ISSN 1344-8803, CSSE-9 May 8, 2000

Effects of Nonsynchronism on Multirate Sampled-Data Systems: $\mathcal{L}^p$ Worst-Case Performance, Robustness and Computation[¶] [§]

Hiroshi Ito[†][‡]

[†]Department of Control Engineering and Science, Kyushu Institute of Technology
680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan
Phone: (+81)948-29-7717, Fax: (+81)948-29-7709
E-mail: hiroshi@ces.kyutech.ac.jp

Abstract: This paper focuses on multi-rate sample-data control with nonsynchronous decentralized controllers whose sampler-and-hold elements in different stations update their state independently of each other. Phases of discrete-time events in decentralized control stations are different because of digital control with multiple processors although each controller in a station is a synchronous single-rate sampled-data controller. The paper analyzes the effect of the nonsynchronous phase distribution on $\mathcal{L}^p$ worst-case performance and stability. New representations of the nonsynchronous sample-data system are presented for analyzing and computing performance degradation caused by the nonsynchronism. The method of analysis is further modified to propose an approach to design of $\mathcal{L}^p$-induced norm suboptimal nonsynchronous multi-rate controllers. Some results on stability and performance robustness for the nonsynchronous closed-loop systems are developed. Furthermore, it is shown that the worst-case performance measure is a continuous function of the phase shift provided that anti-aliasing filters are located appropriately. Slight perturbation of the phase only results in a slight degradation or improvement of the closed-loop performance. The analysis of the continuity property enables us to estimate how robust the performance is against the phase perturbation.

Keywords: nonsynchronism, multirate sampled-data control, continuity, decentralized control, stability, robustness

[¶] Technical Report in Computer Science and Systems Engineering, Log Number CSSE-9, ISSN 1344-8803.

[§] The first version of the paper was completed by March 14, 1997. The current version of the paper was completed by November 7, 1997. The current version of the paper was presented at Control Workshop for Japanese Young Researchers, OVTA, Chiba, Japan on November 8-9, 1997, also at The 3rd FUCOTT Workshop, Fukuoka Garden Palace, Fukuoka, Japan on January 26-28, 1998.

[‡] Author for correspondence

1 Introduction

This paper focuses on multi-rate sampled-data control operating with fixed time intervals of sampling and holding. There are two types of asynchronous mechanism:

- The continuous-time plant is controlled by a single centralized controller. The ratios of sample and hold periods of all the constituent schedules are irrational. The controller is aperiodic itself and so is the closed-loop : Aperiodicity
- The continuous-time plant is controlled by multiple digital controllers. The overall controller consists of individually synchronous sub-controllers whose phases of discrete-time events are independent of each other. The closed-loop system may be periodic although the discrete-time events at different channels may never occur simultaneously : Phase distribution

The former type of sampled-data control excludes standard linear shift-invariant(LSI) discrete-time controllers, while the controller in the latter case consists of individually standard LSI discrete-time systems. The implementation of the former type aperiodic controller is considerably difficult. The latter situation is a realistic case of the asynchronous control which practical engineers often confront. Since distributed(decentralized in other words) multiple processors are introduced in practice to deal with large-scale systems in view of high reliability with low cost[25], this paper focuses on the latter situation defined with phase lags and delays of sampling and holding events in different stations. Kalman and Bertram[20] referred to this type of asynchronism as nonsynchronism

Several results of asynchronous systems concerned with nominal stability [23] and LQG control [28] have been proposed, whereas no results on robust control of asynchronous sampled-data systems appear to be available except a result of $\mathcal{H}^\infty$ control in [24]. In [24], dual-rate sampled-data control is considered and their focus is aperiodicity. The controller is aperiodic so that it is not a pure discrete-time system described by difference equations. The standpoint of [24] is different from this paper. As is stated above, this paper focuses on phase lags of distributed controllers and the controllers are supposed to be individually standard discrete-time systems.

The feature of this paper is that a worst-case performance measure is defined in terms of continuous-time signals. For asynchronous multirate systems, stability and performance analysis based on continuous-time signals is very important since signals operates with different periods and different phases. Besides the continuous time-variable, there is no time-variable which gives a fair judgment for every signal and appreciate the effectiveness of asynchronous multirate control correctly. This standpoint of using continuous-time performance measures is similar to recent research on synchronous sampled-data control, to name a few, [4, 19, 31, 1, 10, 32, 29, 5]. To the author's knowledge, the study of multirate control began in 1950s (see [20] and references therein). Some samples of related recent developments are [2, 9, 6, 18, 8, 29, 5, 16] and references therein. However, analyzing nonsynchronism in terms of continuous-time based worst-case performance measures is a new feature in the multirate control literature.

In this paper, some basic characteristics of the nonsynchronous controller will be investigated with the worst-case performance measure. It will be shown that the degradation of the performance only depends on the relative phase lag among control stations if the multirate system is periodic. Moreover, the worst-case performance remains unchanged even if the starting time of the continuous-time constituents(i.e., plant, disturbance and reference signal) and that of digital controller are not synchronized. The analysis of degradation of the worst-case performance and internal stability is here solved by introducing a delay-advance representation of the nonsynchronous sampled-data system. Roughly speaking, if the overall multirate controller is periodic, an exact solution to the nonsynchronous analysis problem can be obtained by solving a discrete-time $\mathcal{H}^\infty$ control problem in lifted signal spaces. Examples are presented to illustrate the results and to show that the nonsynchronism sometimes improves the $\mathcal{L}^p$ worst-case performance. This paper also analyzes continuity of the closed-loop performance of the nonsynchronous system as a function of the phase shift in individual stations. It will be shown that a slight variation of the phase only results in a slight degradation of the closed-loop performance and we can estimate how robust the performance is against the phase perturbation. Modifying the delay-advance equivalent used in the analysis, an approach to nonsynchronous multirate controller design for the $\mathcal{L}^p$ disturbance attenuation will be proposed. Some robustness results for nonsynchronous multirate systems will be also developed.

For notational simplicity, this paper assumes that each sub-controller is single-rate, while the overall controller is multirate. All results of this paper can be easily modified to deal with the individually multirate case.

Notation used in this paper is standard[7, 30]. $\mathcal{Z}_+$ denotes the set of nonnegative integer. Let ℓ denote the space of one-sided sequences defined on $\mathcal{Z}_+$. For $p \in [0, \infty]$, ℓ^p is the Banach space of p-summable sequences in ℓ. $\mathcal{L}^p$ denotes the Banach space of all Lebesgue measurable functions on $[0, \infty)$ which are p-integrable. The extended space of $\mathcal{L}^p$ is denoted by $\mathcal{L}^{p,e}$. P_T denotes the truncation operator

$$P_T f(t) := \begin{cases} f(t) & t < T \\ 0 & t \geq T \end{cases}$$

for $T \in [0, \infty)$ and $f \in \mathcal{L}^{p,e}$. An operator $H : \mathcal{L}^{p,e} \to \mathcal{L}^{p,e}$ is called causal if H satisfies $P_T H = P_T H P_T$ for all $T \geq 0$. For $f \in \mathcal{L}^{p,e}$ and $\tau \geq 0$, D_τ denotes the time-delay linear operator on $\mathcal{L}^p$:

$$D_\tau f(t) := \begin{cases} 0, & 0 \leq t < \tau \\ f(t-\tau), & t \geq \tau \end{cases} .$$

For a negative real number $\tau < 0$, D_τ represents the time-advance operator on $\mathcal{L}^p$:

$$D_\tau u(t) := u(t-\tau), \quad t \geq 0 .$$

Note that D_τ is not causal for $\tau < 0$. An operator $H : \mathcal{L}^{p,e} \to \mathcal{L}^{p,e}$ is called T-periodic for $T > 0$ if H satisfies $HD_T = D_T H$. The operator H is time-invariant if H satisfies $HD_T = D_T H$ for all $T \geq 0$. For $m \in \mathcal{Z}_+$ and $f \in \ell$, S_m denotes the mth-forward shift operator on ℓ:

$$\sigma_m f(k) := \begin{cases} 0 & 0 \leq k \leq m-1 \\ f(k-m) & k \geq m \end{cases} .$$

In the $m < 0$ case, σ_m denotes the mth-backward shift operator:

$$\sigma_m f(k) := f(k-m), \quad k \in \mathcal{Z}_+ .$$

The backward shift is not a causal operator. Let R_n denote the truncation operator on ℓ defined as

$$R_n f(k) := \begin{cases} f(k) & 0 \leq k \leq n-1 \\ 0 & n \leq k \end{cases}$$

for $n \in \mathcal{Z}_+$ and $f \in \ell$. An operator $H : \ell \to \ell$ is called causal if H satisfies $R_n H = R_n H R_n$ for all $n \in \mathcal{Z}_+$. It is strictly causal if $R_{n+1} H = R_{n+1} H R_n$ holds. H is said to be shift-invariant if H

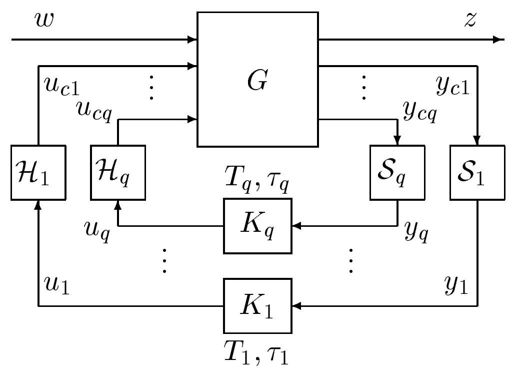

Figure 1: Nonsynchronous multirate sampled-data control.

satisfies $H\sigma_m = \sigma_m H$ for all $m \in \mathcal{Z}_+$. Given a Banach space X, $\|x\|_X$ denotes the norm of x in X. For an operator H from X to Y, $\|H\|_{Y/X}$ denotes the Y/X-induced norm. PC denotes piecewise continuous-time functions on $[0,\infty)$. $\mathcal{F}_\ell(\cdot,\cdot)$ denotes lower linear fractional transformation. A finite dimensional linear time-invariant(FDLTI) system is said to be strictly proper if its transfer function is strictly proper. The transfer function of an FDLTI system G and its state space realization are denoted in the compact form of $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. When this compact form is used, the equal sign $=$ not only denotes equivalence of transfer functions but also means equivalence of state space solutions. We will use $\cong$ for equalities of transfer functions. diag$[K_1, K_2, K_3]$ denotes a block-diagonal matrix with matrix entries K_1, K_2, K_3.

2 Nonsynchronous multirate control system

Consider the multirate sampled-data system with multiple controllers shown in Fig.1 , which is denoted by $\Sigma[G, \mathcal{H}K\mathcal{S}]$. Here, w is the exogenous input, z is the controlled output, and both signals are continuous-time. $u = [u_1^T, u_2^T, \cdots, u_q^T]^T$ is the control input, $y = [y_1^T, y_2^T, \cdots, y_q^T]^T$ is the measurement output, and both signals are discrete-time. G denotes the plant that is a finite dimensional, linear time-invariant(FDLTI), continuous-time system. The signals $u_{c\,i}$ and $y_{c\,i}, i = 1, 2, \ldots, q$ are continuous-time. The integer $q \geq 1$ denotes the number of input-output channels for decentralized control. Let G be described as

$$G : \begin{cases} \dot{x}_G = Ax_G + B_1 w + B_2 u_c \\ z = C_1 x_G + D_{11} w + D_{12} u_c \\ y_c = C_2 x_G \quad . \end{cases} \tag{1}$$

in the state space. We assume that (A, B_2) is stabilizable and that (C_2, A) is detectable. The overall controller $\mathcal{H}K\mathcal{S}$ consists of q sub-controllers:

$$\mathcal{H}K\mathcal{S} := \mathrm{diag}[\mathcal{H}_1 K_1 \mathcal{S}_1, \mathcal{H}_2 K_2 \mathcal{S}_2, \cdots, \mathcal{H}_q K_q \mathcal{S}_q] \ . \tag{2}$$

Each control station $\mathcal{H}_i K_i \mathcal{S}_i$ operates with an independent clock. The relative timing between the sample processes will be called "phase". Let τ_i denote the phase. Then, the nonsynchronous discrete-time events are represented by

$$\mathcal{S} := \mathrm{diag}[\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_q] \tag{3}$$

$$
\begin{array}{c}
y_i = \mathcal{S}_i y_{ci}\,, \quad y_i(k) = y_{ci}(kT_i + \tau_i), \quad i = 1, 2, ..., q \\
\mathcal{H} := \mathrm{diag}[\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_q]\,, \quad u_{ci} = \mathcal{H}_i u_i \\
u_{ci}(kT_i + \tau_i + t) = u_i(k), \;\; 0 < t \leq T_i, \;\; i = 1, 2, ..., q \;,
\end{array} \tag{4}
$$

where $\mathcal{S}_i$'s are samplers and $\mathcal{H}_i$'s are zero-order hold elements. We suppose $-T_i < \tau_i < T_i$ for $i = 1, 2, \ldots, q$. The overall controller $\mathcal{H}K\mathcal{S}$ is multirate. Note that the nonsynchronous multirate controller $\mathcal{H}K\mathcal{S}$ is not periodic if all T_i/T_j, $i, j = 1, 2, \ldots, q$ are not rational numbers. The sampler and hold of each individual control station $\mathcal{H}_i K_i \mathcal{S}_i$ is synchronized itself so that the controller K_i is an FDLSI system as follows:

$$
K_i : \left\{ \begin{array}{l} x_{Ki}(k+1) = A_{Ki} x_{Ki}(k) + B_{Ki} y_i(k) \\ u_i(k) = C_{Ki} x_{Ki}(k) + D_{Ki} y_i(k), \quad k \in \mathcal{Z}_+ \end{array} \right. \tag{5}
$$

The state variable of the overall controller $\mathcal{H}K\mathcal{S}$ is defined by $x_K(k) := [x_{K1}^T(k), x_{K2}^T(k), \cdots, x_{Kq}^T(k)]^T$.

The system $\Sigma[G, \mathcal{H}K\mathcal{S}]$ is called internally stable if there exist positive real constants α_G, α_K, β_G and β_K such that the associated unforced system satisfies

$$
\|x_G(t)\| \leq \|X(0)\| \beta_G e^{-\alpha_G t}, \forall t \geq 0 \tag{6}
$$

$$
\|x_K(k)\| \leq \|X(0)\| \beta_K e^{-\alpha_K k}, \forall k \in \mathcal{Z}_+ \tag{7}
$$

for any initial state $X(0) := [x_G^T(0), x_K^T(0)]^T$. It can be shown that the system is internally stable if and only if it is stable in terms of exponential convergence of the state transition matrix defined for the linear system with jumps[27]. This paper adopts the definition of (6) and (7) since for the closed-loop system consisting of an LTI system and LSI systems the definition matches with intuitive interpretation of stability in practical engineering. For $p \in [1, \infty]$, the worst-case performance of the asynchronous system $\Sigma[G, \mathcal{H}K\mathcal{S}]$ is defined in terms of $\mathcal{L}^p$-induced norm of the linear operator T_{zw} mapping w to z with the starting time $t = 0$ under the zero initial condition $X(0) = 0$:

$$
\|T_{zw}\|_{\mathcal{L}^p/\mathcal{L}^p} = \sup_{w \in \mathcal{L}^p, w \neq 0} \frac{\|z\|_p}{\|w\|_p}, \quad x_G(0) = 0, x_K(0) = 0. \tag{8}
$$

Given a scalar $\gamma > 0$, the system $\Sigma[G, \mathcal{H}K\mathcal{S}]$ is said to have $\mathcal{L}^p$ disturbance attenuation γ if the following conditions are satisfied: (i) It is internally stable, (ii) $\|T_{zw}\|_{\mathcal{L}^p/\mathcal{L}^p} < \gamma$.

