \boldsymbol{y}_n + h\frac{\omega_1}{\omega_0}\boldsymbol{f}(K_0), \\ &K_j \overset{\text{def}}{=} 2\frac{T_{j-1}(\omega_0)}{T_j(\omega_0)}(h\omega_1\boldsymbol{f}(K_{j-1}) + \omega_0 K_{j-1}) - \frac{T_{j-2}(\omega_0)}{T_j(\omega_0)}K_{j-2} \quad (2 \le j \le s), \\ &\boldsymbol{y}_{n+1} = K_s. \end{aligned} \tag{9}$$

Despite giving more robust stability regions, these methods are still of order one only. Suppose now we require

$$R_s(z) = 1 + z + \frac{1}{2}z^2 + \sum_{j=3}^{s} \alpha_{sj} z^j$$

such that $|R_s(z)| \le 1$ for $z \in [-l_s, 0]$ for l_s as large as possible. Riha [16] showed that for a given s such polynomials uniquely exist, satisfy an equal ripple property on $s-1$ points and have exactly two complex zeros. Lebedev [17] gave analytic expressions in terms of elliptic integrals. Abdulle and Medovikov [5] relaxed optimal stability and constructed approximations to these optimal stability polynomials using orthogonal polynomials such that $R_s(x) = w(x)P_{s-2}(x)$, where if we write $w(x) \stackrel{\text{def}}{=} \bar{w}(a_s + x/d_s)$, $P_j(x) \stackrel{\text{def}}{=} \bar{P}_j(a_s + x/d_s)$, and $\bar{w}(x)$ is of degree two with complex zeros and satisfied $\bar{w}(a_s) = 1$, then the orthogonal polynomials $\bar{P}_0(x)$, $\bar{P}_1(x), \ldots, \bar{P}_{s-2}(x)$ are orthogonal with respect to the weight function $\bar{w}^2(x)/\sqrt{1-x^2}$ on $[-1,1]$, $\bar{P}_0(a_s) = \bar{P}_1(a_s) = \cdots \bar{P}_{s-2}(a_s) = 1$, and satisfy a three-term recurrence relation. This leads to the method

$$\begin{aligned}
&K_0 \stackrel{\text{def}}{=} \boldsymbol{y}_n, \qquad K_1 \stackrel{\text{def}}{=} \boldsymbol{y}_n + h\mu_1 \boldsymbol{f}(K_0),\\
&K_j \stackrel{\text{def}}{=} h\mu_j \boldsymbol{f}(K_{j-1}) + (\theta_j + 1)K_{j-1} - \theta_j K_{j-2} \quad (2 \le j \le s-2),\\
&K_{s-1} \stackrel{\text{def}}{=} K_{s-2} + h\sigma_s \boldsymbol{f}(K_{s-2}), \qquad K_s^* \stackrel{\text{def}}{=} K_{s-1} + h\sigma_s \boldsymbol{f}(K_{s-1}),\\
&K_s \stackrel{\text{def}}{=} K_s^* - h\sigma_s(1 - \tau_s/\sigma_s^2)(\boldsymbol{f}(K_{s-1}) - \boldsymbol{f}(K_{s-2})), \qquad \boldsymbol{y}_{n+1} = K_s.
\end{aligned} \tag{10}$$

The computation of K_{s-1}, K_s^* can be viewed as a finishing procedure. When (10) is applied to (3), then

$$K_j = P_j(z)y_n \quad (0 \le j \le s-2), \qquad K_s = w(z)K_{s-2}, \qquad y_{n+1} = R_s(z)y_n,$$

where

$$w(z) = 1 + 2\sigma_s z + \tau_s z^2 \tag{11}$$

and

$$\begin{aligned}
&P_0(z) = 1, \qquad P_1(z) = 1 + \mu_1 z,\\
&P_j(z) = (\mu_j z + \theta_j + 1)P_{j-1}(z) - \theta_j P_{j-2}(z) \quad (2 \le j \le s-2).
\end{aligned} \tag{12}$$

If the zeros of w are $\alpha_s + i\beta_s$ and $\alpha_s - i\beta_s$, then

$$\sigma_s = \frac{a_s - \alpha_s}{d_s\left((a_s - \alpha_s)^2 + \beta_s^2\right)}, \qquad \tau_s = \frac{1}{d_s^2\left((a_s - \alpha_s)^2 + \beta_s^2\right)}, \qquad d_s = \frac{l_s}{1 + a_s}.$$

The value of l_s depends on what damping (10) has. Away from $z = 0$ it is appropriate to require $|R_s(z)| \le \eta_2 < 1$ for $z \le -\varepsilon$ (ε : small positive parameter) and a number of authors set $\eta_2 = 0.95$. In this case the value

Table 1: Zeros of $w(x)$ and parameters

s	α_s	β_s	a_s	d_s	σ_s	τ_s
5	0.876008	0.138447	1.009632	9.48582	0.380486	0.300179
10	0.968456	3.399721D-2	1.001578	39.7252	0.370095	0.281274
20	0.992172	8.455313D-3	1.000433	160.722	0.367831	0.277039
50	0.998801	1.342920D-3	1.000114	1011.69	0.367929	0.276983
100	0.999704	3.355449D-4	1.000032	4049.18	0.367908	0.277012

of l_s is approximately equal to $0.81s^2$ (rather than $0.82s^2$ for $\eta_2 = 1$), and Abdulle and Medovikov [5] have given the values in Table 1.

Finally, we can determine the values of μ_j and θ_j by inserting two different nonzero values, say r_1 and r_2, into z in (12) and solving

$$(\mu_j r_i + \theta_j + 1)P_{j-1}(r_i) - \theta_j P_{j-2}(r_i) = P_j(r_i), \quad i = 1, 2$$

under the assumption that the system is non-singular.

Abdulle [6] extended this idea in the obvious way to construct Chebyshev methods of order four, but we do not extend on this analysis since our SRK methods reduce to the methods of order two in the no noise case.

3. Methods for SDEs

Consider now the autonomous N-dimensional Stratonovich SDE

$$\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{g}_0(\boldsymbol{y}(t))\mathrm{d}t + \sum_{j=1}^{d} \boldsymbol{g}_j(\boldsymbol{y}(t)) \circ \mathrm{d}W_j(t), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{x}_0. \quad (13)$$

Here, the $W_j(t)$, $j = 1, 2, \ldots, d$ are independent Wiener processes and $\boldsymbol{x}_0$ is independent of $W_j(t) - W_j(0)$ for $t > 0$. We assume a global Lipschitz condition is satisfied such that the SDE has exactly one continuous global solution on the entire interval $[0, \infty)$ [18, p. 113]. In addition, suppose that all moments of the initial value $\boldsymbol{x}_0$ exist and any component of $\boldsymbol{g}_j$ is sufficiently smooth [8, p. 474].

When discrete approximations $\boldsymbol{y}_n$ are given by a scheme, we say that the scheme is of weak (global) order q if there exists a constant C_F not depending on h such that $|E[F(y_M)] - E[F(y(T))]| \leq C_F h^q$ with $T = Mh$ and h sufficiently small and for all functions $F : \boldsymbol{R}^n \mapsto \boldsymbol{R}$ that are $2(q+1)$

times continuously differentiable and for which all partial derivatives have polynomial growth [8, p. 327].

For example, if (13) is transformed into the Itô SDE, then the simplest numerical method for simulating it is the Euler-Maruyama (EM) method given by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\tilde{\boldsymbol{g}}_0(\boldsymbol{y}_n) + \sum_{j=1}^{d} \Delta W_j^{(n)} \boldsymbol{g}_j(\boldsymbol{y}_n), \tag{14}$$

where $\tilde{\boldsymbol{g}}_0(\boldsymbol{y}) \overset{\text{def}}{=} \boldsymbol{g}_0(\boldsymbol{y}) + \frac{1}{2}\sum_{j=1}^{d} \boldsymbol{g}_j'(\boldsymbol{y})\boldsymbol{g}_j(\boldsymbol{y})$ and $\Delta W_j^{(n)} \overset{\text{def}}{=} W_j(t_n + h) - W_j(t_n) \sim N(0, h) = \sqrt{h}N(0, 1)$. Note that $N(m, v)$ denotes the normal distribution with mean m and variance v. The EM method is known to be of weak order one [8, p. 457].