3 Delay-advance representation and phase-invariance properties

This section develops a delay-advance equivalent representation of nonsynchronous sampled-data systems. The equivalent representation method is used successfully to characterize the worst-case performance with the nonsynchronism.

Consider a single-channel sampled-data system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ shown in Fig.2. We assume that the starting time of the system is $t = 0$. The system is defined with a sampler $\mathcal{S}^\tau$ and a hold element $\mathcal{H}^\tau$ whose starting times of discrete-time events are delayed for τ:

$$
\begin{array}{l}
y = \mathcal{S}^\tau y_c, \quad y(k) = y_c(kT + \tau), \quad k \in \mathcal{Z}_+ \\
u_c = \mathcal{H}^\tau u, \; \left\{ \begin{array}{ll} u_c(t) = 0, & 0 \leq t \leq \tau \\ u_c(kT + \tau + t) = u(k), & 0 < t \leq T \end{array} \right.
\end{array}
$$

where $0 \leq \tau$. Then, it is obvious that

$$
\mathcal{S}^\tau = \mathcal{S}^0 D_{-\tau}, \quad \mathcal{H}^\tau = D_\tau \mathcal{H}^0, \; 0 \leq \tau \tag{9}
$$

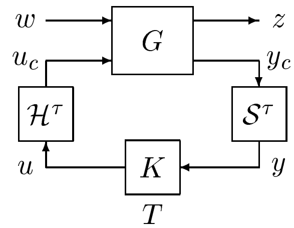

Figure 2: Single-channel sampled-data system.

hold, where $\mathcal{S}^0$ and $\mathcal{H}^0$ are sample and hold elements without any delays. In the case of $\tau < 0$, $\mathcal{S}^\tau$ and $\mathcal{H}^\tau$ are defined as

$$y = \mathcal{S}^\tau y_c, \; \begin{cases} y(k) = 0 & k = 0, 1, 2, \ldots, n-1 \\ y(k) = y_c(kT + \tau), & k = n, n+1, \ldots \end{cases}$$
$$u_c = \mathcal{H}^\tau u, \; \begin{cases} u_c(t) = u(n-1), & 0 < t \leq \tau + nT \\ u_c(kT + \tau + t) = u(k), & 0 < t \leq T, k = n, n+1, \ldots \end{cases} .$$

where n is the smallest integer such that $-\tau \leq nT$. Then, it is not difficult to show

$$\mathcal{S}^\tau = \mathcal{S}^0 D_{-\tau}, \quad \mathcal{L}^{p,e} \cap \mathtt{PC} \mapsto \ell, \quad \tau < 0 \tag{10}$$

$$\mathcal{H}^\tau = D_\tau \mathcal{H}^0 = D_\tau (I - P_{-\tau}) \mathcal{H}^0, \quad \ell \mapsto \mathcal{L}^{p,e}, \; \tau < 0. \tag{11}$$

We will utilize the following properties of nonlinear operators.

Lemma 1 *Let H be an operator on $\mathcal{L}^p$. It satisfies*

$$\|D_\alpha H D_{-\alpha}\|_{\mathcal{L}^p/\mathcal{L}^p} = \|H\|_{\mathcal{L}^p/\mathcal{L}^p}, \quad \forall \alpha > 0 \tag{12}$$

In contrast, Equation (12) does not hold for negative τ in general unless H is time-invariant.

Lemma 2 *Let H be a periodic operator on $\mathcal{L}^p$. Then,*

$$\|H(I - P_\alpha)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|H\|_{\mathcal{L}^p/\mathcal{L}^p}, \quad \forall \alpha > 0 \; . \tag{13}$$

Time-invariant operators also satisfies (13).

Consider the single-channel system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ in Fig.2 again. From (9),(10) and (11), the sampled-data controller can be written as

$$\mathcal{H}^\tau K \mathcal{S}^\tau = D_\tau \mathcal{H}^0 K \mathcal{S}^0 D_{-\tau}. \tag{14}$$

This controller has the following properties.

Lemma 3 *If K is a causal operator on ℓ, $\mathcal{H}^\tau K \mathcal{S}^\tau$ is a causal operator on $\mathcal{L}^{p,e} \cap \mathtt{PC}$ for all $\tau \in (-\infty, \infty)$.*

Lemma 4 *Suppose that K is a causal shift-invariant operator on ℓ. Then,*

$$\mathcal{H}^\tau K \mathcal{S}^\tau D_T = D_T \mathcal{H}^\tau K \mathcal{S}^\tau, \quad \tau \leq 0$$
$$\mathcal{H}^\tau K \mathcal{S}^\tau D_T (I - P_\tau) = D_T \mathcal{H}^\tau K \mathcal{S}^\tau, \quad \tau > 0 \; .$$

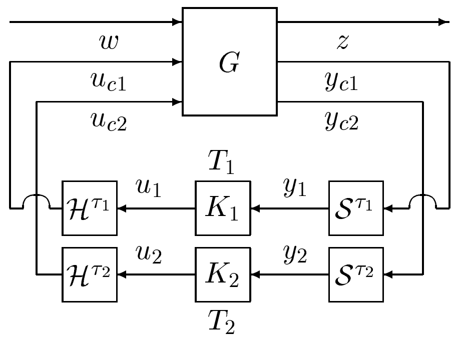

Figure 3: Two-channel sampled-data system

It is interesting to note that the sampled-data controller $\mathcal{H}^\tau K \mathcal{S}^\tau$ is not precisely periodic unless $\tau \leq 0$.

Now, we consider $\mathcal{L}^p$ worst-case performance of the system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ shown in Fig.2. The system is said to have a causal solution if there exists a unique causal operator mapping w to (z, u_c) on $\mathcal{L}^{p,e}$. Henceforth this section assumes that the system has the causal solution. It can be verified straightforwardly that the existence of such a causal solution guarantees the existence of a unique causal map between w to (z, u_c, y_c) on $\mathcal{L}^{p,e}$. This assumption always holds if every K_i is strictly causal. The rest of this section will only focus on the linear operator of the closed loop defined with zero initial conditions. Stability will be considered in the following sections.

Let $T_{zw}(T, \tau) = \mathcal{F}_\ell(G, \mathcal{H}^\tau K \mathcal{S}^\tau)$ denote the operator from w to z in Fig.2 with zero state variables at the initial time $t = 0$. We can prove the following lemma.

Lemma 5 *Consider the sampled-data system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ with zero state conditions at the starting time $t = 0$. Suppose that the system has a causal solution. Then,*

$$T_{zw}(T, \tau)(I - P_\alpha) = \mathcal{F}_\ell(G(I - P_\alpha), \mathcal{H}^\tau K \mathcal{S}^\tau)$$

holds for all $\alpha \geq 0$ and $\tau \in (-\infty, \infty)$.

Theorem 1 *Suppose that the system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ with zero state conditions at the starting time $t = 0$ has a causal solution. Then, $T_{zw}(T, \tau)$ is T-periodic for all $\tau \in (-\infty, T)$.*

Using Lemmas and theorems provided in the above, the following theorem is derived.

Theorem 2 *Consider $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ and $\Sigma[G, \mathcal{H}^0 K \mathcal{S}^0]$ in Fig.2 with zero state conditions at the starting time $t = 0$. If each system has a causal solution,*

$$\|T_{zw}(T, \tau)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T, 0)\|_{\mathcal{L}^p/\mathcal{L}^p}$$

holds for all $\tau \in (-\infty, T)$.

Next, let us consider the multi-channel nonsynchronous multirate sampled-data control system $\Sigma[G, \mathcal{H} K \mathcal{S}]$ shown in Fig.1. Here, we assume $q = 2$ for notational simplicity and the two-channel system is depicted by Fig.3, which is denoted by $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$. The controller K consists of two sub-controllers:

$$K := \mathrm{diag}[K_1, K_2]$$

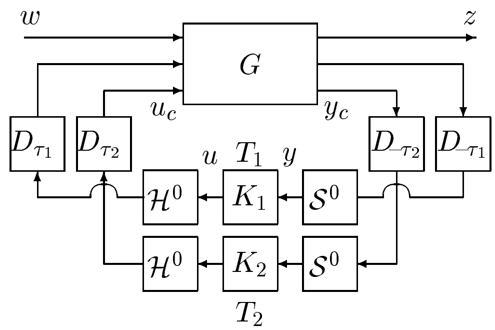

Figure 4: Delay-advance equivalent.

where each controller $K_i, i = 1, 2$ is FDLSI. The multirate sampler $\mathcal{S}$ and the hold element $\mathcal{H}$ are

$$\mathcal{S}^{\tau_1,\tau_2} := \mathrm{diag}[\mathcal{S}^{\tau_1}, \mathcal{S}^{\tau_2}], \quad \mathcal{H}^{\tau_1,\tau_2} := \mathrm{diag}[\mathcal{H}^{\tau_1}, \mathcal{H}^{\tau_2}] \ .$$

The sampling and holding interval of $\mathcal{S}^{\tau_1}$ and $\mathcal{H}^{\tau_1}$ is T_1. The interval of $\mathcal{S}^{\tau_2}$ and $\mathcal{H}^{\tau_2}$ is T_2. Due to (9), (10) and (11), the system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ is equivalent to a system with delay-advance operators shown in Fig.4 for any initial condition of state variables. Using this equivalent system, the following results are obtained.

Lemma 6 *Suppose that every K_i is a causal shift-invariant operator on ℓ. Then,*

$$\mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2} D_{T_C} \bar{P} = D_{T_C} \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}$$
$$\bar{P} := \begin{bmatrix} I - P_{\tau_1} & 0 \\ 0 & I - P_{\tau_2} \end{bmatrix}$$

are satisfied for all $\tau_i \in (-\infty, \infty)$, $i = 1, 2$ if T_1/T_2 is a rational number. Here, T_C denotes the least common multiple among T_i, $i = 1, 2$.

The closed-loop operator between w and z in $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ with zero initial conditions at $t = 0$ is denoted by

$$T_{zw}(T_1, \tau_1, T_2, \tau_2) := \mathcal{F}_\ell(G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}) \ . \tag{15}$$

The following Lemma can be derived in the same manner as Lemma 5.

Lemma 7 *Consider the two-channel sampled-data system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ with zero state conditions at the starting time $t = 0$. Suppose that the system has a causal solution. Then,*

$$T_{zw}(T_1, \tau_1, T_2, \tau_2)(I - P_\alpha) = \mathcal{F}_\ell(G\bar{P}, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2})$$
$$\bar{P} := \begin{bmatrix} I - P_\alpha & 0 & 0 \\ 0 & I - P_{\beta_1} & 0 \\ 0 & 0 & I - P_{\beta_2} \end{bmatrix}$$

holds for all $\alpha \geq 0$, $0 \leq \beta_i \leq \alpha$ and $\tau_i \in (-\infty, \infty)$, $i = 1, 2$.

The proof of the next theorem is based on Lemma 6, which is similar to Theorem 1.

Theorem 3 *Suppose that the system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ with zero state conditions at the starting time $t = 0$ has a causal solution. If T_1/T_2 is a rational number, then $T_{zw}(T_1, \tau_1, T_2, \tau_2)$ is T_C-periodic for all $\tau_i \in (-\infty, T_i)$, $i = 1, 2$.*

The main result in this section is the following.

Theorem 4 *Consider the two-channel multirate sampled-data control systems* $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ *and* $\Sigma[G, \mathcal{H}^{\tau_1+\alpha,\tau_2+\alpha} K \mathcal{S}^{\tau_1+\alpha,\tau_2+\alpha}]$ *in Fig.3 with zero state conditions at the starting time* $t = 0$*. Suppose that each system has a causal solution. If* T_1/T_2 *is a rational number, then*

$$\|T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1, \tau_1 + \alpha, T_2, \tau_2 + \alpha)\|_{\mathcal{L}^p/\mathcal{L}^p}$$

holds for all triplet (τ_1, τ_2, α) *satisfying* $-T_i \le \tau_i < T_i$ *and* $-T_i \le \tau_i + \alpha < T_i$*,* $i = 1, 2$*.*

For any $q > 2$, properties similar to Theorem 4 can be also derived straightforwardly; further details are omitted for the sake of brevity. Theorem 4 shows that only relative phase difference changes the performance of nonsynchronous multi-channel systems while absolute values of the phases do not. In addition, the theorem says that the worst-case performance remains unchanged even if the starting time of the continuous-time constituents(i.e., plant, disturbance and reference signal) and that of digital controller are not synchronized. The following corollary summarizes some of typical characteristics extracted from Theorem 4.

Corollary 1 *Suppose that the two-channel multirate sampled-data control system* $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ *in Fig.3 has a causal solution for zero state conditions at the starting time* $t = 0$*. We assume* T_1/T_2 *to be a rational number and define* $T_{min} := \min\{T_1, T_2\}$*. Then, the following are satisfied.*
(i) Relative phase dependence

$$\|T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1, 0, T_2, \tau_2 - \tau_1)\|_{\mathcal{L}^p/\mathcal{L}^p}, \quad -T_i \le \tau_i < T_i,\ i = 1, 2, \quad -T_2 \le \tau_2 - \tau_1 < T_2$$

(ii) Planar symmetry

$$\|T_{zw}(T_1, \tau, T_2, 0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1, 0, T_2, -\tau)\|_{\mathcal{L}^p/\mathcal{L}^p}, \qquad -T_{min} < \tau < T_{min}$$

(iii) Parallel translation (Periodicity)

$$\|T_{zw}(T_1, -\tau, T_2, 0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1, T_{min} - \tau, T_2, 0)\|_{\mathcal{L}^p/\mathcal{L}^p}, \quad -T_{min} < \tau < T_{min}, \quad -T_1 \le T_{min} - \tau < T_1$$

These properties are not only important to understand the effects of nonsynchronism on the multirate sampled-data system, but also very useful since these properties suggest that we only have to compute the performance in an interval of just one parameter τ_1 or τ_2 instead of the whole range of two parameters τ_1, τ_2 and their combinations.

4 State-space representation of nonsynchronous systems

This section develops state-space representation of nonsynchronous systems. It will be shown that the new representation is useful for assessing stability and performance of nonsynchronous systems.

First, we considers the single-channel sampled-data system shown in Fig.2. We shall derive a state-space realization of the nonsynchronous sampled-data controller $\mathcal{H}^\tau K \mathcal{S}^\tau$. Suppose that the discrete-time controller K is described in the state space as

$$K : \begin{cases} x_K(k+1) = A_K x_K(k) + B_K y(k) \\ u(k) = C_K x_K(k) + D_K y(k), \quad k \in \mathcal{Z}^+ \ . \end{cases} \tag{16}$$

From (9),(10) and (11),

$$\mathcal{H}^{\tau} K \mathcal{S}^{\tau} = D_{\tau} \mathcal{H}^0 K \mathcal{S}^0 D_{-\tau}. \tag{17}$$

holds. Let us define a discrete-time lifting operator $W_n : \ell \to \ell$:

$$\hat{f} = W_n f = \left\{ \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(n-1) \end{bmatrix}, \begin{bmatrix} f(n) \\ f(n+1) \\ \vdots \\ f(2n-1) \end{bmatrix}, \cdots, \right\} \tag{18}$$

for a given positive integer n and $f = \{f(k)\}_{k=0}^{\infty} \in \ell$. Define a fast sampler and a fast holding operator as follows:

$$\begin{aligned} y = \mathcal{S}_F x, &\quad y(k) = x(kT/n), \quad k \in \mathcal{Z}_+ \\ w = \mathcal{H}_F v, &\quad w(kT/n + t) = v(k), \quad 0 < t \leq T/n \end{aligned}$$

for $y, v \in \ell$ and $x, w \in \mathcal{L}^p \cap \mathtt{PC}$. Using the lifting operator W_n we have

$$\mathcal{H}^0 K \mathcal{S}^0 = \mathcal{H}_F W_n^{-1} \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} K \overbrace{\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}}^{n \text{ columns}} W_n \mathcal{S}_F \; . \tag{19}$$

Since any integer m satisfies

$$\mathcal{S}_F D_{mT/n} = \sigma_m \mathcal{S}_F, \quad D_{mT/n} \mathcal{H}_F = \mathcal{H}_F \sigma_m, \tag{20}$$

we obtain

$$\begin{aligned} D_{mT/n} \mathcal{H}^0 K \mathcal{S}^0 D_{-mT/n} &= \mathcal{H}_F \sigma_m W_n^{-1} \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} K \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} W_n \sigma_{-m} \mathcal{S}_F \\ &= \mathcal{H}_F W_n^{-1} \hat{\sigma}_m \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} K \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \hat{\sigma}_{-m} W_n \mathcal{S}_F, \end{aligned} \tag{21}$$

where $\hat{\sigma}_m$ and $\hat{\sigma}_{-m}$ are shift operators on discrete-time lifted spaces defined by

$$\hat{\sigma}_m := W_n \sigma_m W_n^{-1}, \quad \hat{\sigma}_{-m} := W_n \sigma_{-m} W_n^{-1}.$$

Suppose that τ/T is a rational number.

$\tau > 0$ **case :** We can find integers $n > m > 0$ satisfying $\tau = Tm/n$. A state-space realization of the shift operator $\hat{\sigma}_1 : \hat{u}_S \mapsto \hat{y}_S$ is

$$\begin{aligned} x_S(k+1) &= \overbrace{\begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix}}^{n \text{ columns}} \hat{u}_S(k), \quad x_S(0) = 0 \\ \hat{y}_S(k) &= \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_S(k) + \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ I & 0 & \ddots & & \vdots \\ 0 & I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix} \hat{u}_S(k) \end{aligned}$$

so that the m-th right-shift operator is described in the state space by $\hat{\sigma}_m = (\hat{\sigma}_1)^m$ with the state $x_{\sigma m}$ and $x_{\sigma m}(0) = 0$. From the state space descriptions of K and $\hat{\sigma}_m$, we obtain

$$
\begin{aligned}
\hat{\sigma}_m \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} K \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \hat{\sigma}_{-m} &= \left[\begin{array}{ccc|c} 0 & \cdots & 0 & I \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & I \\ \hline & I_m & & 0 \\ & & & I \\ & 0 & & \vdots \\ & & & I \end{array}\right] \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \begin{bmatrix} \overbrace{0 \cdots 0}^{m \text{ times}} & I & 0 & \cdots & 0 \end{bmatrix} \\
&= F_{l+} K_{a+} F_{r+}, \quad x_{\sigma m}(0) = 0, \forall x_K(0) \qquad (22)
\end{aligned}
$$

$$
F_{l+} := \begin{bmatrix} & I_m & & 0 \\ 0 & \cdots & 0 & I \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & I \end{bmatrix}, \; F_{r+} := \begin{bmatrix} \underbrace{\begin{matrix} I & 0 & 0 \\ 0 & \cdots & 0 \end{matrix}}_{m \text{ columns}} & \begin{matrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \end{matrix} \end{bmatrix},
$$

where I_m denotes the $m \times m$(blocks) identity. K_{a+} is defined in the state space by

$$
K_{a+} := \left[\begin{array}{cccc|cc} 0 & \cdots & 0 & C_K & 0 & D_K \\ \vdots & \ddots & \vdots & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & C_K & 0 & D_K \\ 0 & \cdots & 0 & A_K & 0 & B_K \\ \hline & I_m & & 0 & 0 & 0 \\ 0 & \cdots & 0 & C_K & 0 & D_K \end{array}\right] \qquad (23)
$$

with the state variable $x_{Ka+} = [x_{\sigma m}^T, x_K^T]^T$.

$\tau < 0$ **case :** With integers $-n < m < 0$ satisfying $\tau = Tm/n$, we have

$$
\begin{aligned}
\hat{\sigma}_m \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} K \begin{bmatrix} I & 0 \cdots 0 \end{bmatrix} \hat{\sigma}_{-m} &= \hat{\sigma}_m \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} \left[\begin{array}{cccc|ccccccc} A_K & B_K & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 & 0 & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & I \\ \hline C_K & D_K & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{array}\right] \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} (-m) \text{ rows} \\ \\ \end{array} \\
&= F_{l-} K_{a-} F_{r-}, \quad x_{\sigma m}(0) = 0, \forall x_K(0) \qquad (24)
\end{aligned}
$$

$$
F_{l-} := \begin{bmatrix} I & 0 \\ \vdots & \vdots \\ I & 0 \\ 0 & I \\ \vdots & \vdots \\ 0 & I \end{bmatrix} \begin{array}{l} \\ \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} (-m) \text{ 's} \end{array} \qquad F_{r-} := \begin{bmatrix} I & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & & I_{(-m)} & \end{bmatrix}.
$$

The system K_{a-} is defined in the state space by

$$
K_{a-} := \left[\begin{array}{cccc|cccc} A_K & B_K & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & I & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & I \\ \hline C_K & D_K & 0 & 0 & 0 & 0 & \cdots & 0 \\ C_K A_K & C_K B_K & 0 & 0 & 0 & D_K & 0 & 0 \end{array}\right] \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} (-m) \text{ rows} \\ \\ \\ \end{array} \qquad (25)
$$

with the state variable $x_{Ka-} = [x_K^T, x_{\sigma m}^T]^T$.

The input-output property of $\mathcal{H}^\tau K \mathcal{S}^\tau$ is obviously identical with that of $\mathcal{H}_F W_n^{-1} F_{l+} K_{a+} F_{r+} W_n \mathcal{S}_F$ (or $\mathcal{H}_F W_n^{-1} F_{l-} K_{a-} F_{r-} W_n \mathcal{S}_F$ for $\tau < 0$). Because of the additional state variable $x_{\sigma m}$, the equivalence of internal stability of the closed-loop system defined with $\mathcal{H}^\tau K \mathcal{S}^\tau$ and the closed-loop with

$\mathcal{H}_F W_n^{-1} F_{l+} K_{a+} F_{r+} W_n \mathcal{S}_F$ (or $\mathcal{H}_F W_n^{-1} F_{l-} K_{a-} F_{r-} W_n \mathcal{S}_F$) is not straightforward consequence unless $x_{\sigma m}(0)$ is restricted to be zero. We, however, can prove the following.

Theorem 5 *Suppose that τ/T is a rational number. The system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ in Fig.2 is internally stable if and only if*

$$\Sigma[G, \mathcal{H}_F W_n^{-1} F_{l\pm} K_{a\pm} F_{r\pm} W_n \mathcal{S}_F] \text{ with the state variable } x(t)$$
$$\text{where } \begin{cases} F_{l\pm} := F_{l+},\ K_{a\pm} := K_{a+},\ F_{r\pm} := F_{r+},\ x = [x_G^T, x_{\sigma m}^T, x_K^T]^T & \text{for } \tau > 0 \\ F_{l\pm} := F_{l-},\ K_{a\pm} := K_{a-},\ F_{r\pm} := F_{r-},\ x = [x_G^T, x_K^T, x_{\sigma m}^T]^T & \text{for } \tau < 0 \end{cases}$$

is internally stable in terms of $\forall x_{\sigma m}(0)$, $\forall x_K(0)$, $\forall x_G(0)$. Moreover, if $x_{\sigma m}(0)$ is restricted to zero, the state transition $(x_G(t), x_K(k))$ of both the systems are identical for the same w and $(x_G(0), x_K(0))$.

Hence, we obtain the following theorem.

Theorem 6 *Suppose that τ/T is a rational number. Given a scalar $\gamma > 0$, the system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ in Fig.2 has $\mathcal{L}^p$ disturbance attenuation γ if and only if $\Sigma[G, \mathcal{H}_F W_n^{-1} F_{l\pm} K_{a\pm} F_{r\pm} W_n \mathcal{S}_F]$ has $\mathcal{L}^p$ disturbance attenuation γ.*

This theorem is very useful for solving analysis problems. In fact, the $\mathcal{L}^p$ disturbance attenuation γ of $\Sigma[G, \mathcal{H}_F W_n^{-1} F_{l\pm} K_{a\pm} F_{r\pm} W_n \mathcal{S}_F]$ can be easily assessed by using [29, 5, 16, 14]. It should be noted that to obtain the least dimensional input/output spaces involved in the computation, we should choose n and m such that they are coprime.

Next, consider the two-channel nonsynchronous sampled-data control system shown in Fig.3 defined with the multirate sampler $\mathcal{S}^{\tau_i}$ and hold $\mathcal{H}^{\tau_i}$, $i = 1, 2$. Let $\bar{K}$ be

$$\bar{K} := \mathrm{diag}[\mathcal{H}_{F1} W_{n_1}^{-1} F_{l1} K_{a1} F_{r1} W_{n_1} \mathcal{S}_{F1}, \mathcal{H}_{F2} W_{n_2}^{-1} F_{l2} K_{a2} F_{r2} W_{n_2} \mathcal{S}_{F2}]$$

where $\mathcal{H}_{Fi}, F_{li}, K_{ai}, F_{ri}$ and $\mathcal{S}_{Fi}$ are defined appropriately as $\mathcal{H}_F, F_{l\pm}, K_{a\pm}, F_{r\pm}$ and $\mathcal{S}_F$ in Theorem 6 for each i. Now, the main result of this section is given by the following , which can be derived straightforwardly from Theorem 6.

Theorem 7 *Suppose that τ_i/T_i is a rational number for all $i = 1, 2$ and that integers m_i and n_i satisfy $\tau_i = T_i m_i / n_i$ and $n_i > 0$ for every i. Given a scalar $\gamma > 0$, the system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ in Fig.3 has $\mathcal{L}^p$ disturbance attenuation γ if and only if $\Sigma[G, \bar{K}]$ has $\mathcal{L}^p$ disturbance attenuation γ.*

Several methods of assessing $\mathcal{L}^p$ disturbance attenuation of $\Sigma[G, \bar{K}]$ is readily available in [29, 5, 16, 14]. Theorem 6 and Theorem 7 restrict the phase τ_i to a number for which τ_i/T_i is rational. Section 8 of this paper focuses on the irrational case.

5 Examples

Consider the nonsynchronous multirate sampled-data control of the continuous-time plant G:

$$P = \left[\begin{array}{cc|cc} 1 & 0.2 & 1 & 0 \\ 0.1 & -10 & 0.2 & 1 \\ \hline -1 & 1 & 0 & 0 \\ 0.1 & -7 & 0 & 0 \end{array}\right], \quad G = \begin{bmatrix} V & P \\ V & P \end{bmatrix},$$

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix},\ V_1 = \frac{10(s+2)}{(s+0.2)(s+10)},\ V_2 = \frac{10(s+20)}{(s+2)(s+10)},$$

Table 1: $\mathcal{L}^2$-induced norm vs. phase (τ_1, τ_2)

Delay	-0.375	-0.25	-0.125	0	$+0.125$	$+0.25$	$+0.375$
$(\tau_1, 0)$	66.5	67.8	67.0	64.1	66.5	67.8	67.0
$(0, \tau_2)$	67.0	67.8	66.6	64.1	67.0	67.8	66.6

Table 2: $\mathcal{L}^2$-induced norm vs. phase (τ_1, τ_2)

Delay	-0.45	-0.3	-0.15	0	$+0.15$	$+0.3$	$+0.45$
$(\tau_1, 0)$	10.2	9.4	10.2	13.2	10.2	9.4	10.2
$(0, \tau_2)$	10.2	9.4	10.2	13.2	10.2	9.4	10.2

which represents an $\mathcal{L}^2$ worst-case minimization problem of the sensitivity at the output of P. Table 1 shows the result of $\mathcal{L}^2$-induced norm analysis of the closed-loop system with the two-channel multirate controller K given by

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad T_1 = 1, \quad T_2 = 0.5, \quad K_1 = \frac{4.5}{(z+0.5)}, \quad K_2 = \frac{-0.7}{z-0.01} .$$

All the resulting closed-loop systems are internally stable. It can be seen that in Table 1, the $\mathcal{L}^2$-induced norms for various τ_i are obtained in the order of Theorem 4.

Next, consider the continuous-time plant G given by

$$G = \left[\begin{array}{cc|ccc} 0.2 & 0.2 & -1 & -1 & -2 \\ 0.1 & -1 & 1 & 0.1 & 0.3 \\ \hline 1 & -2 & 0 & 0 & 0 \\ -1 & 0.8 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 \end{array}\right] .$$

The two-channel controller is chosen as

$$T_1 = T_2 = 0.6, \quad K_1 = \frac{-2}{z+0.5}, \quad K_2 = \frac{0.2}{z+0.1} .$$

The $\mathcal{L}^2$-induced norms are shown in Table 2. Every sampled-data controller internally stabilizes the closed-loop. The table shows that the phase shift not only degrades the performance but also improves it.

Figure.5 (Fig.6) is an illustration of degradation (improvement) of the performance computed in the first example (the second example, respectively). The computation was performed only at a finite number of rational relative phases. Nevertheless, owing to the results in the next section, it is reasonable to draw a continuous curve as shown in Fig.5 and 6 if the rational numbers picked are sufficiently dense.

6 Worst-case performance design

We shall derive another type of the state-space equivalents for nonsynchronous systems which have the less number of state variables and we shall deal with the controller synthesis problem in this section.

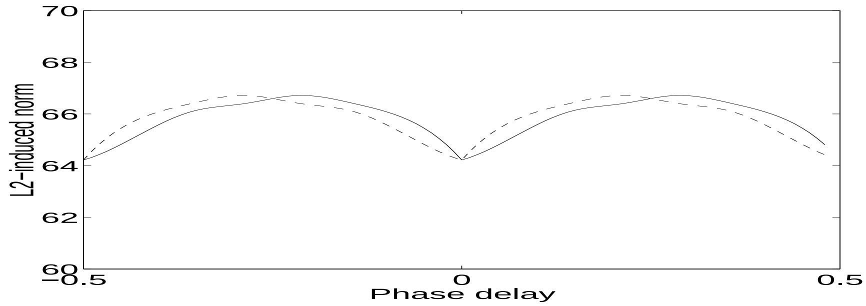

Figure 5: $\mathcal{L}^2$-induced norm vs. phase delay τ_i :
function of τ_1 with $\tau_2 = 0$ (solid); function of τ_2 with $\tau_1 = 0$ (dashed).

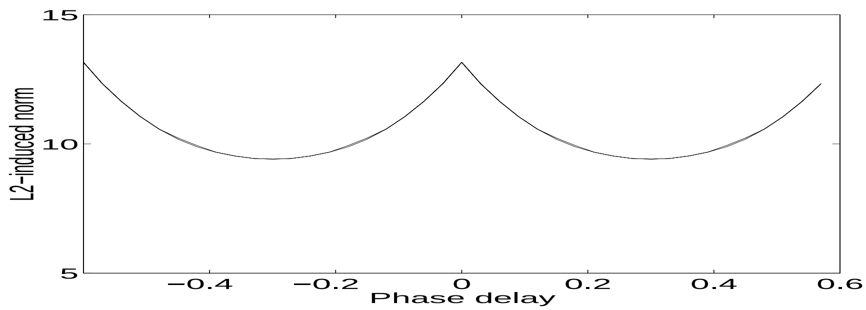

Figure 6: $\mathcal{L}^2$-induced norm vs. phase delay τ_i :
function of τ_1 with $\tau_2 = 0$(solid); function of τ_2 with $\tau_1 = 0$ (dashed).

To design nonsynchronous controllers by using Theorem 6 or Theorem 7, one might first find K_{a+} or K_{a-}, then K is extracted from K_{a+} or K_{a-}. However, in general, K_{a+} and K_{a-} does not have the desired structure in the state space to extract K. This is why Theorem 6 and Theorem 7 are not suitable for controller design. To overcome this difficulty, we will develop input-output equivalents of K_{a+} and K_{a-}.