As with the deterministic case, the quality of a stochastic method can be partly characterised by its stability region, associated with the scalar linear test equation

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sum_{j=1}^{d} \sigma_j y(t) \circ \mathrm{d}W_j(t), \qquad t > 0, \qquad y(0) = x_0, \tag{15}$$

where $\lambda, \sigma_1, \ldots, \sigma_d \in \boldsymbol{C}$ and where $x_0 \neq 0$ with probability one (w. p. 1). The solution is $y(t) = \exp(\lambda t + \sum_{j=1}^{d} \sigma_j W_j(t))x_0$ [8, p. 158] and it is MS stable ($\lim_{t\to\infty} E[|y(t)|^2] = 0$) if $\Re(\lambda) + \sum_{j=1}^{d} (\Re(\sigma_j))^2 < 0$ [19].

If an SRK method is applied to (15),

$$E[|y_{n+1}|^2] = \hat{R}(h\lambda, \sqrt{h}\sigma_1, \ldots, \sqrt{h}\sigma_d)E[|y_n|^2],$$

where R is a multinomial in $h\lambda$ and $\sqrt{h}\sigma_j$ ($j = 1, 2, \ldots, d$) if the method is explicit. Analogous to the deterministic case, the MS stability region of a method is defined as $S = \{(h\lambda, \sqrt{h}\sigma_1, \ldots, \sqrt{h}\sigma_d) : \hat{R}(h\lambda, \sqrt{h}\sigma_1, \ldots, \sqrt{h}\sigma_d) \leq 1\}$. For example, if λ and σ_1 are real values when $d = 1$ and (15) is transformed into the Itô SDE, for the EM method we have

$$\hat{R}(h\lambda, \sqrt{h}\sigma_1) = |1 + a|^2 + |b|^2,$$

where $a \overset{\text{def}}{=} h\lambda + \frac{1}{2}h\sigma_1^2$ and $b \overset{\text{def}}{=} \sqrt{h}\sigma_1$. In the (a, b) plane, thus, the stability region is simply represented by a circle of radius 1 centred on $(-1, 0)$ [20].

In general, it is difficult to construct methods that can cope with stiff SDEs. Very recently, one effective approach has been proposed by Abdulle and Cirilli [7] who derived a family of explicit s-stage SROCK methods with extended MS stability regions. By making the number of stages large, stiff problems can be effectively solved without resource to the linear algebra overheads associated with implicit or drift-implicit methods. When there is no noise, these methods reduce to the Chebyshev RK methods of order one (either undamped or damped). However, the drawbacks of these SROCK methods is that they are of weak order one. We extend these ideas to construct a family of s-stage SROCK2 methods that are of weak order two for non-commutative Stratonovich SDEs and that reduce to the family of second order Chebyshev methods (ROCK2) presented in [5].

4. A general SRK framework

For solving (13), we consider the following framework [13]:

$$
\begin{aligned}
Y_i^{(0,0)} &= hg_0\left(y_n + {\boldsymbol{\alpha}_i^{(0)}}^\top \boldsymbol{Y}^{(0,0)} + {\boldsymbol{\alpha}_i^{(2)}}^\top \sum_{j=1}^{d} \boldsymbol{Y}^{(j,j)}\right),\\
Y_i^{(j,j)} &= \zeta_i^{(j,j)} g_j\left(y_n + {\boldsymbol{\alpha}_i^{(1)}}^\top \boldsymbol{Y}^{(0,0)} + {\boldsymbol{\alpha}_i^{(3)}}^\top \boldsymbol{Y}^{(j,j)} + {\boldsymbol{\alpha}_i^{(4)}}^\top \sum_{\substack{l=1\\ l\neq j}}^{d} \boldsymbol{Y}^{(l,l)}\right),\\
Y_i^{(j,l)} &= \zeta_i^{(j,l)} g_l\left(y_n + {\boldsymbol{\alpha}_i^{(5)}}^\top \boldsymbol{Y}^{(0,0)} + {\boldsymbol{\alpha}_i^{(6)}}^\top \sum_{\substack{m=1\\ m\neq l}}^{d} \boldsymbol{Y}^{(l,m)}\right),\\
y_{n+1} &= y_n + \boldsymbol{b}_0^\top \boldsymbol{Y}^{(0,0)} + \boldsymbol{b}_1^\top \sum_{j=1}^{d} \boldsymbol{Y}^{(j,j)} + \boldsymbol{b}_2^\top \sum_{l=1}^{d} \boldsymbol{Y}^{(k(l),l)}
\end{aligned}
\tag{16}
$$

for $i = 1, 2, \ldots, s$ and $l \neq j$ $(j, l = 1, 2, \ldots, d)$. Here, the $k(l)$ is a value in $\{1, 2, \ldots, l-1, l+1, \ldots, d\}$, the $\boldsymbol{\alpha}_i^{(r_a)}$ $(0 \leq r_a \leq 6)$ and $\boldsymbol{b}_{r_b}$ $(r_b = 0, 1, 2)$ are column vectors of length s and the $\zeta_i^{(j,l)}$ is a random variable independent of y_n. Note that we have made the interpretation simpler by assuming a scalar problem to avoid tensor notations.

In order to construct weak second order methods the $\zeta_i^{(j,l)}$ are chosen as follows [11, 10, 13]:

$$\zeta_i^{(j,l)} = \begin{cases} \Delta\hat{W}_l & (j = l), \\ \Delta\hat{W}_j \Delta\tilde{W}_l/\sqrt{h} & (l > j > 0 \text{ and } i = s-2), \\ -\Delta\tilde{W}_j \Delta\hat{W}_l/\sqrt{h} & (j > l > 0 \text{ and } i = s-2), \\ \sqrt{h} & (j \neq l \text{ and } i \neq s-2), \end{cases} \tag{17}$$

where the $\Delta\tilde{W}_l$ are independent two-point distributed random variables with $P(\Delta\tilde{W}_j = \pm\sqrt{h}) = 1/2$ and the $\Delta\hat{W}_j$ are independent three-point distributed random variables with $P(\Delta\hat{W}_j = \pm\sqrt{3h}) = 1/6$ and $P(\Delta\hat{W}_j = 0) = 2/3$ [8, p. 225]. In the sequel, we will make the number of nonzero roles concerning the stochastic parts as small as possible. For this, in addition to the assumption for $\zeta_i^{(j,l)}$ we suppose

$$\alpha_{i,i_a}^{(6)} = 0 \quad (i, i_a < s-2 \text{ or } i \leq i_a), \qquad b_{2,i_a} = 0 \quad (i_a < s-2) \tag{18}$$

for elements of $\boldsymbol{\alpha}_i^{(6)}$ $(1 \leq i \leq s)$ and $\boldsymbol{b}_2$. Moreover, we define

$$A^{(r_a)} \overset{\text{def}}{=} \left[\boldsymbol{\alpha}_1^{(r_a)}\ \boldsymbol{\alpha}_2^{(r_a)}\ \cdots\ \boldsymbol{\alpha}_s^{(r_a)}\right]^\top, \quad \boldsymbol{c}^{(r_a)} \overset{\text{def}}{=} A^{(r_a)}\boldsymbol{e},$$
$$C^{(r_a)} \overset{\text{def}}{=} \operatorname{diag}\left(c_1^{(r_a)}, c_2^{(r_a)}, \ldots, c_s^{(r_a)}\right)$$

for $r_a = 0, 1, \ldots, 6$. With these conditions we give, for completeness, the weak second order conditions for the scalar Wiener process case and for the completely general multi-dimensional Wiener process case [13]: for the scalar Wiener process case $(d = 1)$

1. $\boldsymbol{b}_0^\top \boldsymbol{e} = 1$,
2. $\boldsymbol{b}_0^\top \boldsymbol{c}^{(0)} = 1/2$,
3. $\boldsymbol{b}_0^\top \boldsymbol{c}^{(2)} = 1/2$,
4. $\boldsymbol{b}_0^\top C^{(2)}\boldsymbol{c}^{(2)} = 1/2$,
5. $\boldsymbol{b}_0^\top A^{(2)}\boldsymbol{c}^{(3)} = 1/4$,
6. $\boldsymbol{b}_1^\top \boldsymbol{e} = 1$,
7. $\boldsymbol{b}_1^\top \boldsymbol{c}^{(1)} = 1/2$,
8. $\boldsymbol{b}_1^\top \boldsymbol{c}^{(3)} = 1/2$,
9. $\boldsymbol{b}_1^\top A^{(3)}\boldsymbol{c}^{(1)} = 1/4$,
10. $\boldsymbol{b}_1^\top A^{(1)}\boldsymbol{c}^{(2)} = 0$,
11. $\boldsymbol{b}_1^\top C^{(1)}\boldsymbol{c}^{(3)} = 1/4$,
12. $\boldsymbol{b}_1^\top A^{(3)}\boldsymbol{c}^{(3)} = 1/6$,
13. $\boldsymbol{b}_1^\top A^{(3)}A^{(3)}\boldsymbol{c}^{(3)} = 1/24$,
14. $\boldsymbol{b}_1^\top A^{(3)}C^{(3)}\boldsymbol{c}^{(3)} = 1/12$,
15. $\boldsymbol{b}_1^\top C^{(3)}\boldsymbol{c}^{(3)} = 1/3$,
16. $\boldsymbol{b}_1^\top C^{(3)}A^{(3)}\boldsymbol{c}^{(3)} = 1/8$,
17. $\boldsymbol{b}_1^\top C^{(3)}C^{(3)}\boldsymbol{c}^{(3)} = 1/4$,

additionally for the multi-dimensional Wiener process case $(d > 1)$

18. $\boldsymbol{b}_1^\top \boldsymbol{c}^{(4)} = 1/2$,
19. $\boldsymbol{b}_1^\top C^{(4)}A^{(4)}\boldsymbol{c}^{(4)} = 0$,
20. $\boldsymbol{b}_1^\top C^{(4)}\boldsymbol{c}^{(4)} = 1/2$,
21. $\boldsymbol{b}_1^\top A^{(3)}A^{(4)}\boldsymbol{c}^{(3)} = 1/8$,
22. $\boldsymbol{b}_1^\top A^{(4)}A^{(4)}\boldsymbol{c}^{(4)} = 0$,
23. $\boldsymbol{b}_1^\top A^{(4)}A^{(3)}\boldsymbol{c}^{(4)} = 0$,
24. $\boldsymbol{b}_1^\top A^{(3)}C^{(4)}\boldsymbol{c}^{(4)} = 1/4$,
25. $\boldsymbol{b}_1^\top A^{(4)}C^{(3)}\boldsymbol{c}^{(4)} = 0$,
26. $\boldsymbol{b}_1^\top A^{(3)}\boldsymbol{c}^{(4)} = 1/4$,

27. $\boldsymbol{b}_1^\top C^{(3)} A^{(4)} \boldsymbol{c}^{(3)} = 1/8$, 28. $\boldsymbol{b}_1^\top C^{(4)} A^{(3)} \boldsymbol{c}^{(4)} = 1/4$, 29. $\boldsymbol{b}_1^\top A^{(4)} \boldsymbol{c}^{(3)} = 1/4$,
30. $\boldsymbol{b}_1^\top C^{(3)} C^{(4)} \boldsymbol{c}^{(4)} = 1/4$, 31. $\boldsymbol{b}_1^\top A^{(4)} \boldsymbol{c}^{(4)} = 0$, 32. $\boldsymbol{b}_1^\top C^{(3)} \boldsymbol{c}^{(4)} = 1/4$,
33. $b_{2,s-2} = 0$, 34. $\boldsymbol{b}_2^\top \boldsymbol{e} = 0$, 35. $\boldsymbol{b}_2^\top \boldsymbol{c}^{(5)} = 0$,
36. $\alpha_{s,s-1}^{(6)} = 0$, 37. $\boldsymbol{b}_2^\top \boldsymbol{c}^{(6)} = 1/2$, 38. $\boldsymbol{b}_2^\top C^{(6)} \boldsymbol{c}^{(6)} = 0$.

Since ROCK2 methods are embedded in (16) when there is no noise, $A^{(0)}$ and $\boldsymbol{b}_0$ are given by the Chebyshev formulation in (10). We now assume that the $A^{(r_a)}$ takes the partitioned form

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{\mathbf{0}} \\ \hline A_1^{(r_a)} & A_2^{(r_a)} \end{array}\right]$$

for $r_a = 1, 2, \dots, 6$, where the big zero denotes a $(s-4) \times s$ zero matrix and where $A_1^{(r_a)}$ or $A_2^{(r_a)}$ denotes a $(s-4) \times (s-4)$ or 4×4 square matrix, respectively. Similarly, we assume that $\boldsymbol{b}_{r_b}^\top$ takes the form

$$\begin{bmatrix} \mathbf{0}_{s-4}^\top & * & * & * & * \end{bmatrix}$$

for $r_b = 1, 2$. Here, $\mathbf{0}_{s-4}$ denotes a zero column vector of length $s-4$, whereas $*$ denotes, possibly, a nonzero element. In fact, for $A^{(6)}$ and $\boldsymbol{b}_2$ we have already taken

$$A_1^{(6)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad A_2^{(6)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \end{bmatrix},$$
$$\boldsymbol{b}_2^\top = \begin{bmatrix} \mathbf{0}_{s-4}^\top & 0 & * & * & * \end{bmatrix}.$$

If we want to make the number of nonzero roles in $A^{(r_a)}$ as small as possible for $r_a = 2, 3, 4$, we can assume

$$A_1^{(r_a)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad A_2^{(r_a)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Then, there is a unique solution [21] so that

$$A_2^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 \\ 1/12 & 1/4 & 0 & 0 \\ -5/4 & 1/4 & 2 & 0 \end{bmatrix}, \qquad A_2^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{b}_1^\top = \begin{bmatrix} \mathbf{0}_{s-4}^\top & 1/8 & 3/8 & 3/8 & 1/8 \end{bmatrix}.$$

Finally, in order to achieve good stability properties, we will assume

$$A_1^{(1)} = \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \end{bmatrix}, \qquad A_2^{(1)} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

as well as

$$A_1^{(5)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha_{s-2,1}^{(0)} & \alpha_{s-2,2}^{(0)} & \cdots & \alpha_{s-2,s-4}^{(0)} \\ \alpha_{s-2,1}^{(0)} & \alpha_{s-2,2}^{(0)} & \cdots & \alpha_{s-2,s-4}^{(0)} \\ \alpha_{s-2,1}^{(0)} & \alpha_{s-2,2}^{(0)} & \cdots & \alpha_{s-2,s-4}^{(0)} \end{bmatrix}, \qquad A_2^{(5)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha_{s-2,s-3}^{(0)} & 0 & 0 & 0 \\ \alpha_{s-2,s-3}^{(0)} & 0 & 0 & 0 \\ \alpha_{s-2,s-3}^{(0)} & 0 & 0 & 0 \end{bmatrix}.$$

It is remarkable that Condition 35 is automatically satisfied from Conditions 33, 34 and the assumptions on $\boldsymbol{b}_2$ and $A^{(5)}$.

5. MS stability analysis

Let us apply our SROCK2 method to (15) and for simplicity assume that $\lambda, \sigma_1, \ldots, \sigma_d$ are real values in the sequel. Because of the structure we can easily see that

$$Y_i^{(0,0)} = P_{i-1}(h\lambda)y_0 \quad (1 \le i \le s-3).$$

We now compute successively $Y_i^{(0,0)}, Y_i^{(j,j)}, Y_i^{(j,l)}$ for $i = s-2, s-1, s$ and y_{n+1}, using the order conditions to get a simple form for these expressions. Once we have found the form $y_{n+1} = Ry_n$, the MS stability function is given by

$$\hat{R} = E[R^2].$$

Here, $\hat{R}$ will be a function of $p \overset{\text{def}}{=} h\lambda$, $q_j \overset{\text{def}}{=} h\sigma_j^2$ $(1 \le j \le d)$.