$\tau > 0$ **case :** Suppose that integers $n > m > 0$ satisfy $\tau = Tm/n$. Using the state transformation matrix

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ -I & I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and removing uncontrollable state variables we have

$$K_{a+} \cong \left[\begin{array}{cc|cc} 0 & C_K & 0 & D_K \\ 0 & A_K & 0 & B_K \\ \hline I & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ I & 0 & 0 & 0 \\ 0 & C_K & 0 & D_K \end{array}\right] \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} m \text{ times} \qquad = \begin{bmatrix} I & 0 \\ \vdots & \vdots \\ I & 0 \\ 0 & I \end{bmatrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} m \text{ times} \quad \begin{bmatrix} z^{-1} I \\ I \end{bmatrix} K \begin{bmatrix} 0 & I \end{bmatrix}. \qquad (26)$$

Here, z^{-1} denotes the z-transform of the unit delay.

$\tau < 0$ **case :** Suppose that integers $-n < m < 0$ satisfy $\tau = Tm/n$. Removing unobservable state variables, we obtain

$$K_{a-} \cong \left[\begin{array}{cc|cccc} A_K & B_K & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \hline C_K & D_K & 0 & 0 & \cdots & 0 \\ C_K A_K & C_K B_K & 0 & D_K & 0 & 0 \end{array}\right] = \begin{bmatrix} z^{-1}I \\ I \end{bmatrix} K \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} 0 & \overbrace{0 \quad \cdots \quad 0}^{(-m)\text{ columns}} \\ 0 & I \quad 0 \quad 0 \end{bmatrix}. \tag{27}$$

Since uncontrollable modes and unobservable modes removed are stable, Theorem 5 yields the following.

Theorem 8 *Suppose that τ/T is a rational number. Define*

- *for $\tau > 0$*

$$F_l := F_{l+} \begin{bmatrix} I & 0 \\ \vdots & \vdots \\ I & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m \text{ times} \\ \\ \end{matrix} \begin{bmatrix} z^{-1}I \\ I \end{bmatrix}, \qquad F_r := \begin{bmatrix} 0 & I \end{bmatrix} F_{r+} \tag{28}$$

- *for $\tau < 0$*

$$F_l := F_{l-} \begin{bmatrix} z^{-1}I \\ I \end{bmatrix}, \qquad F_r := \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} 0 & \overbrace{0 \quad \cdots \quad 0}^{(-m)\text{ columns}} \\ 0 & I \quad 0 \quad 0 \end{bmatrix} F_{r-} \tag{29}$$

- *for $\tau = 0$*

$$F_l := \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}, \qquad F_r := \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}. \tag{30}$$

Given a scalar $\gamma > 0$, the system $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ in Fig.2 has $\mathcal{L}^p$ disturbance attenuation γ if and only if $\Sigma[G, \mathcal{H}_F W_n^{-1} F_l K F_r W_n \mathcal{S}_F]$ has $\mathcal{L}^p$ disturbance attenuation γ. Moreover, if the initial state of z^{-1} in F_l is restricted to zero, the state transition $(x_G(t), x_K(k))$ of both the systems are identical for the same w and $(x_G(0), x_K(0))$.

One advantage of using Theorem 8 is that the theorem only adds one state variable associated with z^{-1} to the original system for obtaining the equivalent. This is a contrast to Theorem 6 which requires additional m state variables. This is a great improvement in computation since dimensions of Riccati equations involved in the analysis and synthesis are equal to the size of the state vector. Another advantage of Theorem 8 is that the transformed system is amenable to proposing a controller design method. Now define the hybrid plant including the fast sampler and hold:

$$G_h := \begin{bmatrix} I & 0 \\ 0 & F_r W_n \mathcal{S}_F \end{bmatrix} G \begin{bmatrix} I & 0 \\ 0 & \mathcal{H}_F W_n^{-1} F_l \end{bmatrix}. \tag{31}$$

In the $p = 2$ case, the following theorem is obtained.

Theorem 9 *Suppose that τ/T is a rational number. Given a scalar $\gamma > 0$, the sampled-data system $\Sigma[G, \mathcal{H}_F W_n^{-1} F_l K F_r W_n \mathcal{S}_F]$ has $\mathcal{L}^2$ disturbance attenuation γ if and only if $\Sigma[\hat{G}, K]$ has ℓ^2 disturbance attenuation γ, where $\hat{G}$ is the discrete-time worst-case equivalent of G_h.*

Theorem 9 is proved easily using Theorem 8 and results shown in [29, 5, 16, 14]. The transformed system $\hat{G}$ can be computed directly from G_h (see [29, 5, 16, 14] for details). Note that $\hat{G}$ is dependent on γ. Since K is a standard FDLSI controller (without causality constraints[29, 5]), the solution K can be obtained as a standard solution to discrete-time $\mathcal{H}^\infty$ control of $\hat{G}$.

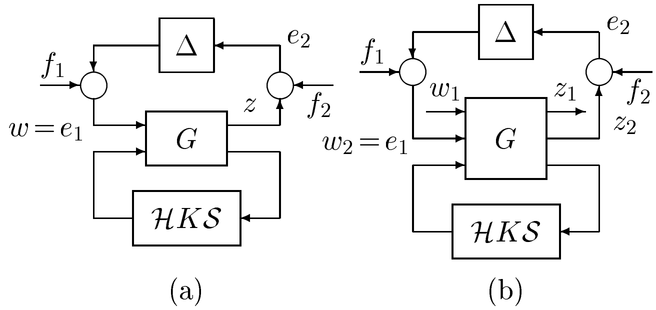

Figure 7: Stability and performance robustness.

Consider the two-channel nonsynchronous sampled-data control system in Fig.3 again. Let $\tilde{K}$ be defined by

$$\tilde{K} := \mathrm{diag}[\mathcal{H}_{F1} W_{n_1}^{-1} F_{l1} K_1 F_{r1} W_{n_1} \mathcal{S}_{F1}, \mathcal{H}_{F2} W_{n_2}^{-1} F_{l2} K_2 F_{r2} W_{n_2} \mathcal{S}_{F2}]$$

where $\mathcal{H}_{Fi}, F_{li}, F_{ri}$ and $\mathcal{S}_{Fi}$ are defined appropriately as $\mathcal{H}_F, F_l, F_r$ and $\mathcal{S}_F$ in Theorem 8 for each i. The following can be derived in the same way as Theorem 8.

Theorem 10 *Suppose that τ_i/T_i is rational for all $i = 1, 2$ and that integers m_i and n_i satisfy $\tau_i = T_i m_i/n_i$ and $n_i > 0$ for every i. Given a scalar $\gamma > 0$, the system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ in Fig.3 has $\mathcal{L}^p$ disturbance attenuation γ if and only if $\Sigma[G, \tilde{K}]$ has $\mathcal{L}^p$ disturbance attenuation γ.*

Again, the discrete-time worst-case equivalent $\hat{G}$ is computed using [29, 16, 14] when T_i/T_j is rational for all $i, j = 1, 2, \ldots, q$. If T_i/T_j are rational, the integers m_i and n_i can be chosen as

$$n_i = T_i/T_D, \quad m_i = \tau_i/T_D, i = 1, 2, \ldots, q \tag{32}$$

where T_D is the greatest common divisor among $\{T_i\}$ and $\{\tau_i\}$. A desired controller K can be obtained as a decentralized control solution to a popular problem of discrete-time $\mathcal{H}^\infty$ control of $\Sigma[\hat{G}, K]$ (e.g. [16, 17] and references therein). Note again that each controller K_i is a standard FDLSI controller without causality constraints.

7 Nonsynchronous robustness analysis

This section tackles stability and performance robustness of nonsynchronous multirate sampled-data systems.

Consider the nonsynchronous system shown in Fig.1. Making use of a result in [15], we can prove the following.

Lemma 8 *Suppose that that all the T_i/T_j and τ_i/T_i, $i, j = 1, \ldots, q$ are rational numbers. If the nonsynchronous sampled-data system $\Sigma[G, \mathcal{H}K\mathcal{S}]$ shown in Fig.1 is internally stable, there exists a unique causal linear operator T_{zw} and it is bounded on $\mathcal{L}^p$ for all $p \in [1, \infty]$.*

Consider the uncertain nonsynchronous multirate sampled-data system shown in Fig.7(a). The system Δ belongs to

$$\boldsymbol{B\Delta}_{PTV} := \{\Delta : \text{linear}, T_C\text{-periodic}, \text{causal}, \|\Delta\|_{\mathcal{L}^p/\mathcal{L}^p} < 1/\gamma\} \ ,$$

where T_C is a positive real number.

Definition 1 *The system shown in Fig.7(a) is said to be robustly $\mathcal{L}^p$ stable with respect to $\boldsymbol{B\Delta}_{PTV}$ if the mapping from $[f_1^T, f_2^T]^T$ to $[e_1^T, e_2^T]^T$ is bounded on $\mathcal{L}^p \times \mathcal{L}^p$ for all $\Delta \in \boldsymbol{B\Delta}_{PTV}$.*

With the aid of Lemma 8 and Theorem 10, the next theorem is obtained.

Theorem 11 *Suppose that all the T_i/T_j and τ_i/T_i, $i, j = 1, \ldots, q$ are rational numbers and T_C denotes the least common multiple among $\{T_i\}$. Assume that $\Sigma[G, \mathcal{H}K\mathcal{S}]$ is internally stable. Then, the nonsynchronous system shown in Fig.7(a) is robustly $\mathcal{L}^2$ stable with respect to $\boldsymbol{B\Delta}_{PTV}$ if and only if $\Sigma[G, \mathcal{H}K\mathcal{S}]$ has $\mathcal{L}^2$ disturbance attenuation less than or equal to γ. If Δ is FDLTI and if it is stabilizable and detectable in the state space, the robust $\mathcal{L}^2$ stability implies internal stability.*

The above theorem states that the $\mathcal{L}^p$ disturbance attenuation is sufficient for robust $\mathcal{L}^p$ stabilization. It can be claimed that whatever the uncertain system is, the sufficiency is straightforward from the small-gain theorem. However, it should be noted that Theorem 11 demonstrates that the input-output stability guarantees the internal stability even in the nonsynchronous case if the sample-and-hold schedule is periodic.

Another uncertain system is shown in Fig.7(b). Here, Δ belongs to $\boldsymbol{B\Delta}_{TV}$:

$$\boldsymbol{B\Delta}_{TV} := \{\Delta : \text{Linear, causal}, \|\Delta\|_{\mathcal{L}^p/\mathcal{L}^p} < 1/\gamma\} \ .$$

Definition 2 *The system shown in Fig.7(b) is said to achieve robust $\mathcal{L}^p$ performance with respect to $\boldsymbol{B\Delta}_{TV}$ if the system is robustly $\mathcal{L}^p$ stable and if the operator from w_1 to z_1 has $\mathcal{L}_p$ disturbance attenuation less than or equal to γ for all $\Delta \in \boldsymbol{B\Delta}_{TV}$.*

The following statement is true even if T_i/T_j and τ_i/T_i, $i, j = 1, \ldots, q$ are not rational numbers.

Theorem 12 *Suppose that the nonsynchronous system $\Sigma[G, \mathcal{H}K\mathcal{S}]$ has a unique causal linear operator T_{zw} on $\mathcal{L}^p$. Then, the system shown in Fig.7(b) achieves robust $\mathcal{L}^p$ performance with respect to $\boldsymbol{B\Delta}_{TV}$ if $\Sigma[G, \mathcal{H}K\mathcal{S}]$ has $\mathcal{L}^p$ disturbance attenuation less than or equal to γ.*

Note that this theorem allows T_i/T_j and τ_i/T_i to be irrational numbers so that the system is no longer periodic. Therefore, to prove the robust performance, we cannot use the argument of the equivalence between the stability and performance robustness on which Khammash [21] and Sivashankar and Khargonekar[26] rely for dealing with periodic synchronous systems. In fact, for periodic nonsynchronous systems, Theorem 11 enables us to establish the equivalence between the stability and performance robustness, while Theorem 12 does not establish the equivalence.

8 Continuity properties of closed-loop performance

Consider the two-channel sampled-data system depicted in Fig.3. We suppose that the continuous-time plant G is described by (1) in the state space. In this section, the performance of the closed-loop system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ is measured by $\|T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^p/\mathcal{L}^p}$ for $p \in [1, \infty]$. This section analyzes continuity of the closed-loop performance as a function of the phase shift τ_i. Is the closed-loop measure is a continuous function of an open-loop change caused by the phase variation? In other words, the question is whether or not the closed-loop performance of a nonsynchronous sampled-data system can be always approximated to any degree of accuracy by that of the same sampled-data system having a slight different phase. If it is discontinuous, the slight variation of the phase may cause a significant degradation of the closed-loop performance. If it has the continuity property, there will be only a slight deviation and we can estimate how robust the performance is against the phase perturbation. Another advantage of having the continuity is that a reasonable approximation of the performance of a system defined with irrational phase shift can be obtained by analyzing rationally-shifted systems belonging to an appropriately small neighborhood of the irrational system.

8.1 Phase Perturbation Operators

This subsection presents the key lemma in this section and collects properties pertinent to the input-output operators representing phase perturbation in sampling and holding operators. These perturbation operators in themselves will be shown to be unbounded. However, preceded or followed by a low-pass filter, they are bounded and well-behaved as functions of the phase τ.

Consider the following sampled-data controller of the phase τ:

$$\mathcal{H}^{\tau} K \mathcal{S}^{\tau}, \quad -T < \tau < T \ . \tag{33}$$

A sampled-data controller defined with the phase shift ν is represented by

$$\mathcal{H}^{\tau+\nu} K \mathcal{S}^{\tau+\nu}, \quad -T < \tau + \nu < T, \tag{34}$$

where ν is the difference of phases of these two controllers. From (9), (10) and (11), we have

$$\mathcal{H}^{\tau+\nu} K \mathcal{S}^{\tau+\nu} = D_{\nu} \mathcal{H}^{\tau} K \mathcal{S}^{\tau} D_{-\nu}. \tag{35}$$

The difference of two hold operators of these controllers is denoted by

$$\Lambda_O := \mathcal{H}^{\tau+\nu} - \mathcal{H}^{\tau} = (D_{\nu} - I)\mathcal{H}^{\tau}, \quad -T < \tau + \nu < T \tag{36}$$

which maps ℓ to $\mathtt{PC}$. In the same manner, the difference of two sampling operators mapping $\mathtt{PC}$ to ℓ is

$$\Lambda_I := \mathcal{S}^{\tau+\nu} - \mathcal{S}^{\tau} = \mathcal{S}^{\tau}(D_{-\nu} - I), \quad -T < \tau + \nu < T \tag{37}$$

Note that these phase perturbation operators Λ_I and Λ_O satisfy $\Lambda_I = 0$ and $\Lambda_O = 0$ if $\nu = 0$. The sampled-data controller having the phase $\tau + \nu$ is written as

$$\mathcal{H}^{\tau+\nu} K \mathcal{S}^{\tau+\nu} = (\mathcal{H}^{\tau} + \Lambda_O) K (\mathcal{S}^{\tau} + \Lambda_I). \tag{38}$$

This equation is depicted by a block diagram shown in Fig.8. Before moving on to continuity properties of Λ_I and Λ_O, we need the following lemmas.