5.1. How to determine $\boldsymbol{\alpha}_i^{(1)}$ and $\boldsymbol{\alpha}_i^{(2)}$

In order to determine the vector values of $\boldsymbol{\alpha}_i^{(1)}$ and $\boldsymbol{\alpha}_i^{(2)}$ when $s-3 \le i \le s$, let us begin with the scalar Wiener process case. By applying (16) to (15) when $d = 1$, we obtain

$$\begin{aligned} R &= R(p, \Delta\hat{W}_1, \sigma_1) \\ &= \left(1 + 2\sigma_s p + \tau_s p^2\right) P_{s-2}(p) + \Delta\hat{W}_1\sigma_1 \left(\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3\right) \\ &\quad + (\Delta\hat{W}_1\sigma_1)^2 \left(\beta_{20} + \beta_{21}p + \beta_{22}p^2\right) + (\Delta\hat{W}_1\sigma_1)^3 \left(\beta_{30} + \beta_{31}p\right) + (\Delta\hat{W}_1\sigma_1)^4 \beta_{40}, \end{aligned}$$

and thus

$$
\begin{aligned}
\hat{R} &= \hat{R}(p, q_1) \\
&= \left(1 + 2\sigma_s p + \tau_s p^2\right)^2 \left(P_{s-2}(p)\right)^2 \\
&\quad + q_1 \Big\{ 2\left(\beta_{20} + \beta_{21} p + \beta_{22} p^2\right)\left(1 + 2\sigma_s p + \tau_s p^2\right) P_{s-2}(p) \\
&\qquad\quad + \left(\beta_{10} + \beta_{11} p + \beta_{12} p^2 + \beta_{13} p^3\right)^2 \Big\} \\
&\quad + 3q_1^2 \Big\{ 2\beta_{40}\left(1 + 2\sigma_s p + \tau_s p^2\right) P_{s-2}(p) + \left(\beta_{20} + \beta_{21} p + \beta_{22} p^2\right)^2 \\
&\qquad\quad + 2\left(\beta_{10} + \beta_{11} p + \beta_{12} p^2 + \beta_{13} p^3\right)\left(\beta_{30} + \beta_{31} p\right) \Big\} \\
&\quad + 9q_1^3 \left\{ 2\left(\beta_{20} + \beta_{21} p + \beta_{22} p^2\right)\beta_{40} + \left(\beta_{30} + \beta_{31} p\right)^2 \right\} + 27 q_1^4 \beta_{40}^2,
\end{aligned}
\tag{19}
$$

where, for example,

$$
\beta_{13} \stackrel{\text{def}}{=} b_{0,s}\alpha^{(0)}_{s,s-1}\alpha^{(0)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3} Q_{s-3}(p), \qquad Q_{s-3}(p) \stackrel{\text{def}}{=} 1 + \sum_{j=1}^{s-3} \alpha^{(1)}_{s-3,j}\, p P_{j-1}(p).
$$

The others are given in Appendix A.

In order to make $Q_{s-3}(p)$ a shifted orthogonal polynomial whose degree is as large as possible, let us assume

$$
\alpha^{(1)}_{s-3,i_a} = \alpha^{(0)}_{s-2,i_a} \qquad (1 \le i_a \le s-3). \tag{20}
$$

Then, we have $Q_{s-3}(p) = P_{s-3}(p)$ because of the equation:

$$
P_{i-1}(p) = 1 + \sum_{i_a=1}^{i-1} \alpha^{(0)}_{i,i_a}\, p P_{i_a-1}(p) \qquad (1 \le i \le s-1),
$$

which is obtained from the assumption on $A^{(0)}$. Similarly, let us assume

$$
\begin{aligned}
&\alpha^{(1)}_{s-2,i_a} = \alpha^{(0)}_{s-1,i_a}, \quad \alpha^{(1)}_{s-1,i_a} = \alpha^{(1)}_{s,i_a} = \alpha^{(0)}_{s-2,i_a} \quad (1 \le i_a \le s-3), \\
&\alpha^{(2)}_{s-1,s-2} = \alpha^{(2)}_{s-2,s-3}.
\end{aligned}
\tag{21}
$$

Then, $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$, $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ and $\beta_{30}+\beta_{31}p$ are expressed by shifted polynomials $P_{s-3}(p)$ or $P_{s-2}(p)$ multiplied by a polynomial of p with degree two at most. For details, see Appendix B.

In fact, $\beta_{10} + \beta_{11}p + \beta_{12}p^2 + \beta_{13}p^3$ has terms $p^2 P_{s-3}(p)$ and $p^2 P_{s-2}(p)$, whereas $\beta_{20} + \beta_{21}p + \beta_{22}p^2$ has a term $p^2 P_{s-3}(p)$. In order to make the coefficient of $p^2 P_{s-3}(p)$ vanish in the latter, let us assume for $s \ge 4$

$$
\alpha^{(2)}_{s,s-1} = -\frac{\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3}}{\alpha^{(1)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3} - \alpha^{(3)}_{s-1,s-2}\alpha^{(0)}_{s-1,s-2}}. \tag{22}
$$

Further, in order to make the absolute values of the coefficients of $p^2P_{s-3}(p)$ and $p^2P_{s-2}(p)$ small in $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$, let us assume

$$\alpha^{(2)}_{s-2,s-3} = \frac{b_{0,s-2} - \sqrt{b_{0,s-2}\left(b_{0,s-1}+b_{0,s}\right)\gamma_1}}{b_{0,s-2}(\gamma_1+1)} \tag{23}$$

when $b_{0,s-2}\left(b_{0,s-1}+b_{0,s}\right)\gamma_1 \geq 0$ where $\gamma_1 \stackrel{\text{def}}{=} 2\left(b_{0,s-2}+b_{0,s-1}+b_{0,s}\right)-1$. For details, see Appendix B.

After all, from (22) and the system of the order conditions in the scalar Wiener process case, we obtain a final solution for $\boldsymbol{\alpha}^{(1)}_i$ and $\boldsymbol{\alpha}^{(2)}_i$ $(s-3 \leq i \leq s)$:

$$\begin{aligned}
\alpha^{(1)}_{s-1,s-2} &= 1 - \frac{1}{2}\left(3c^{(0)}_{s-2} + c^{(0)}_{s-1} - \alpha^{(0)}_{s-1,s-2}\right), \\
\alpha^{(1)}_{s,s-2} &= -\frac{2(b_{0,s-1}+b_{0,s})\alpha^{(2)}_{s-2,s-3}\gamma_3}{(\gamma_1+1)\alpha^{(2)}_{s-2,s-3}-1} + 1 - \frac{1}{2}\left(c^{(0)}_{s-2} + 3c^{(0)}_{s-1} - 3\alpha^{(0)}_{s-1,s-2}\right), \\
\alpha^{(1)}_{s,s-1} &= -\frac{2(b_{0,s-1}+b_{0,s})\alpha^{(2)}_{s-2,s-3}\gamma_3}{(\gamma_1+1)\alpha^{(2)}_{s-2,s-3}-1}, \qquad \alpha^{(2)}_{s-1,s-3} = \frac{1-(\gamma_1+1)\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1}+b_{0,s}\right)}, \\
\alpha^{(2)}_{s,s-3} &= \frac{1-2b_{0,s-2}\alpha^{(2)}_{s-2,s-3}}{2\left(b_{0,s-1}+b_{0,s}\right)} - \frac{3-8b_{0,s-1}\alpha^{(2)}_{s-2,s-3}}{8b_{0,s}} + \frac{4\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}}{3\gamma_2}, \\
\alpha^{(2)}_{s,s-2} &= \frac{3-8b_{0,s-1}\alpha^{(2)}_{s-2,s-3}}{8b_{0,s}} + \frac{4\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}}{3\gamma_2}, \qquad \alpha^{(2)}_{s,s-1} = -\frac{8\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}}{3\gamma_2}
\end{aligned}$$

under the assumptions (20), (21) and (23), where

$$\begin{aligned}
\gamma_2 &\stackrel{\text{def}}{=} 2\left(2-3c^{(0)}_{s-2} - c^{(0)}_{s-1} + \alpha^{(0)}_{s-1,s-2}\right)\alpha^{(2)}_{s-2,s-3} - \alpha^{(0)}_{s-1,s-2}, \\
\gamma_3 &\stackrel{\text{def}}{=} 4 - 5c^{(0)}_{s-2} - 3c^{(0)}_{s-1} + 3\alpha^{(0)}_{s-1,s-2}.
\end{aligned}$$

For details, see Appendix C.

By applying Abdulle's parameter values[1] to this solution, we obtain Figure 2. The solid, dash or dotted line means the behaviour of $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$, $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ or $\beta_{30}+\beta_{31}p$, respectively. On the other hand, since β_{40} is very small, it is omitted. Here, note that $\eta_2 = 0.95$.

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[1] Readers can get them from a fortran code "rock2.f" in http://www.unige.ch/~hairer/software.html.

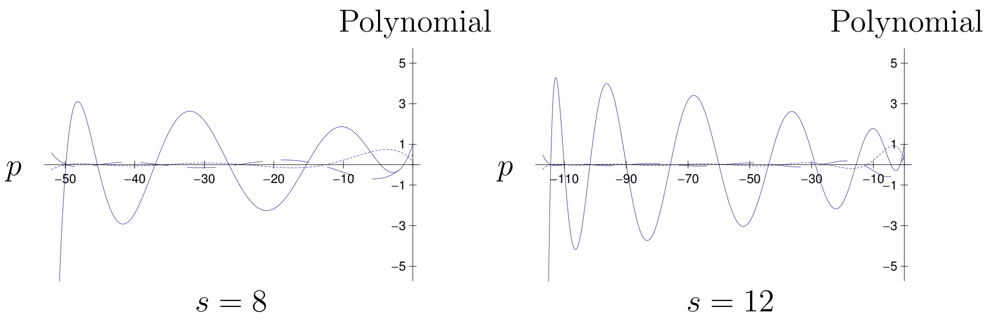

Figure 2: Behaviour of $\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$, $\beta_{20}+\beta_{21}p+\beta_{22}p^2$ or $\beta_{30}+\beta_{31}p$

5.2. The multi-dimensional Wiener process case

In this subsection let us deal with the multi-dimensional Wiener process case. By applying (16) to (15) and by using Condition 33 and the assumption on $A^{(5)}$, we obtain

$$R = R\left(p, \{\Delta\hat{W}_j\}_{j=1}^{d}, \{\Delta\tilde{W}_l\}_{l=2}^{d}, \{\sigma_j\}_{j=1}^{d}\right) = \left(1+2\sigma_s p+\tau_s p^2\right) P_{s-2}(p) + \sum_{j=1}^{d} G_j$$

and thus

$$\begin{aligned}\hat{R} &= \hat{R}(p, \{q_j\}_{j=1}^{d}) \\ &= \left(1+2\sigma_s p+\tau_s p^2\right)^2 \left(P_{s-2}(p)\right)^2 \\ &\quad +2\left(1+2\sigma_s p+\tau_s p^2\right) P_{s-2}(p) \\ &\quad \times \left\{ \sum_{j=1}^{m} q_j \left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right) + 3\sum_{j=1}^{d} q_j^2 \beta_{40} + \sum_{j=1}^{d} q_j \sum_{\substack{l=1 \\ l\neq j}}^{d} q_l \delta_{220} \right\} \\ &\quad + \sum_{j=1}^{d} E\left[G_j^2\right] + 2\sum_{j=1}^{d-1} \sum_{l=j+1}^{d} E\left[G_j G_l\right],\end{aligned} \tag{24}$$

where δ_{220} and G_j are given in Appendix A.

Our $\boldsymbol{b}_1$, $A^{(3)}$ and $A^{(4)}$ satisfy Conditions 18–32 [10, 21]. In addition, as we have said, $A^{(5)}$ satisfies Condition 35. Thus, all we need to do is to seek a solution for Conditions 34, 37 and 38 under the Conditions 33 and 36. From these, we have

$$\alpha^{(6)}_{s,s-2} = \frac{1}{4b_{2,s}}, \qquad \alpha^{(6)}_{s-1,s-2} = -\frac{1}{4b_{2,s}}, \qquad b_{2,s-1} = -b_{2,s}.$$

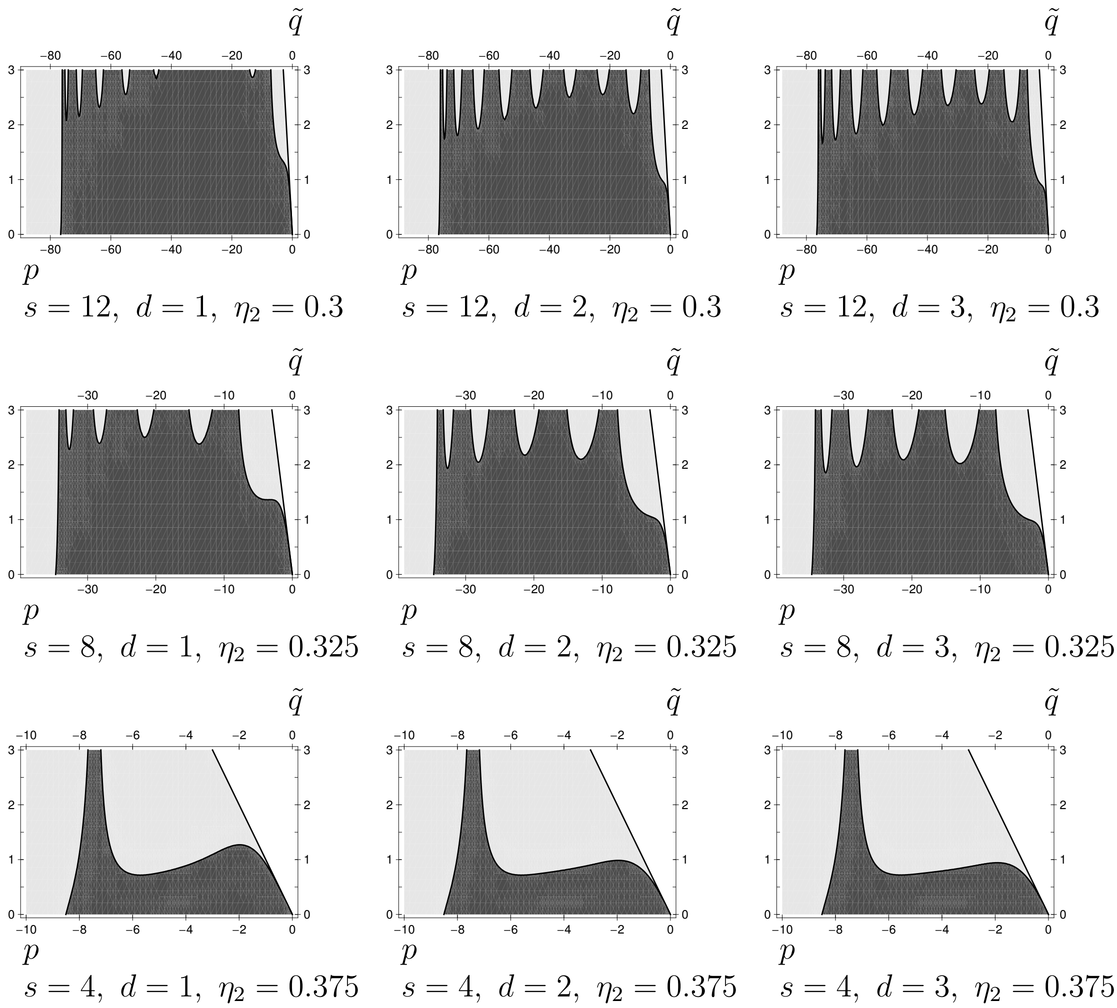

Figure 3: MS stability region of the SROCK2 schemes for some s, d and η_2

Here, note that $\hat{R}$ in (24) does not depend on the free parameters $b_{2,s}$.