Lemma 9 *For $-T < \tau < 0$, it holds that $\mathcal{S}^{\tau}(D_{-\nu} - I) = \sigma_1 \mathcal{S}^{T+\tau}(D_{-\nu} - I)$.*

Lemma 10 *Let H be an operator mapping ℓ^p to $\mathcal{L}^p$, $p \in [1, \infty]$. Suppose that there exists a positive number α such that $H\sigma_1 = D_{\alpha} H$ holds. Then, H satisfies*

$$\|H(I - R_n)\|_{\mathcal{L}^p/\ell^p} = \|H\|_{\mathcal{L}^p/\ell^p}, \quad n \in \mathcal{Z}_+ \ .$$

Lemma 11 *Suppose that $-T < \tau < 0$ and $\nu \geq 0$. Let F be a time-invariant operator on $\mathcal{L}^{p,e}$. Then,*

$$\|F(D_{\nu} - I)\mathcal{H}^{\tau}\|_{\mathcal{L}^p/\ell^p} = \|F(D_{\nu} - I)\mathcal{H}^{T+\tau}\|_{\mathcal{L}^p/\ell^p}, \quad p \in [0, \infty] \quad .$$

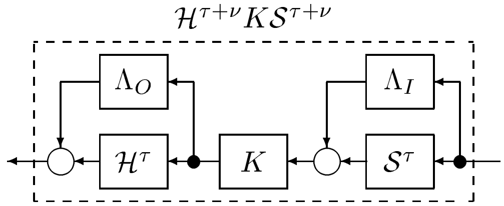

Figure 8: Phase perturbation operators.

In the rest of this subsection, we focus on functions $\|\mathcal{S}^\tau(D_{-\nu}-I)F\|_{\mathcal{L}^p/\ell^p}$, $\|F(D_\nu-I)\mathcal{H}^\tau\|_{\mathcal{L}^p/\ell^p}$ and their continuity as functions of ν. Here, F represents the identity I or any time-invariant operator to be specified later on. For this purpose, Lemma 9 and Lemma 11 suggest that properties of $\mathcal{S}^\tau$ and $\mathcal{H}^\tau$ in the $-T<\tau<0$ case is obtained precisely by using the properties of $\mathcal{S}^{T+\tau}$ and $\mathcal{H}^{T+\tau}$ with $0\le T+\tau<T$. In addition, it should be noted that this approach of replacing τ can be applied to the $\nu<0$ case of $\|F(D_\nu-I)\mathcal{H}^\tau\|_{\mathcal{L}^p/\ell^p}$ although Lemma 11 assumes $\nu\ge 0$. As it will be described, this is successful since continuity properties are obtained as functions which are independent of τ.

The following lemma clarifies the continuity property of Λ_O with respect to ν.

Lemma 12 *For any $p\in[1,\infty)$, the linear operator $(D_\nu-I)\mathcal{H}^\tau$ is $\mathcal{L}^p/\ell^p$-induced norm bounded for all $\nu\in(-T-\tau,T-\tau)$. For each $p\in[1,\infty)$, there exists a positive number δ for any positive number ϵ such that $\|(D_\nu-I)\mathcal{H}^\tau\|_{\mathcal{L}^p/\ell^p}<\epsilon$ holds for all $\nu\in(-\delta,\delta)$.*

The situation with $p=\infty$ is more delicate.

Lemma 13 *The linear operator $(D_\nu-I)\mathcal{H}^\tau$ is $\mathcal{L}^\infty/\ell^\infty$-induced norm bounded for all $\nu\in(-T-\tau,T-\tau)$. For every positive number ϵ, there exists no positive number δ guaranteeing $\|(D_\nu-I)\mathcal{H}^\tau\|_{\mathcal{L}^\infty/\ell^\infty}<\epsilon$ for all $\nu\in(-\delta,\delta)$.*

In the $p=\infty$ case, followed by a low-pass filter, the operator Λ_O has a continuity property.

Lemma 14 *Suppose that the system F is FDLTI, strictly proper and stable. The linear operator $F(D_\nu-I)\mathcal{H}^\tau$ is $\mathcal{L}^\infty/\ell^\infty$-induced norm bounded for all $\nu\in(-T-\tau,T-\tau)$. For any positive number ϵ, there exists a positive number δ such that $\|F(D_\nu-I)\mathcal{H}^\tau\|_{\mathcal{L}^\infty/\ell^\infty}<\epsilon$ holds for all $\nu\in(-\delta,\delta)$,*

The next result shows that the situation with the sampler is more complicated.

Lemma 15 *Restrict the domain of the linear operator $\mathcal{S}^\tau(D_{-\nu}-I):\mathtt{PC}\to\ell$ to $\mathtt{PC}\cap\mathcal{L}^p$. Then,*

$$\sup_{u\in\mathtt{PC}\cap\mathcal{L}^p,u\neq 0}\frac{\|\mathcal{S}^\tau(D_{-\nu}-I)u\|_{\ell^p}}{\|u\|_{\mathcal{L}^p}}\ge 1,\quad p\in[1,\infty]$$

holds for all ν but $\nu=0$.

In order to guarantee the continuity of the phase perturbation operator with respect to the phase variation, we introduce a low-pass filter preceding the sampling operator. However, by contrast with the phase perturbation of hold operators, the behavior in the $p=1$ case is completely different from the other cases.

Lemma 16 *Suppose that the system F is FDLTI, strictly proper and stable. The linear operator $\mathcal{S}^\tau(D_{-\nu}-I)F$ is $\ell^p/\mathcal{L}^p$-induced norm bounded for all $\nu \in (-T-\tau, T-\tau)$ and for every $p \in [1,\infty]$. Moreover, for each $p \in (1,\infty]$, there exists a positive number δ for any positive number ϵ such that $\|\mathcal{S}^\tau(D_{-\nu}-I)F\|_{\ell^p/\mathcal{L}^p} < \epsilon$ holds. for all $\nu \in (-\delta,\delta)$.*

Here, it should be noted that for the $p=1$ case, the existence of δ was not proved in lemma 16. In fact, the following shows that the operator is not a continuous function of ν around zero unless the roll-off rate of the filter F is large enough.

Lemma 17 *Let the system F be a first-order delay described by the transfer function $F(s) = b/(s-a)$, $a<0$. Assume that $\nu \neq 0$. Then, there exists a real constant $\epsilon > 0$ such that*

$$\sup_{u \in \mathcal{L}^1, u \neq 0} \frac{\|\mathcal{S}^\tau(D_{-\nu}-I)Fu\|_{\ell^1}}{\|u\|_{\mathcal{L}^1}} \geq \epsilon$$

holds independently of ν.

The continuity property can be recovered if the gain of the filter F falls at not less than 40dB/decade.

Lemma 18 *Suppose that the system F_1 is causal, FDLTI and stable and that it has relative degree more than or equal to 2. The linear operator $\mathcal{S}^\tau(D_{-\nu}-I)F_1$ is $\ell^1/\mathcal{L}^1$-induced norm bounded for all $\nu \in (-T-\tau, T-\tau)$. Moreover, for any positive number ϵ, there exists a positive number δ such that $\|\mathcal{S}^\tau(D_{-\nu}-I)F_1\|_{\ell^1/\mathcal{L}^1} < \epsilon$ holds for all $\nu \in (-\delta,\delta)$.*

8.2 Continuity of closed-loop measure

Consider the two-channel sampled-data system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ shown in Fig.3. Subdivide G into a new block matrix form as

$$\begin{bmatrix} z \\ y_c \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u_c \end{bmatrix}.$$

Since the continuous-time plant G is supposed to be described by (1), the LTI operator G can be always decomposed as

$$G = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix} E, \quad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}. \tag{39}$$

The size of E_{ij} is the same as that of G_{ij}. Here, F is an FDLTI strictly proper system and it has the form of

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}.$$

Each system F_i is square in size, which is compatible with the measurement output y_{ci} of G. The system F_i is considered as an anti-aliasing low-pass filter of any order with any bandwidth. Then, the system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ is identical with $\Sigma[E, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}F]$ shown in Fig.9 since

$$T_{zw}(T_1,\tau_1,T_2,\tau_2) = \mathcal{F}_\ell(G, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}) = \mathcal{F}_\ell(E, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}F) \ .$$

In the rest of this subsection, we assume that τ_i/T_i and T_i/T_j are rational numbers for every $i,j = 1,2$.

Lemma 19 *Suppose that the multirate system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ is internally stable. Consider the system $\Sigma[E, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}F]$ in Fig 9, where G is decomposed as (39). Then, the mapping from $[w^T, h^T, g^T, f^T, d^T]^T$ to $[z^T, v_c^T, u_c^T, y^T, u^T]^T$ is a bounded linear operator on $\mathcal{L}^p \times \mathcal{L}^p \times \ell^p \times \ell^p$.*

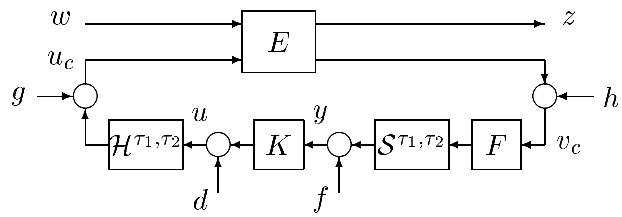

Figure 9: Stability of hybrid system

The main result of this section establishes the continuity property of the closed-loop performance as a function of the phase.

Theorem 13 *Consider the two-channel sampled-data system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ in Fig.3. Suppose that $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ is internally stable and that F in (39) is stable and strictly proper. Let p be an arbitrary number in $(1, \infty)$. Then, given any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that the system has a unique causal operator $T_{zw}(T_1, \tau_1 + \nu_1, T_2, \tau_2 + \nu_2)$ mapping from w to z on $\mathcal{L}^p$ and $\|T_{zw}(T_1, \tau_1+\nu_1, T_2, \tau_2+\nu_2) - T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^p/\mathcal{L}^p} < \epsilon$ holds for all $\nu_1 \in (-\delta_1, \delta_1)$ and $\nu_2 \in (-\delta_2, \delta_2)$.*

Now, as $\|\Lambda_{I,i}\|_{\ell^1/\mathcal{L}^1}$ is not a continuous function of ν around zero, using an anti-aliasing filter with an appropriate roll-off rate we can prove the continuity of the closed-loop measure T_{zw} in terms of the $\mathcal{L}^1$-induced norm. The proof of the following theorem requires Lemma 18 instead of Lemma 16. The rest of the proof is the same as Theorem 13.

Theorem 14 *Consider the two-channel sampled-data system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ in Fig.3. Suppose that $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ is internally stable and that F in (39) is a stable system whose transfer function has relative degree more than or equal to 2. Then, there exist $\delta_1, \delta_2 > 0$ such that the system has a unique causal operator $T_{zw}(T_1, \tau_1 + \nu_1, T_2, \tau_2 + \nu_2)$ mapping from w to z on $\mathcal{L}^1$ and $\|T_{zw}(T_1, \tau_1+\nu_1, T_2, \tau_2+\nu_2) - T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^1/\mathcal{L}^1} < \epsilon$ holds for all $\nu_1 \in (-\delta_1, \delta_1)$ and $\nu_2 \in (-\delta_2, \delta_2)$.*

In the $\mathcal{L}^\infty$ case, as described in Lemma 13, $\|\Lambda_{O,i}\|_{\mathcal{L}^\infty/\ell^\infty}$ is independent of ν. To obtain the continuity of $\|T_{zw}\|_{\mathcal{L}^\infty/\mathcal{L}^\infty}$ with respect to ν, the operator $\Lambda_{O,i}$ should be followed by a low pass filter. We assume that the LTI operator G is given by

$$G = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix} E \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix} . \tag{40}$$

Here, Q is an FDLTI strictly proper system and without loss of generality, the operator has the block-diagonal form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

Each operator Q_i is square in size, which is compatible with the control input u_{ci} of G. Making use of Lemma 14, we can prove the following.

Theorem 15 *Consider the two-channel sampled-data system $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ in Fig.3. Suppose that $\Sigma[G, \mathcal{H}^{\tau_1,\tau_2} K \mathcal{S}^{\tau_1,\tau_2}]$ is internally stable and that F and Q in (40) are stable and strictly proper. Then, there exist $\delta_1, \delta_2 > 0$ such that the system has a unique causal operator $T_{zw}(T_1, \tau_1+\nu_1, T_2, \tau_2+\nu_2)$ mapping from w to z on $\mathcal{L}^\infty$ and $\|T_{zw}(T_1, \tau_1 + \nu_1, T_2, \tau_2 + \nu_2) - T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^\infty/\mathcal{L}^\infty} < \epsilon$ holds for all $\nu_1 \in (-\delta_1, \delta_1)$ and $\nu_2 \in (-\delta_2, \delta_2)$.*

9 Concluding remarks

This paper has shown that the $\mathcal{L}^p$ worst-case performance of nonsynchronous multirate sampled-data systems can be computed exactly whenever the phase lag τ_i and the sampling period T_i are rationally related. Since the set of rational numbers is dense in $(-\infty, \infty)$, in any case, an irrational relative phase τ_i/T_i can always be approximated to any degree of accuracy by a rational number. However, the relationship between the $\mathcal{L}^p$ performance and the rational approximation needed further theoretical study. For this purpose, the paper has succeeded in proving that the performance measure is a continuous function of the phase shift. Therefore, a reasonable approximation of the performance of a system operating with irrational phase shift can be obtained by analyzing a rationally-shifted system belonging to a sufficiently small neighborhood of the irrational system. We can also estimate how robust the performance is against the phase perturbation. This paper has also developed an approach to the problem of nonsynchronous multirate controller design.

An important feature of the analysis and synthesis in this paper is that each controller is restricted to be a standard synchronous FDLSI system. The paper has not enlarged the class of controllers to include genuine aperiodic controllers since engineers hardly implement a genuine aperiodic controller in a decentralized control station on purpose although the overall decentralized controller may not be synchronized.

Finnally, the author makes some comments on the design approach presented in Section 6. As mentioned in the section, the approach is not directly applicable to the systems in which τ_i/T_i is an irrational number. Nevertheless, given any degree of accuracy, there exists a tractable system defined with rational τ_i/T_i such that the solution to the performance design of the rational system yields a controller that achieves almost the same performance for the irrrational τ_i/T_i in terms of the $\mathcal{L}^p$-induced norm. Explicit bounds for the accuracy can be obtained by making use of the results in Section 8. Furthermore, the robustness result in Section 7 tells us that the controller obtained using the rational approximation of τ_i/T_i guarantees almost the same level of stability and performance robustness of the actual system operating with irrational τ_i/T_i as the approximated system achieves in the design procedure.

References

[1] B. A. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with application to sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 418–435, 1992.

[2] M. C. Berg, N. Amit and J. Powell, "Multirate digital control system design," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 1139–1150, 1988.

[3] M. J. Chen and C. A. Desoer, "Necessary and sufficient condition for robust stability of linear distributed feedback systems," *Int. J. Contr.*, vol. 35, pp. 255–267, 1982.

[4] T. Chen and B. A. Francis, "$\mathcal{H}^2$-optimal sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 387–397, 1991.

[5] T. Chen and L.Qiu, "$\mathcal{H}^\infty$ design of general multirate sampled-data control systems," *Automatica*, vol. 30, pp. 1139-1152, 1994.

[6] P. Colaneri, R. Scattolini and N. Schiavoni, "Stabilization of multirate sampled-data linear systems," *Automatica*, vol. 26, pp. 377–380, 1990.

[7] C.A. Desoer and M. Vidyasagar, *Feedback Systems : Input-Output Properties*, New York: Academic, 1975.

[8] M. J. Er, B. D. O. Anderson and W. Yan, "Gain margin improvement using generalized sampled-data hold function based multirate output compensator," *Automatica*, vol. 30, pp. 461–470, 1994.

[9] T. Hagiwara and M. Araki, "Design of a stable feedback controller based on the multirate sampling of the plant output," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 812–819, 1988.

[10] Y. Hayakawa, S. Hara and Y. Yamamoto, "$\mathcal{H}^\infty$ type problem for sampled-data control systems – A solution via minimum energy characterization," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2278–2284, 1994.

[11] H. Ito, "Analysis and synthesis of asynchronous sampled-data control for large-scale systems," in *Proc. SICE Symp. on Dynamical Systems Theory*, Hamamatsu, Japan, 1996, pp. 29–34.