Finally, we show MS-stability regions, in which $\hat{R} < 1$. In general, however, such a region lies in the $d+1$-dimensional space with respect to p and q_j $(1 \le j \le d)$. For this, let us assume $q_1 = q_2 = \cdots = q_d$ and denote $d \times q_1$ by $\tilde{q}$. Then, in Figure 3 a dark-colored part indicates an MS-stability region, whereas the part enclosed by the two straight lines $\tilde{q} = -p$ and $\tilde{q} = 0$ indicates the region in which the test SDE is stable in mean square. It is remarkable that $s = 4$ is the minimum stage number because our SROCK2 methods are of weak order two [10, 21].

6. Numerical experiments

In the previous section we have derived the SROCK2 methods, which have the free parameters $b_{2,s}$. Now let us set it at 1 and confirm its performance in two numerical experiments.

The first experiment comes from the last example in [12]. That is, we apply numerical schemes to the following SDE:

$$\mathrm{d}y(t) = \left(y(t) - \frac{1}{4}\sum_{j=1}^{10}\sigma_j^2 \right)\mathrm{d}t + \sum_{j=1}^{10}\sigma_j\sqrt{y(t)+k_j} \circ \mathrm{d}w_j(t), \qquad y(0) = x_0, \tag{25}$$

where

$$\sigma_1 = k_4 = k_9 = \frac{1}{10}, \quad \sigma_2 = \sigma_8 = \frac{1}{15}, \quad \sigma_3 = \sigma_7 = \sigma_9 = k_5 = k_{10} = \frac{1}{20},$$
$$\sigma_4 = \sigma_6 = \sigma_{10} = \frac{1}{25}, \quad \sigma_5 = \frac{1}{40}, \quad k_1 = k_6 = \frac{1}{2}, \quad k_2 = k_7 = \frac{1}{4}, \quad k_3 = k_8 = \frac{1}{5}.$$

The fourth moment of its solution is given by

$$\begin{aligned} E\left[(X(t))^4\right] = &(74342479604283 + 1749302625065840\mathrm{e}^t - 2479888554641521 8\mathrm{e}^{2t} \\ &-263952793100784216\mathrm{e}^{3t} + 1531088033542529311\mathrm{e}^{4t}) \\ &/\left(124416 \times 10^{13}\right) \end{aligned}$$

when $x_0 = 1$ (w. p. 1) [13]. We simulate 64×10^6 independent trajectories for a given h. In Monte Carlo simulation for SDEs, statistical independence properties in pseudo random numbers are very important [22]. In addition, their period needs to be very long. For this, we use the Mersenne twister [23]. By it, for example, we generate a pseudo random number for $\Delta\tilde{W}_l/\sqrt{h}$ which takes ± 1.

The results are indicated in Fig. 4. The solid, dash or dotted lines denote the SROCK2 scheme with four stages ($\eta_2 = 0.375$), the SROCK scheme with three stages [7] or the RS1 scheme [11], respectively. The RS1 scheme is of weak order two and is computationally efficient. That is, only the SROCK scheme is of weak order one. In addition, S_a stands for the sum of the number of evaluations on the drift or diffusion coefficients and the number of generated pseudo random numbers. We can see that the SROCK2 scheme

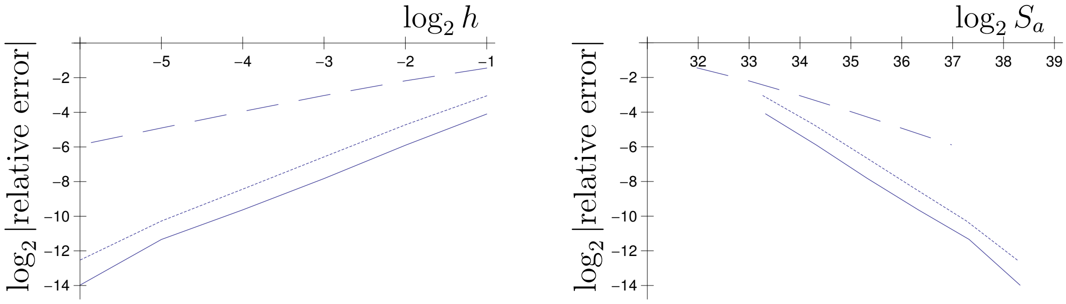

Figure 4: Relative errors about the fourth moment at $t = 1$. (Solid: SROCK2, dash: SROCK, dotted: RS1.)

shows good performance not only with respect to relative errors, but also in terms of the computational costs.

The second experiment comes from the following heat equation with noise:

$$\mathrm{d}u(t,x) = (D\Delta u(t,x))\mathrm{d}t + ku(t,x) \circ \mathrm{d}w_1(t), \qquad (t,x) \in [0,T] \times [0,1], \quad (26)$$

which was dealt with in [7]. Here, Δ is the Laplacian operator, D is the diffusion coefficient, and k is a noise parameter.

Let us suppose that $u(0,x) = 1$ as an initial condition and $u(t,0) = \frac{\partial u(t,x)}{\partial x}|_{x=1} = 0$ as mixed boundary conditions, and set $D = k = 1$ for simplicity. If we discretize the space interval by $N+1$ equidistant points x_{i_a} $(0 \le i_a \le N)$ and define a vector-valued function by $\boldsymbol{y}(t) \overset{\text{def}}{=} [u(t,x_1)\ u(t,x_2)\ \cdots\ u(t,x_N)]^\top$, then we obtain

$$\mathrm{d}\boldsymbol{y}(t) = A\boldsymbol{y}(t)\mathrm{d}t + \boldsymbol{y}(t) \circ \mathrm{d}w_1(t), \qquad \boldsymbol{y}(0) = [1\ 1\ \cdots 1]^\top \quad \text{(w. p. 1)} \quad (27)$$

by applying the central difference scheme to (26) and by using the relationship $u(t,x_{N-1}) = u(t,x_{N+1})$ from the boundary conditions, where

$$A \overset{\text{def}}{=} N^2 \begin{bmatrix} -2 & 1 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \mathbf{0} & & & 2 & -2 \end{bmatrix}.$$

It is known that the eigenvalues of A are distributed around the negative real axis in the interval $(-4N^2, 0)$ [7]. Thus, remark that normal explicit SRK schemes need a very small step size for stability when N is large.

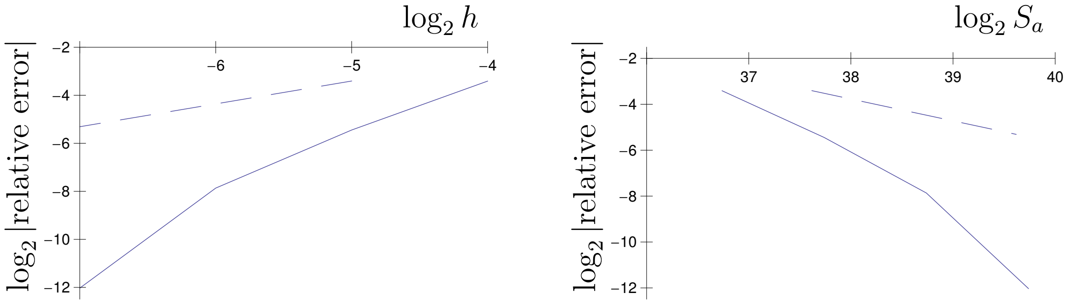

Figure 5: Relative errors about the variance at $t = 1$. (Solid: SROCK2, dash: SROCK.)

Because (27) is linear, we can get a system of ODEs with respect to the mean and variance of $\boldsymbol{y}(t)$. In fact, they are given by $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = \tilde{A}E[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(t) = \tilde{A}\Phi(t) + \Phi(t)\tilde{A}^\top + \Phi(t) + E[\boldsymbol{y}(t)](E[\boldsymbol{y}(t)])^\top,$$

where

$$\tilde{A} \stackrel{\text{def}}{=} A + \frac{1}{2}\mathrm{diag}(1, 1, \ldots, 1), \qquad \Phi(t) = E\left[(\boldsymbol{y}(t) - E[\boldsymbol{y}(t)])(\boldsymbol{y}(t) - E[\boldsymbol{y}(t)])^\top\right].$$

Some results are indicated in Fig. 5. Because $\boldsymbol{y}(t)$ is a vector, the Euclidean norm has been used. In order to obtain the results, 64×10^6 independent trajectories have been simulated for a given h. The solid or dash lines denote the SROCK2 scheme with 104 stages ($\eta_2 = 0.285$) or the SROCK scheme (of weak order one) with 100 stages [7], respectively. Both schemes solve the SDE without reducing the step size too much, but our weak second order scheme is clearly superior.

On the other hand, in the RS1 scheme the step size has to be reduced significantly. That is, it can not solve the SDE numerically stably for $h = 2^{-i}$ $(1 \leq i \leq 14)$. Results for $h = 2^{-15}$ are given in Table 2.

Table 2: Results by the RS1 scheme about the variance at $t = 1$.

$\log_2 h$	num of trajectories	$\log_2 S_a$	$\log_2 \lvert\text{relative error}\rvert$
-15	32×10^5	39.7796	-8.47558

After all, the RS1 scheme spends much computational efforts due to a very small step size required for stability, and can not spare them to reduce statistical errors at a magnitude of $\log_2 S_a$. Furthermore, the SROCK scheme does not need a very small step size, but it suffers from low convergence order. Thus, we can see again that the SROCK2 scheme has good performance not only with respect to relative errors, but also in terms of the computational costs.

7. Conclusions

We have derived explicit s-stage SROCK2 schemes of weak order two for non-commutative Stratonovich SDEs. The SROCK2 schemes have the following features.

- The schemes have large MS stability regions along the negative real axis because they are equivalent to the ROCK2 schemes with a small $\eta_2 < 1$ when they are applied to ODEs and their parameter values are carefully chosen for stability.

- The schemes are based on efficient SRK methods [13], and are efficient in terms of not only the number of generated pseudo random numbers but also the number of evaluations on the diffusion coefficients.

In the numerical experiments we have confirmed advantages which come from these facts. In the first experiment where the SDE has a 10-dimensional Wiener process, our schemes' efficiency in computational costs have been clearly shown in comparison with the SROCK and RS1 schemes. The second experiment has highlighted the advantages of the SROCK2 schemes in accuracy and stability. That is, whereas the RS1 scheme or the SROCK scheme has suffered from poor stability properties or low convergence order respectively, our schemes have shown high performance in accuracy, computational costs and stability.

Finally, the following should be also remarked:

- Although the stability region of our schemes is large along the negative real axis, it is not so wide, compared with that of the SROCK schemes [7].

- For Itô SDEs a different version of SROCK schemes exists [24]. An extension of our present approach for Itô SDEs is a future work.

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Appendix A. Some notations

Notations in (19) are defined as follows:

$$
\begin{aligned}
&Q_i(p) \overset{\text{def}}{=} 1 + \sum_{j=1}^{i-1} \alpha_{ij}^{(1)} p P_{j-1}(p) \quad (s-2 \le i \le s), \\
&\beta_{10} \overset{\text{def}}{=} \sum_{i=s-3}^{s} b_{1,i} Q_i(p), \qquad \beta_{11} \overset{\text{def}}{=} \sum_{i=s-2}^{s} \sum_{j=s-3}^{i-1} b_{0,i} \alpha_{ij}^{(2)} Q_j(p), \\
&\beta_{12} \overset{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} b_{0,i} \alpha_{ij}^{(0)} \alpha_{jk}^{(2)} Q_k(p), \quad \beta_{20} \overset{\text{def}}{=} \sum_{i=s-2}^{s} \sum_{j=s-3}^{i-1} b_{1,i} \alpha_{ij}^{(3)} Q_j(p), \\
&\beta_{21} \overset{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} \left(b_{0,i} \alpha_{ij}^{(2)} \alpha_{jk}^{(3)} + b_{1,i} \alpha_{ij}^{(1)} \alpha_{jk}^{(2)} \right) Q_k(p), \\
&\beta_{22} \overset{\text{def}}{=} \Big(b_{1,s} \alpha_{s,s-1}^{(1)} \alpha_{s-1,s-2}^{(0)} \alpha_{s-2,s-3}^{(2)} + b_{0,s} \alpha_{s,s-1}^{(0)} \alpha_{s-1,s-2}^{(2)} \alpha_{s-2,s-3}^{(3)} \\
&\qquad\qquad + b_{0,s} \alpha_{s,s-1}^{(2)} \alpha_{s-1,s-2}^{(1)} \alpha_{s-2,s-3}^{(2)} \Big) Q_{s-3}(p), \\
&\beta_{30} \overset{\text{def}}{=} \sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} \sum_{k=s-3}^{j-1} b_{1,i} \alpha_{ij}^{(3)} \alpha_{jk}^{(3)} Q_k(p), \\
&\beta_{31} \overset{\text{def}}{=} \Big(b_{0,s} \alpha_{s,s-1}^{(2)} \alpha_{s-1,s-2}^{(3)} \alpha_{s-2,s-3}^{(3)} + b_{1,s} \alpha_{s,s-1}^{(1)} \alpha_{s-1,s-2}^{(2)} \alpha_{s-2,s-3}^{(3)} \\
&\qquad\qquad + b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(1)} \alpha_{s-2,s-3}^{(2)} \Big) Q_{s-3}(p), \\
&\beta_{40} \overset{\text{def}}{=} b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(3)} \alpha_{s-2,s-3}^{(3)} Q_{s-3}(p).
\end{aligned}
$$

Notations in (24) are defined as follows:

$$G_j \stackrel{\text{def}}{=} \Delta\hat{W}_j\sigma_j\left(\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3\right)+(\Delta\hat{W}_j\sigma_j)^2\left(\beta_{20}+\beta_{21}p+\beta_{22}p^2\right)$$
$$+(\Delta\hat{W}_j\sigma_j)^3\left(\beta_{30}+\beta_{31}p\right)+(\Delta\hat{W}_j\sigma_j)^4\beta_{40}$$
$$+\Delta\hat{W}_j\sigma_j\sum_{\substack{l=1\\l\neq j}}^{d}\left[\Delta\hat{W}_l\sigma_l\left(\delta_{110}+\delta_{111}p+\delta_{112}p^2\right)+(\Delta\hat{W}_l\sigma_l)^2\left(\delta_{120}+\delta_{121}p\right)\right]$$
$$+(\Delta\hat{W}_j\sigma_j)^2\sum_{\substack{l=1\\l\neq j}}^{d}\left[\Delta\hat{W}_l\sigma_l\left(\delta_{210}+\delta_{211}p\right)+(\Delta\hat{W}_l\sigma_l)^2\delta_{220}\right],$$

$$\delta_{110}\stackrel{\text{def}}{=}\sum_{i=s-1}^{s}\sum_{j=s-3}^{s-2}b_{1,i}\alpha_{ij}^{(4)}Q_j(p),$$
$$\delta_{111}\stackrel{\text{def}}{=}\sum_{i=s-1}^{s}\sum_{j=s-2}^{i-1}\sum_{k=s-3}^{j-1}b_{1,i}\alpha_{ij}^{(1)}\alpha_{jk}^{(2)}Q_k(p)+b_{0,s}\alpha_{s,s-1}^{(2)}\sum_{i=s-3}^{s-2}\alpha_{s-1,i}^{(4)}Q_i(p),$$
$$\delta_{112}\stackrel{\text{def}}{=}\left(b_{1,s}\alpha_{s,s-1}^{(1)}\alpha_{s-1,s-2}^{(0)}+b_{0,s}\alpha_{s,s-1}^{(2)}\alpha_{s-1,s-2}^{(1)}\right)\alpha_{s-2,s-3}^{(2)}Q_{s-3}(p),$$
$$\delta_{120}\stackrel{\text{def}}{=}\left(\sum_{i=s-1}^{s}b_{1,i}\alpha_{i,s-2}^{(4)}\right)\alpha_{s-2,s-3}^{(3)}Q_{s-3}(p),$$
$$\delta_{121}\stackrel{\text{def}}{=}\left(b_{1,s}\alpha_{s,s-1}^{(1)}\alpha_{s-1,s-2}^{(2)}+b_{0,s}\alpha_{s,s-1}^{(2)}\alpha_{s-1,s-2}^{(4)}\right)\alpha_{s-2,s-3}^{(3)}Q_{s-3}(p),$$
$$\delta_{210}\stackrel{\text{def}}{=}b_{1,s}\alpha_{s,s-1}^{(3)}\sum_{i=s-3}^{s-2}\alpha_{s-1,i}^{(4)}Q_i(p),\quad \delta_{211}\stackrel{\text{def}}{=}b_{1,s}\alpha_{s,s-1}^{(3)}\alpha_{s-1,s-2}^{(1)}\alpha_{s-2,s-3}^{(2)}Q_{s-3}(p),$$
$$\delta_{220}\stackrel{\text{def}}{=}b_{1,s}\alpha_{s,s-1}^{(3)}\alpha_{s-1,s-2}^{(4)}\alpha_{s-2,s-3}^{(3)}Q_{s-3}(p).$$