[12] H Ito, "Worst-case performance and stability of multirate sampled-data systems with asynchronous decentralized controllers", in *Proc. IEEE Amer. Contr. Conf.*, New Mexico, USA, 1997, pp. 778–783.

[13] H Ito, "Continuity property of worst-case performance measure of nonsynchronous multirate systems", in *Proc. SICE Symp. on Control Theory*, Chiba, Japan, 1997, pp. 301–306.

[14] H. Ito, T. Chuman, H. Ohmori and A. Sano, "An approach to multirate control design with multiple objectives," in *Proc. 13th IFAC World Congress*, 1996, Vol-vol. C, pp. 325–330.

[15] H. Ito and H. Ohmori and A. Sano, "Stability analysis of multirate sampled-data control systems," *IMA J. Math. Contr. Inform.*, vol. 11, pp. 341–354, 1994.

[16] H. Ito, H. Ohmori, and A. Sano, "A subsystem design approach to continuous-time performance of decentralized multirate sampled-data systems," *Int. J. Syst. Science*, vol. 26 pp. 1263–1287, 1995.

[17] H. Ito, H. Ohmori and A. Sano, "Robust performance of decentralized control systems by expanding sequential designs," *Int. J. Control*, vol. 61, pp. 1297–1311, 1995.

[18] P. T. Kabamba, "Control of linear systems using generalized hold functions," *IEEE Trans. Automat. Contr.*, vol. 32, pp. 772–783, 1987.

[19] P. T. Kabamba and S. Hara, "Worst-case analysis and design of sampled-data control systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1337–1357, 1993.

[20] R. E. Kalman and J. E. Bertram, "A unified approach to the theory of sampling systems," *J. Franklin Inst.*, vol. 267, pp. 405–436, 1959.

[21] M. H. Khammash, "Necessary and sufficient conditions for the robustness of time-varying systems with applications to sampled-data system," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 49–57, 1993.

[22] Y. Oishi, "A bound of conservativeness in sampled-data robust stabilization and its dependency on sampling period," *Technical Report 95-08*, Dept. of Math. Eng. and Inform. Physics, University of Tokyo, Japan, 1995.

[23] V.S. Ritchey and G.F. Franklin, "A stability criterion for asynchronous multirate linear systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 529–535, 1993.

[24] M. F. Sågfors and H. T. Toivonen, "Optimal $\mathcal{H}^\infty$ and LQG control of asynchronous sampled data systems," in *Proc. 13th IFAC World Congress*, 1996, vol. C, pp. 337–342.

[25] N. R. Sandell, Jr., P. Varaiya, M. Athans and M. G. Safonov, "Survey of decentralized control methods for large scale systems," *IEEE Trans. Automat. Contr.*, vol. 23, pp. 108–128, 1978.

[26] N. Sivashankar and P. P. Khargonekar, "Robust stability and performance analysis of sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 58–69, 1993.

[27] N. Sivashankar and P.P. Khargonekar, "Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems," *SIAM J. Contr. Optimization*, vol. 32, pp. 1128–1150, 1995.

[28] P.G. Voulgaris, "Control of asynchronous sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1451–1455, 1994.

[29] P.G. Voulgaris and B. Bamieh, "Optimal $\mathcal{H}^\infty$ and $\mathcal{H}^2$ control of hybrid multirate systems," *Syst. Contr. Lett.*, vol. 20, pp. 249–261, 1993.

[30] J.C. Willems, *The Analysis of Feedback Systems*, Cambridge, MA: MIT Press, 1971.

[31] Y. Yamamoto, "A function space approach to sampled data control systems and tracking problems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 703–713, 1994.

[32] Y. Yamamoto and M. Araki, "Frequency responses for sampled-data systems – Their equivalence and relationships," *Linear Algebra and its applications*, vol. 205-206, pp. 1319–1339, 1994.

Proof of Lemma 1

Since $w = D_{-\tau}D_\tau w$ holds for all $w \in \mathcal{L}^p$, we have $\{w : w \in \mathcal{L}^p\} = \{D_{-\tau}\bar{w} : \bar{w} \in \mathcal{L}^p\}$ and $\|D_{-\tau}\bar{w}\|_{\mathcal{L}^p} \leq \|\bar{w}\|_{\mathcal{L}^p}$, $\forall \bar{w} \in \mathcal{L}^p$ holds. Therefore, the $\mathcal{L}^p$ sequence of the worst-case input signal for the gain of $HD_{-\tau}$ can be constructed by $\bar{w}^* = D_\tau w^*$, where w^* is the worst-case signal for H. Then, Equation (12) follows from $\|D_\tau z\|_{\mathcal{L}^p} = \|z\|_{\mathcal{L}^p}$, $\forall z \in \mathcal{L}^p$. ∎

Proof of Lemma 2

Suppose H is T-periodic. Then, $D_{nT}H = HD_{nT}$ holds for all $n \in \mathcal{Z}_+$. From $(I - P_{nT})D_{nT} = D_{nT}$, we obtain

$$\begin{aligned}
\|H(I-P_{nT})\|_{\mathcal{L}^p/\mathcal{L}^p} &\geq \sup_v \frac{\|H(I-P_{nT})D_{nT}v\|_{\mathcal{L}^p}}{\|D_{nT}v\|_{\mathcal{L}^p}} \\
&= \sup_v \frac{\|HD_{nT}v\|_{\mathcal{L}^p}}{\|D_{nT}v\|_{\mathcal{L}^p}} \\
&= \sup_v \frac{\|D_{nT}Hv\|_{\mathcal{L}^p}}{\|D_{nT}v\|_{\mathcal{L}^p}} \\
&= \sup_v \frac{\|Hv\|_{\mathcal{L}^p}}{\|v\|_{\mathcal{L}^p}}
\end{aligned}$$

for all $n \in \mathcal{Z}_+$. Now suppose $0 \le \alpha \le \beta$. By using $\|v\| \ge \|(I-P_\beta)v\|_{\mathcal{L}^p}$ and

$$\{(I-P_\beta)v : v \in \mathcal{L}^p\} \subseteq \{(I-P_\alpha)w : w \in \mathcal{L}^p\},$$

it is shown that

$$\|H(I-P_\beta)\|_{\mathcal{L}^p/\mathcal{L}^p} \le \sup_v \frac{\|H(I-P_\beta)v\|_{\mathcal{L}^p}}{\|(I-P_\beta)v\|_{\mathcal{L}^p}} \le \sup_w \frac{\|H(I-P_\alpha)w\|_{\mathcal{L}^p}}{\|(I-P_\alpha)w\|_{\mathcal{L}^p}} = \sup_w \frac{\|H(I-P_\alpha)w\|_{\mathcal{L}^p}}{\|w\|_{\mathcal{L}^p}}$$

hold. Hence, the operator H satisfies

$$\|H\|_{\mathcal{L}^p/\mathcal{L}^p} \quad \le \quad \|H(I-P_{nT})\|_{\mathcal{L}^p/\mathcal{L}^p} \le \|H(I-P_\tau)\|_{\mathcal{L}^p/\mathcal{L}^p} \le \|H\|_{\mathcal{L}^p/\mathcal{L}^p}$$

for $0 \le \tau \le nT$ and for all $n \in \mathcal{Z}_+$. This completes the proof. ∎

Proof of Lemma 3

It can be verified that $P_{nT}\mathcal{H}^0 = \mathcal{H}^0 R_n$ holds for any $n \in \mathcal{Z}_+$. On the other hand, the sampler satisfies $\mathcal{S}^0 P_\alpha = R_n \mathcal{S}^0$, where n is the smallest integer satisfying $\alpha \le nT$. Since

$$\begin{aligned} P_\alpha \mathcal{H}^0 K \mathcal{S}^0 P_\alpha &= P_\alpha P_{nT} \mathcal{H}^0 K R_n \mathcal{S}^0 \\ &= P_\alpha \mathcal{H}^0 R_n K R_n \mathcal{S}^0 \\ &= P_\alpha \mathcal{H}^0 R_n K \mathcal{S}^0 \\ &= P_\alpha \mathcal{H}^0 K \mathcal{S}^0 \ , \end{aligned}$$

the operator $\mathcal{H}^0 K \mathcal{S}^0$ is causal in continuous time. We also have

$$P_\alpha D_\tau \mathcal{H}^0 K \mathcal{S}^0 D_{-\tau} P_\alpha = D_\tau P_{\alpha-\tau} \mathcal{H}^0 K \mathcal{S}^0 P_{\alpha-\tau} D_{-\tau} = D_\tau P_{\alpha-\tau} \mathcal{H}^0 K \mathcal{S}^0 D_{-\tau} = P_\alpha D_\tau \mathcal{H}^0 K \mathcal{S}^0 D_{-\tau} \ .$$

Hence, $\mathcal{H}^\tau K \mathcal{S}^\tau = D_\tau \mathcal{H}^0 K \mathcal{S}^0 D_{-\tau}$ proves the claim. ∎

Proof of Lemma 4

First, $\mathcal{H}^0 K \mathcal{S}^0$ is T-periodic since

$$D_T \mathcal{H}^0 K \mathcal{S}^0 = \mathcal{H}^0 \sigma_1 K \mathcal{S}^0 = \mathcal{H}^0 K \sigma_1 \mathcal{S}^0 = \mathcal{H}^0 K \mathcal{S}^0 D_T \ .$$

Next, let $H := \mathcal{H}^0 K \mathcal{S}^0$. For $\tau > 0$, from (9) we have

$$D_T \mathcal{H}^\tau K \mathcal{S}^\tau = D_T D_\tau H D_{-\tau} = D_\tau D_T H D_{-\tau} = D_\tau H D_T D_{-\tau} = D_\tau H D_{-\tau} D_T (I - P_\tau) \ .$$

When $\tau \le 0$, (10) and (11) yield

$$D_T D_\tau H D_{-\tau} = D_\tau D_T (I - P_{-\tau}) H D_{-\tau} = D_\tau D_T H D_{-\tau} - D_\tau D_T P_{-\tau} H P_{-\tau} D_{-\tau}$$

since H is causal by Lemma 3. Hence, we obtain $D_T D_\tau H D_{-\tau} = D_\tau H D_{-\tau} D_T$ for $\tau > 0$. ∎

Proof of Lemma 5

Given a causal operator H on $\mathcal{L}^{p,e}$, $(I-P_\alpha)H(I-P_\alpha) = H(I-P_\alpha)$ holds for any $\alpha \ge 0$. Since it is assumed that the closed-loop $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ with $X(0) = 0$ has a causal solution on $\mathcal{L}^{p,e}$ as a mapping from w to (z, u_c), $u_c(t) = ((I-P_\alpha)u_c)(t), \quad \forall t \in [0, \infty)$ is satisfied for all $\tau \in (-\infty, \infty)$. Hence, the claim is proved. ∎

Proof of Theorem 1

Since $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ with $X(0) = 0$ has a causal operator between w and (z, u_c),

$$T_{zw}(T, \tau) D_T = \mathcal{F}_\ell(G, (I - P_T)\mathcal{H}^\tau K \mathcal{S}^\tau) D_T \ . \tag{41}$$

Subdivide G into a new block matrix form as

$$\begin{bmatrix} z \\ y_c \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u_c \end{bmatrix}.$$

Let $H := \mathcal{H}^\tau K \mathcal{S}^\tau$. Using (41) we obtain

$$T_{zw}(T,\tau)D_T = G_{11}D_T + G_{12}(I-(I-P_T)HG_{22})^{-1}(I-P_T)HG_{21}D_T \ .$$

First, suppose $\tau \leq 0$. Then, from Lemma 4 and $(I-P_T)D_T = D_T$ it follows that

$$\begin{aligned}(I-(I-P_T)HG_{22})^{-1}D_T &= (I-D_TD_{-T}(I-P_T)HG_{22})^{-1}D_T \\ &= D_T(I-D_{-T}(I-P_T)D_THG_{22})^{-1} \\ &= D_T(I-HG_{22})^{-1} \ .\end{aligned}$$

Thus, using $(I-P_T)D_T = D_T$ again, we obtain

$$T_{zw}(T,\tau)D_T = D_TG_{11} + G_{12}D_T(I-HG_{22})^{-1}HG_{21} = D_TT_{zw}(T,\tau) \ .$$

Next, suppose $0 < \tau < T$. By assumption, $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ with $X(0) = 0$ has a causal mapping from w to y_c, so that

$$T_{zw}(T,\tau)D_T = \mathcal{F}_\ell(G,(I-P_T)\mathcal{H}^\tau K \mathcal{S}^\tau(I-P_{T+\tau}))D_T \ . \tag{42}$$

Making use of Lemma 4 and $(I-P_T)D_T = D_T$,

$$\begin{aligned}(I-(I-P_T)H(I-P_{T+\tau})G_{22})^{-1}D_T &= D_T(I-D_{-T}(I-P_T)H(I-P_{T+\tau})G_{22}D_T)^{-1} \\ &= D_T(I-D_{-T}(I-P_T)HD_T(I-P_\tau)G_{22})^{-1} \\ &= D_T(I-D_{-T}(I-P_T)D_THG_{22})^{-1} \\ &= D_T(I-HG_{22})^{-1} \ .\end{aligned}$$

Finally, it follows from (42) that

$$\begin{aligned}T_{zw}(T,\tau)D_T &= G_{11}D_T + G_{12}(I-(I-P_T)H(I-P_{T+\tau})G_{22})^{-1}(I-P_T)H(I-P_{T+\tau})G_{21}D_T \\ &= D_TG_{11} + G_{12}(I-(I-P_T)H(I-P_{T+\tau})G_{22})^{-1}(I-P_T)D_THG_{21} \\ &= D_TT_{zw}(T,\tau) \ . \qquad \blacksquare\end{aligned}$$

Proof of Theorem 2

Due to (9), (10) and (11), for all τ, we have

$$\mathcal{F}_\ell(G,\mathcal{H}^\tau K \mathcal{S}^\tau) = \mathcal{F}_\ell(G_{D1},\mathcal{H}^0 K \mathcal{S}^0), \qquad G_{D1} := \begin{bmatrix} I & 0 \\ 0 & D_{-\tau}\end{bmatrix} G \begin{bmatrix} I & 0 \\ 0 & D_\tau\end{bmatrix} \ .$$

(i) $\tau < 0$ Case
Due to Lemma 1,

$$\|\mathcal{F}_\ell(G_{D1},\mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|D_{-\tau}\mathcal{F}_\ell(G_{D1},\mathcal{H}^0 K \mathcal{S}^0)D_\tau\|_{\mathcal{L}^p/\mathcal{L}^p}$$

is true for all $\tau < 0$. Now, the closed-loop map is rewritten as

$$D_{-\tau}\mathcal{F}_\ell(G_{D1},\mathcal{H}^0 K \mathcal{S}^0)D_\tau = \mathcal{F}_\ell(D_{-\tau}GD_\tau,\mathcal{H}^0 K \mathcal{S}^0).$$

Then, it can be shown that

$$D_{-\tau}GD_\tau = D_{-\tau}GD_\tau(I-P_{-\tau}) = GD_{-\tau}D_\tau(I-P_{-\tau}) = G(I-P_{-\tau}).$$

Therefore, we have

$$\|\mathcal{F}_\ell(G_{D1},\mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|\mathcal{F}_\ell(G(I-P_{-\tau}),\mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} \ .$$