Appendix B. Some expressions

From (20) and (21), we obtain

$$\beta_{10}+\beta_{11}p+\beta_{12}p^2+\beta_{13}p^3$$

$$
\begin{aligned}
&= \Bigg[\sum_{\substack{i=s-3\\ i\neq s-2}}^{s} b_{1,i} + \Bigg(-b_{1,s-2}\alpha^{(0)}_{s-1,s-2} + \sum_{i=s-1}^{s} b_{1,i}\alpha^{(1)}_{i,s-2} \\
&\qquad\qquad + \sum_{i=s-2}^{s} b_{0,i}\alpha^{(2)}_{i,s-3} + b_{0,s}\alpha^{(2)}_{s,s-1} \Bigg) p \\
&\quad + b_{0,s} \Bigg(\sum_{i=s-2}^{s-1} \alpha^{(0)}_{s,i}\alpha^{(2)}_{i,s-3} + \alpha^{(2)}_{s,s-1}\alpha^{(1)}_{s-1,s-2} \\
&\qquad\qquad - \alpha^{(2)}_{s,s-2}\alpha^{(0)}_{s-1,s-2} \Bigg) p^2 \Bigg] P_{s-3}(p) \\
&\quad + \Bigg[b_{1,s-2} + \Bigg(b_{1,s}\alpha^{(1)}_{s,s-1} + \sum_{i=s-1}^{s} b_{0,i}\alpha^{(2)}_{i,s-2} \Bigg) p \\
&\qquad + b_{0,s}\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-2,s-3}p^2 \Bigg] P_{s-2}(p),
\end{aligned}
$$

$$
\begin{aligned}
&\beta_{20} + \beta_{21}p + \beta_{22}p^2 \\
&\quad = \Bigg\{ \sum_{i=s-2}^{s} b_{1,i}\alpha^{(3)}_{i,s-3} + b_{1,s}\alpha^{(3)}_{s,s-1} \\
&\qquad + \Bigg[- \sum_{i=s-1}^{s} b_{1,i}\alpha^{(3)}_{i,s-2}\alpha^{(0)}_{s-1,s-2} + b_{1,s}\alpha^{(3)}_{s,s-1}\alpha^{(1)}_{s-1,s-2} \\
&\qquad\quad + \sum_{i=s-1}^{s}\sum_{j=s-2}^{i-1} \Big(b_{1,i}\alpha^{(1)}_{i,j}\alpha^{(2)}_{j,s-3} + b_{0,i}\alpha^{(2)}_{i,j}\alpha^{(3)}_{j,s-3} \Big) \Bigg] p \\
&\qquad + b_{0,s} \Big[\alpha^{(0)}_{s,s-1}\alpha^{(2)}_{s-1,s-2}\alpha^{(3)}_{s-2,s-3} + \alpha^{(2)}_{s,s-1}\alpha^{(1)}_{s-1,s-2}\alpha^{(2)}_{s-2,s-3} \\
&\qquad\qquad - \alpha^{(2)}_{s,s-1}\alpha^{(3)}_{s-1,s-2}\alpha^{(0)}_{s-1,s-2} \Big] p^2 \Bigg\} P_{s-3}(p),
\end{aligned}
$$

$$\beta_{30} + \beta_{31}p$$

$$
\begin{aligned}
&= \left[\sum_{i=s-1}^{s} \sum_{j=s-2}^{i-1} b_{1,i} \alpha_{i,j}^{(3)} \alpha_{j,s-3}^{(3)} \right. \\
&\quad + \left(b_{1,s} \alpha_{s,s-1}^{(1)} \alpha_{s-1,s-2}^{(2)} \alpha_{s-2,s-3}^{(3)} + b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(1)} \alpha_{s-2,s-3}^{(2)} \right. \\
&\qquad \left. \left. + b_{0,s} \alpha_{s,s-1}^{(2)} \alpha_{s-1,s-2}^{(3)} \alpha_{s-2,s-3}^{(3)} - b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(3)} \alpha_{s-1,s-2}^{(0)} \right) p \right] \\
&\quad \times P_{s-3}(p) + b_{1,s} \alpha_{s,s-1}^{(3)} \alpha_{s-1,s-2}^{(3)} P_{s-2}(p).
\end{aligned}
$$

Appendix C. How to solve the order conditions

Let us solve the system of the order conditions for the scalar Wiener process case. Since $A^{(0)}$, $A^{(3)}$, $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ are given, we can solve it as follows [21]:

1) From Conditions 7 and 9, seek $c_{s-1}^{(1)}$ and $c_s^{(1)}$. Then, Condition 11 is automatically satisfied.
2) From Conditions 3 and 4, seek seek $c_{s-1}^{(2)}$ and $c_s^{(2)}$.
3) Substitute the results in 2) into Condition 10, and seek $\alpha_{s,s-2}^{(1)}$.
4) Substitute the results in 2) into Condition 5, and seek $\alpha_{s,s-2}^{(2)}$.

Noting that

$$
-4b_{0,s-2} \left(\sum_{i=s-2}^{s} b_{0,i} \right) \left(\alpha_{s-2,s-3}^{(2)} \right)^2 + 4b_{0,s-2} \alpha_{s-2,s-3}^{(2)} + 2 \sum_{i=s-1}^{s} b_{0,i} - 1 = 0
$$

because of (23), thus, we have

$$
\begin{aligned}
c_{s-1}^{(1)} &= 1 - \frac{1}{2} \left(c_{s-3}^{(1)} + c_{s-2}^{(1)} \right), \qquad c_s^{(1)} = 1 + \frac{1}{2} \left(c_{s-3}^{(1)} - 3c_{s-2}^{(1)} \right), \\
c_{s-1}^{(2)} &= \frac{1 - 2b_{0,s-2} \alpha_{s-2,s-3}^{(2)}}{2 \left(b_{0,s-1} + b_{0,s} \right)}, \qquad c_s^{(2)} = \frac{1 - 2b_{0,s-2} \alpha_{s-2,s-3}^{(2)}}{2 \left(b_{0,s-1} + b_{0,s} \right)}, \\
\alpha_{s,s-2}^{(1)} &= -\frac{\alpha_{s,s-1}^{(1)}}{2 \left(b_{0,s-1} + b_{0,s} \right) \alpha_{s-2,s-3}^{(2)}} + \frac{b_{0,s-2} \alpha_{s,s-1}^{(1)} - 3 \left(b_{0,s-1} + b_{0,s} \right) \alpha_{s-1,s-2}^{(1)}}{b_{0,s-1} + b_{0,s}}, \\
\alpha_{s,s-2}^{(2)} &= \frac{3 - 8b_{0,s-1} \alpha_{s-1,s-2}^{(2)} - 4b_{0,s} \alpha_{s,s-1}^{(2)}}{8b_{0,s}}.
\end{aligned}
$$

From $A^{(3)}$, (20), (21) and the equations above, we can obtain the final solution for $\boldsymbol{\alpha}_i^{(1)}$ and $\boldsymbol{\alpha}_i^{(2)}$ $(s - 3 \le i \le s)$ in Subsection 5.1.

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