Finally, we obtain

$$\|\mathcal{F}_\ell(G_{D1},\mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|\mathcal{F}_\ell(G,\mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} \ .$$

by applying Lemma 5, Theorem 1 and Lemma 2.
(ii) $\tau \geq 0$ Case
Due to periodicity guaranteed by Theorem 1, Lemma 2 implies

$$\|\mathcal{F}_\ell(G_{D1}, \mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|\mathcal{F}_\ell(G_{D1}, \mathcal{H}^0 K \mathcal{S}^0)(I - P_\tau)\|_{\mathcal{L}^p/\mathcal{L}^p}$$

for all $\tau \geq 0$. Since G is time-invariant,

$$\begin{aligned}
\begin{bmatrix} I & 0 \\ 0 & D_{-\tau} \end{bmatrix} G \begin{bmatrix} I - P_\tau & 0 \\ 0 & D_\tau \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & D_{-\tau} \end{bmatrix} G D_\tau \begin{bmatrix} D_{-\tau}(I - P_\tau) & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ 0 & D_{-\tau} \end{bmatrix} D_\tau G \begin{bmatrix} D_{-\tau} & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} D_\tau & 0 \\ 0 & I \end{bmatrix} G \begin{bmatrix} D_{-\tau} & 0 \\ 0 & I \end{bmatrix} .
\end{aligned}$$

From this equivalence it follows

$$\|\mathcal{F}_\ell(G_{D1}, \mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|D_\tau \mathcal{F}_\ell(G, \mathcal{H}^0 K \mathcal{S}^0) D_{-\tau}\|_{\mathcal{L}^p/\mathcal{L}^p} = \|\mathcal{F}_\ell(G, \mathcal{H}^0 K \mathcal{S}^0)\|_{\mathcal{L}^p/\mathcal{L}^p} .$$

Here, Lemma 1 is used to obtain the last equation. This completes the proof. ∎

Proof of Lemma 6

Let M_i be an integer satisfying $T_C = M_i T_i$. Then, as obtained in Lemma 4, we can prove

$$\begin{aligned}
\mathcal{H}^{\tau_i} K_i \mathcal{S}^{\tau_i} D_{M_i T_i} &= D_{M_i T_i} \mathcal{H}^{\tau_i} K_i \mathcal{S}^{\tau_i}, \quad \tau_i \leq 0 \\
\mathcal{H}^{\tau_i} K_i \mathcal{S}^{\tau_i} D_{M_i T_i}(I - P_{\tau_i}) &= D_{M_i T_i} \mathcal{H}^{\tau_i} K_i \mathcal{S}^{\tau_i}, \quad \tau_i > 0 .
\end{aligned}$$

for each $i = 1, 2$. Combining these equations, the required equation is obtained. ∎

Proof of Theorem 4

It suffices to prove the claim for $\tau_1 > 0$, $\tau_2 \leq 0$ and $\alpha \geq 0$. The other cases of (τ_1, τ_2) are straightforward from the following argument and the $\alpha < 0$ case can be proved by replacing τ_i with $\tau_i - \alpha$. Suppose $\tau_1 > 0$, $\tau_2 \leq 0$ and $\alpha \geq 0$. According to (9), (10) and (11),

$$T_{zw}(T_1, \tau_1, T_2, \tau_2) = \mathcal{F}_\ell(G_{D1}, \mathcal{H}^{0,0} K \mathcal{S}^{0,0})$$

holds, where G_{D1} is defined by

$$G_{D1} := \begin{bmatrix} I & 0 & 0 \\ 0 & D_{-\tau_1} & 0 \\ 0 & 0 & D_{-\tau_2} \end{bmatrix} G \begin{bmatrix} I & 0 & 0 \\ 0 & D_{\tau_1} & 0 \\ 0 & 0 & D_{\tau_2} \end{bmatrix} .$$

Lemma 1 is applied to obtain

$$\|\mathcal{F}_\ell(G_{D1}, \mathcal{H}^{0,0} K \mathcal{S}^{0,0})\|_{\mathcal{L}^p/\mathcal{L}^p} = \|\mathcal{F}_\ell(G_{D2}, \mathcal{H}^{0,0} K \mathcal{S}^{0,0})\|_{\mathcal{L}^p/\mathcal{L}^p},$$

where G_{D2} is given by

$$G_{D2} := \begin{bmatrix} D_\alpha & 0 & 0 \\ 0 & D_{-\tau_1} & 0 \\ 0 & 0 & D_{-\tau_2} \end{bmatrix} G \begin{bmatrix} D_{-\alpha} & 0 & 0 \\ 0 & D_{\tau_1} & 0 \\ 0 & 0 & D_{\tau_2} \end{bmatrix} .$$

This operator G_{D2} can be rewritten as follows:

$$\begin{aligned}
G_{D2} &= \begin{bmatrix} I & 0 & 0 \\ 0 & D_{-\tau_1-\alpha} & 0 \\ 0 & 0 & D_{-\tau_2-\alpha} \end{bmatrix} G D_\alpha \begin{bmatrix} D_{-\alpha} & 0 & 0 \\ 0 & D_{\tau_1} & 0 \\ 0 & 0 & D_{\tau_2} \end{bmatrix} \\
&= \begin{bmatrix} I & 0 & 0 \\ 0 & D_{-\tau_1-\alpha} & 0 \\ 0 & 0 & D_{-\tau_2-\alpha} \end{bmatrix} G \begin{bmatrix} I - P_\alpha & 0 & 0 \\ 0 & D_{\tau_1+\alpha} & 0 \\ 0 & 0 & (I - P_\alpha) D_{\tau_2+\alpha} \end{bmatrix} .
\end{aligned}$$

Thus, using (9), (10) and (11) we have

$$\mathcal{F}_\ell(G_{D2},\mathcal{H}^{0,0}K\mathcal{S}^{0,0}) = \mathcal{F}_\ell(G_{D3},\mathcal{H}^{\tau_1+\alpha,\tau_2+\alpha}K\mathcal{S}^{\tau_1+\alpha,\tau_2+\alpha})$$

where G_{D3} is

$$G_{D3} := G\begin{bmatrix} I-P_\alpha & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I-P_\alpha \end{bmatrix} .$$

Now, by assumption, $\Sigma[G,\mathcal{H}^{\tau_1+\alpha,\tau_2+\alpha}K\mathcal{S}^{\tau_1+\alpha,\tau_2+\alpha}]$ has a causal solution, so that

$$\mathcal{F}_\ell(G_{D3},\mathcal{H}^{\tau_1+\alpha,\tau_2+\alpha}K\mathcal{S}^{\tau_1+\alpha,\tau_2+\alpha}) = T_{zw}(T_1,\tau_1+\alpha,T_2,\tau_2+\alpha)(I-P_\alpha)$$

holds by Lemma 7. Recalling that T_1/T_2 is rational, Theorem 3 shows that $T_{zw}(T_1,\tau_1+\alpha,T_2,\tau_2+\alpha)$ is periodic. Therefore, it follows from Lemma 2 that

$$\|T_{zw}(T_1,\tau_1+\alpha,T_2,\tau_2+\alpha)(I-P_\alpha)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1,\tau_1+\alpha,T_2,\tau_2+\alpha)\|_{\mathcal{L}^p/\mathcal{L}^p} .$$

This completes the proof. ∎

Proof of Corollary 1

(i) The equivalence is obtained by substituting $\alpha = -\tau_1$ into Theorem 4.
(ii) Let $\tau_1 = \tau$, $\tau_2 = 0$ and $\alpha = -\tau$. Then, Theorem 4 proves the equation.
(iii) Owing to the equivalence proved in (ii), it suffices to prove

$$\|T_{zw}(T_1,0,T_2,\tau)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1,T_{min}-\tau,T_2,0)\|_{\mathcal{L}^p/\mathcal{L}^p} .$$

Let T_C denote the least common multiple among T_i, $i=1,2$. From $-T_1 \le T_{min}-\tau < T_1$ and Theorem 3 it follows that $T_{zw}(T_1,T_{min}-\tau,T_2,0)$ is T_C-periodic. Then, Lemma 2 is used to obtain

$$\|T_{zw}(T_1,T_{min}-\tau,T_2,0)\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1,T_{min}-\tau,T_2,0)(I-P_{T_C-\tau})\|_{\mathcal{L}^p/\mathcal{L}^p} .$$

Now, using the initial condition $X(0)=0$, it is not difficult to verify that

$$\|T_{zw}(T_1,T_{min}-\tau,T_2,0)(I-P_{T_C-\tau})\|_{\mathcal{L}^p/\mathcal{L}^p} = \|T_{zw}(T_1,0,T_2,\tau)\|_{\mathcal{L}^p/\mathcal{L}^p} .$$

holds. This proves the claim. ∎

Proof of Theorem 5

(i) $\tau > 0$ case
By construction,

$$\mathcal{H}^\tau K\mathcal{S}^\tau = \mathcal{H}_F W_n^{-1}F_{l+}K_{a+}F_{r+}W_n\mathcal{S}_F, \quad x_{\sigma m}(0)=0, \forall x_K(0)$$

is satisfied and the state transition of K on both sides are identical. Then, it is obvious that $\Sigma[G,\mathcal{H}^\tau K\mathcal{S}^\tau]$ is internally stable in terms of $x=[x_G^T,x_K^T]^T$ if $\Sigma[G,\mathcal{H}_F W_n^{-1}F_{l+}K_{a+}F_{r+}W_n\mathcal{S}_F]$ is internally stable in terms of $x=[x_G^T,x_{\sigma m}^T,x_K^T]^T$. Conversely, exponential convergence of x_K and x_G implies that $x_{\sigma m}$ is also exponentially convergent since it can be shown that the additional state satisfies

$$x_{\sigma m}(k+1) = \begin{bmatrix} C_K x_K(k) + D_K C_2 x_G(kT+\frac{mT}{n}) \\ C_K x_K(k) + D_K C_2 X_G(kT+\frac{mT}{n}) \\ \vdots \\ C_K x_K(k) + D_K C_2 X_G(kT+\frac{mT}{n}) \end{bmatrix}, \ \forall k\in\mathcal{Z}_+\setminus\{0\} .$$

(i) $\tau < 0$ case
In this case, the state $x_{\sigma m}$ can be represented by

$$x_{\sigma m}(k+1) = \begin{bmatrix} C_2 x(kT+\frac{(n-m)T}{n}) \\ \vdots \\ C_2 x(kT+\frac{(n-2)T}{n}) \\ C_2 x(kT+\frac{(n-1)T}{n}) \end{bmatrix}, \ \forall k\in\mathcal{Z}_+\setminus\{0\} .$$

Then, the claim can be proved in the same way as $\tau > 0$ case. ∎

Proof of Theorem 6

It follows directly from Theorem 5 and the input-output equivalence between $\mathcal{H}_F W_n^{-1} F_l K F_r W_n \mathcal{S}_F$ and $\mathcal{H}^\tau K \mathcal{S}^\tau$.
∎

Proof of Theorem 8

Equation (26) and Equation (27) show that the $\mathcal{L}^p$ worst-case performance of $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ is the same as that of $\Sigma[G, \mathcal{H}_F W_n^{-1} F_l K F_r W_n \mathcal{S}_F]$. According to Theorem 5, internal stability of $\Sigma[G, \mathcal{H}^\tau K \mathcal{S}^\tau]$ is equivalent to that of $\Sigma[G, \mathcal{H}_F W_n^{-1} F_{l\pm} K_{a\pm} F_{r\pm} W_n \mathcal{S}_F]$. Since $\Sigma[G, \mathcal{H}_F W_n^{-1} F_{l\pm} K_{a\pm} F_{r\pm} W_n \mathcal{S}_F]$ is multirate sampled-data control system without any phase lags and leads, internal stability of the control system remains unchanged if uncontrollable modes and unobservable modes removed from the LSI controller are stable[15]. Furthermore, since the modes removed correspond to $x_{\sigma m}$, the state transitions of the two systems are proved to be the same.
∎

Proof of Lemma 8

The internal stability of $\Sigma[G, \mathcal{H} K \mathcal{S}]$ implies that the equivalent representation $\Sigma[G, \tilde{K}]$ in Theorem 10 is also internally stable. According to [15], the internal stability guarantees the uniqueness and boundedness of $T_{zw} = \mathcal{F}_\ell(G, \tilde{K})$. ∎

Proof of Theorem 11

By proper scaling, we can always take $\gamma = 1$.
(i) Sufficiency for robust $\mathcal{L}^p$ stability: According to Lemma 8, the assumption of internal stability implies that the operator from w to z is $\mathcal{L}^p$ induced-norm bounded. From the small-gain theorem it follows that $\|\mathcal{F}_\ell(G, \mathcal{H} K \mathcal{S})\|_{\mathcal{L}^p/\mathcal{L}^p} \leq 1$ is sufficient for the $\mathcal{L}^2$-boundedness of the linear map from $[f_1^T, f_2^T]^T$ to $[e_1^T, e_2^T]^T$ for any $\Delta \in \boldsymbol{B\Delta}_{PTV}$.
(ii) Necessity for robust $\mathcal{L}^2$ stability: Assume that there holds $\|\mathcal{F}_\ell(G, \mathcal{H} K \mathcal{S})\|_{\mathcal{L}^2/\mathcal{L}^2} > 1$. From Theorem 10, it is equivalent to $\|\mathcal{F}_\ell(G, \tilde{K})\|_{\mathcal{L}^2/\mathcal{L}^2} > 1$. Following an idea in [26], it can be shown that there exist contractive linear operators $\mathcal{U} : \ell^2 \to \mathcal{L}^2$ and $\mathcal{V} : \mathcal{L}^2 \to \ell^2$ such that

$$\|\mathcal{V}\,\mathcal{F}_\ell(G, \tilde{K})\,\mathcal{U}\|_{\ell^2/\ell^2} > 1 \tag{43}$$

holds, and the operators are in the form of

$$\mathcal{U} \;:\; \ell_s^2 \to \mathcal{L}^2, \quad (\mathcal{U} f)(t) := (U f_j)(t - jT_D) \qquad \text{for} \;\; jT_D < t \leq (j+1)T_D \tag{44}$$

$$\mathcal{V} \;:\; \mathcal{L}^2 \to \ell_r^2, \quad (\mathcal{V} z)(j) := V z_{j-1} \qquad \text{for} \;\; j \geq 0, \quad (\mathcal{V} z)(0) := 0 \;, \tag{45}$$

where U and V are linear operators:

$$U : \mathcal{R}^s \to \mathcal{L}^2[0, T_D], \quad V : \mathcal{L}^2[0, T_D] \to \mathcal{R}^r \tag{46}$$

An important thing which is different from the single-rate case shown in [26] is that we here consider a multirate system, $\mathcal{V}\mathcal{F}_\ell(G, \tilde{K})\mathcal{U}$ is not an LSI, but a linear T_C/T_D-periodic operator mapping $\ell^2 \to \ell^2$. Hence, we introduce a discrete-time lifting operator W_{T_C/T_D} which stacks up its input sequence every T_C/T_D points, and $W_{T_C/T_D} \mathcal{V}\mathcal{F}_\ell(G, \tilde{K})\mathcal{U} W_{T_C/T_D}^{-1}$ becomes shift-invariant. Since W_{T_C/T_D} is bijective isometory, (43) is equivalent to

$$\|W_{T_C/T_D} \mathcal{V}\mathcal{F}_\ell(G, \tilde{K})\mathcal{U} W_{T_C/T_D}^{-1}\|_{\ell^2/\ell^2} > 1 \;\; . \tag{47}$$

Then, there is an FDLSI and strictly-causal system Δ_{Nd} satisfying $\|\Delta_{Nd}\|_{\ell^2/\ell^2} < 1$ and destabilizing the pair $(W_{T_C/T_D} \mathcal{V}\mathcal{F}_\ell(G, \tilde{K})\mathcal{U} W_{T_C/T_D}^{-1}, \Delta_{Nd})$ in feedback. Actually, one can find a strictly-causal destabilizer by using the technique in [3] and using the bilinear transformation. A biproper FDLSI system $\Delta_\beta(z)$ is found as a destabilizing admissible perturbation with respect to $\sigma_1 M$ for $M := W_{T_C/T_D} \mathcal{V}\mathcal{F}_\ell(G, \tilde{K})\mathcal{U} W_{T_C/T_D}^{-1}$. Then, set $\Delta_{Nd} := \Delta_\beta \sigma_1$, which becomes a strictly proper destabilizer of M and $\|\Delta_{Nd}\|_{\ell^2/\ell^2} < 1$ holds. Now define $\Delta_d := W_{T_C/T_D}^{-1} \Delta_{Nd} W_{T_C/T_D}$ which is a causal and T_C/T_D-periodic linear operator which meets $\|\Delta_d\|_{\ell^2/\ell^2} < 1$. Then, the feedback connection of $\mathcal{V}\mathcal{F}_\ell(G, \tilde{K})\mathcal{U}$ and Δ_d is not stable in the sense of an ℓ^2 input-output property. Now put $\Delta := \mathcal{U}\Delta_d\mathcal{V}$ in Fig.7(a). Since $\mathcal{V}$ and $\mathcal{U}$ are contractive, $\|\Delta\|_{\mathcal{L}^p/\mathcal{L}^p} < 1$ is clear. From the definition of $\mathcal{U}$ and $\mathcal{V}$ in (44) and (45), it follows that Δ is T_C-periodic, and Δ belongs to $\boldsymbol{B\Delta}_{PTV}$. Furthermore, it is easy

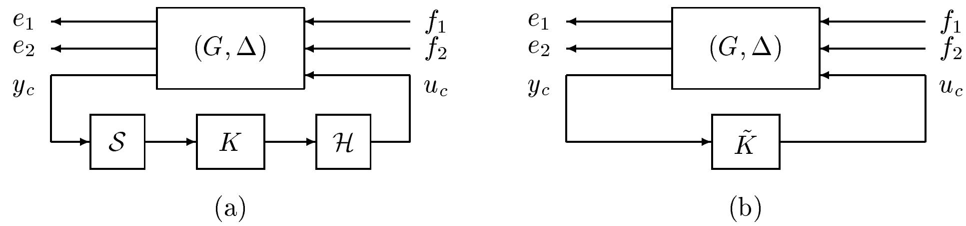

Figure 10: Equivalent problems.

to see that $\mathcal{V}$ and $\mathcal{U}$ are bounded and bijective when their inputs and outputs are restricted to spaces which the destabilizing signals constructed above belong to. Therefore, we conclude that if $\|\mathcal{F}_\ell(G,\mathcal{H}K\mathcal{S})\|_{\mathcal{L}^2/\mathcal{L}^2} > 1$, there is an uncertainty Δ in $\boldsymbol{B\Delta}_{PTV}$ such that the operator from $[f_1^T, f_2^T]^T$ to $[e_1^T, e_2^T]^T$ is not $\mathcal{L}^2$-induced norm bounded.

(iii) Sufficiency for robust internal stability: It suffices to prove that robust $\mathcal{L}^2$ stability of the system shown in Fig.7(a) implies internal stability. First, rewrite the system as shown in Fig.10(a). Here, the system (G,Δ) has a realization with the state (x_G, x_d) in the state-space, where x_d is the state of Δ. Keeping in mind $\|\mathcal{F}_\ell(G,\mathcal{H}K\mathcal{S})\|_{\mathcal{L}^2/\mathcal{L}^2} \le 1$, the uncertain system (G,Δ) is input-output stabilizable by a continuous-time controller K_c because of the existence of K_c satisfying $\|\mathcal{F}_\ell(G,K_c)\|_\infty \le 1$ (see also [22]). Since each of state-space realizations G and Δ is stabilizable and detectable, the input-output stabilizability and finite-dimensionality of (G,Δ) and K_c imply the realizations of (G,Δ) is stabilizable and detectable from u_c and y_c. Now, applying Theorem 10 to $\Sigma[(G,\Delta),\mathcal{H}K\mathcal{S}]$, the nonsynchronous system in Fig.10(a) is transformed into the system shown in Fig.10(b), which preserves internal stability and $\|\mathcal{F}_\ell((G,\Delta),\mathcal{H}K\mathcal{S})\|_{\mathcal{L}^2/\mathcal{L}^2} = \|\mathcal{F}_\ell((G,\Delta),\tilde{K})\|_{\mathcal{L}^2/\mathcal{L}^2}$. Here, note that the direct feedthrough matrix of the linear map (G,Δ) from $[f_1^T, f_2^T, u_c^T]^T$ to y_c is zero. Then, $\mathcal{L}^2$-hybrid stability (see [15, 4] for definition) of the system in Fig.10(b) is equivalent to robust $\mathcal{L}^2$ stability of Fig.7(a). Due to Theorem 4.1 in [15] and Theorem 10, the system in Fig.10(b) is $\mathcal{L}^2$-hybrid stable if and only if it is internally stable. Hence, the system in Fig.7(a) is robustly internally stable since the state of (G,Δ) is (x_G, x_d). ∎

Proof of Theorem 12

Making use of proper scaling, it suffices to prove the claim for $\gamma = 1$. Clearly, the small gain theorem proves that the system shown in Fig.7(b) is robustly $\mathcal{L}^p$ stable with respect to $\boldsymbol{B\Delta}_{TV}$. Let $w = [w_1^T, w_2^T]^T$ and $z = [z_1^T, z_2^T]^T$. The $\mathcal{L}^p$ disturbance attenuation less than or equal to 1 is equivalent to

$$\sup_{w \in \mathcal{L}^p} (\|z\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p \|w\|_{\mathcal{L}^p}^p) \le 0 \tag{48}$$

for $\tilde{\gamma} \le 1$. Noticing that

$$\|\Delta z_2\|_{\mathcal{L}^p}^p - \|z_2\|_{\mathcal{L}^p}^p \le 0, \ \forall z_2 \in \mathcal{L}^p, \ \forall \Delta \in \boldsymbol{B\Delta}_{TV} \ .$$

and $w_2 = \Delta z_2$, for $p \in [1,\infty)$ we obtain

$$\|z\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p\|w\|_{\mathcal{L}^p}^p = \|z_1\|_{\mathcal{L}^p}^p + \|z_2\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p\|w_1\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p\|w_2\|_{\mathcal{L}^p}^p \ge \|z_1\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p\|w_1\|_{\mathcal{L}^p}^p + (1-\tilde{\gamma}^p)\|z_2\|_{\mathcal{L}^p}^p$$

for all $w_1 \in \mathcal{L}^p$. From (48) it follows that for $\tilde{\gamma} \le 1$,

$$0 \ge \|z\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p\|w\|_{\mathcal{L}^p}^p \ge \|z_1\|_{\mathcal{L}^p}^p - \tilde{\gamma}^p\|w_1\|_{\mathcal{L}^p}^p \ .$$

For the $p = \infty$ case,

$$0 \ge \|z\|_{\mathcal{L}^\infty} - \tilde{\gamma}\|w\|_{\mathcal{L}^\infty} \ge \|z_1\|_{\mathcal{L}^\infty} - \tilde{\gamma}\max\{\|w_1\|_{\mathcal{L}^\infty}, \|z_2\|_{\mathcal{L}^\infty}\} \ge \|z_1\|_{\mathcal{L}^\infty} - \tilde{\gamma}\|w_1\|_{\mathcal{L}^\infty} \ .$$

This completes the proof. ∎

Proof of Lemma 9

By $T > 0$, $\tau < 0$ and (10), it can be shown that $\mathcal{S}^\tau = \mathcal{S}^0 D_{-\tau} = \mathcal{S}^0 D_{-(T+\tau)} D_T = \mathcal{S}^{T+\tau} D_T$. Since $\mathcal{S}^{T+\tau} D_T = \sigma_1 \mathcal{S}^{T+\tau}$ holds for $-T < \tau$, we have $\mathcal{S}^\tau = \sigma_1 \mathcal{S}^{T+\tau}$ which proves the claim. ∎

Proof of Lemma 10

The proof is similar to that of Lemma 2. Thus it is omitted. ∎

Proof of Lemma 11

From Equation (11), $T > 0$ and $\tau < 0$, we obtain $\mathcal{H}^\tau = D_\tau \mathcal{H}^0 = D_{-T} D_{T+\tau} \mathcal{H}^0 = D_{-T} \mathcal{H}^{T+\tau}$. It is straightforward to verify that $D_{-T}\mathcal{H}^{T+\tau}(I - R_1) = \mathcal{H}^{T+\tau}\sigma_{-1}$. Hence, we have

$$F(D_\nu - I)\mathcal{H}^\tau (I - R_1) = F(D_\nu - I)\mathcal{H}^{T+\tau}\sigma_{-1} \ . \tag{49}$$

Since $D_T \mathcal{H}^\tau = \mathcal{H}^\tau \sigma_1$ holds, time-invariance of F results in $F(D_\nu - I)\mathcal{H}^\tau\sigma_1 = D_T F(D_\nu - I)\mathcal{H}^\tau$ for $\nu \geq 0$. Finally, due to Lemma 10 and (49), we have

$$\|F(D_\nu - I)\mathcal{H}^\tau\|_{\mathcal{L}^p/\ell^p} = \|F(D_\nu - I)\mathcal{H}^\tau(I - R_1)\|_{\mathcal{L}^p/\ell^p} = \|F(D_\nu - I)\mathcal{H}^{T+\tau}\|_{\mathcal{L}^p/\ell^p} \ . \quad ∎$$

Proof of Lemma 12

We first assume $v \geq 0$ and $\tau \geq 0$. Let $p \in [0, \infty)$. For any $u \in \ell^p$, define $m := (D_\nu - I)\mathcal{H}^\tau u$. Then, the function $m(t)$ is

$$m(t) = \begin{cases} u(0) & \tau < t \leq \tau + \nu \\ u(k) - u(k-1) & kT + \tau < t \leq kT + \tau + \nu, \quad k = 1, 2, \ldots \\ 0 & kT + \tau + \nu < t \leq (k+1)T + \tau, \quad k \in \mathcal{Z}_+ \\ 0 & 0 \leq t \leq \tau \quad . \end{cases}$$

Thus, using the Hölder's inequality, we obtain

$$\begin{aligned} \|m\|_{\mathcal{L}^p} &= \left\{ \nu |u(0)|^p + \nu \sum_{k=1}^{\infty} |u(k) - u(k-1)|^p \right\}^{1/p} \\ &\leq \nu^{1/p} \left\{ |u(0)|^p + 2^{p-1} \sum_{k=1}^{\infty} (|u(k)|^p + |u(k-1)|^p) \right\}^{1/p} \\ &\leq 2\nu^{1/p} \|u\|_{\ell^p} \ . \end{aligned}$$

Hence, taking $\nu < (\epsilon/2)^p$, we have

$$\|(D_\nu - I)\mathcal{H}^\tau\|_{\mathcal{L}^p/\ell^p} < \epsilon \quad . \tag{50}$$

Lemma 11 implies that even in the $\tau < 0$ case, (50) is guaranteed to be satisfied since the radius $(\epsilon/2)^p$ is independent of τ. Finally, replacing τ with $\tau - \nu$ in the above argument it is proved easily that $-(\epsilon/2)^p < \nu < 0$ implies (50). ∎

Proof of Lemma 13

Since $\|\mathcal{H}^\tau u\|_{\mathcal{L}^\infty} = \|u\|_{\ell^\infty}$ hold for any $u \in \ell^\infty$, it follows from $\|D_\nu - I\|_{\mathcal{L}^\infty/\mathcal{L}^\infty} \leq 2$ that $\|(D_\nu - I)\mathcal{H}^\tau u\|_{\mathcal{L}^\infty} \leq 2\|u\|_{\ell^\infty}$. It is straightforward to see that $u_w \in \ell^\infty$ defined by $u_w(k) = (-1)^k$, $k \in \mathcal{Z}_+$ achieves $\|(D_\nu - I)\mathcal{H}^\tau u_w\|_{\mathcal{L}^\infty} = 2\|u_w\|_{\ell^\infty}$ for $\nu \neq 0$. Hence, the induced norm is 2, which does not depends on ν. ∎

Proof of Lemma 14

Assume that $\nu \geq 0$ and $\tau \geq 0$. Suppose $u \in \ell^\infty$. By Lemma 13, $m := (D_\nu - I)\mathcal{H}^\tau u$ is in $\mathcal{L}^\infty$. Since F is a stable FDLTI system, $y = F(D_\nu - I)\mathcal{H}^\tau u$ is also in $\mathcal{L}^\infty$. Let $f(t), t \geq 0$ denote the impulse response of F. Then, we have $y(t) = \int_0^t f(t - \xi)m(\xi)d\xi$. Since $m(t) = 0$ for $0 \leq t \leq \tau$ and for $kT + \tau + \nu < t \leq (k+1)T + \tau$ with $k \in \mathcal{Z}_+$,

$$y(t) = \sum_{i=0}^{N(t)} \int_{iT+\tau}^{iT+\tau+\nu} f(t - \tau)m(\xi)d\xi,$$

where $N(t)$ is the largest integer satisfying $N(t) \leq (t - \tau - \nu)/T$. Noting that there exist $a < 0$ and $b > 0$ satisfying $|f(t)| \leq be^{at}$ for all $t \geq 0$, for $t \geq 0$, $(k-1)T + \tau + \nu \leq t < kT + \tau + \nu$ and $k \in \mathcal{Z}_+$, we obtain

$$|y(t)| \leq \sum_{i=0}^{k} \hat{f}(k - i) \int_{iT+\tau}^{iT+\tau+\nu} |m(\xi)|d\xi \ , \quad \hat{f}(k) := be^{a(k+1)T} \ .$$

Now, define

$$v(i) := \int_{iT+\tau}^{iT+\tau+\nu} |m(\xi)| d\xi, \quad i \in \mathcal{Z}_+ \ , \quad \bar{y}(k) := \sum_{i=0}^{k} \hat{f}(k-i)v(i), \quad k \in \mathcal{Z}_+.$$

Since obviously $\hat{f}$ is in ℓ^1 and since $\|v\|_{\ell^\infty} \le \nu \|m\|_{\mathcal{L}^\infty}$ holds, $\bar{y}(k)$ satisfies

$$\|\bar{y}\|_{\ell^\infty} \le \|\hat{f}\|_{\ell^1} \|v\|_{\ell^\infty} \le \nu \|\hat{f}\|_{\ell^1} \|m\|_{\mathcal{L}^\infty}$$

From $\|y\|_{\mathcal{L}^\infty} \le \|\bar{y}\|_{\ell^\infty}$ and Lemma 13 it follows that

$$\|y\|_{\ell^\infty} \le 2\nu \|\hat{f}\|_{\ell^1} \|u\|_{\ell^\infty} \tag{51}$$

holds. This prove the claim for the case of $\nu \ge 0$ and $\tau \ge 0$. Since $2\nu\|\hat{f}\|_{\ell^1}$ in (51) is independent of $\tau \in [0, T)$, Lemma 11 guarantees (51) to be satisfied for $\tau \in (-T, 0)$. Finally, again Equation (51) is independent of τ, replacing τ with $\tau - \nu$, it is straightforward to verify that the $\nu < 0$ case results in the same as the $\nu > 0$ case. ∎

Proof of Lemma 15

Suppose that $\nu > 0$ and $\tau \ge 0$. Consider the function $u(t)$ defined by

$$u(t) = \begin{cases} \left(\frac{2^{-k}}{h}\right)^{1/p} & \text{if } kT + \tau \le t < kT + \tau + h, \quad k \in \mathcal{Z}_+ \\ 0 & \text{otherwise} \ , \end{cases}$$

where h is an arbitrary positive number satisfying $h < \max\{\nu, 1\}$. Here, u belongs to $\mathtt{PC}$ and $\|u\|_{\mathcal{L}^p} = 2^{1/p}$. Then, $\|\mathcal{S}^\tau(D_{-\nu} - I)u\|_{\ell^p} = (2/h)^{1/p}$ holds for all $p \in [1, \infty]$. Hence,

$$\frac{\|\mathcal{S}^\tau(D_{-\nu} - I)u\|_{\ell^p}}{\|u\|_{\mathcal{L}^p}} = (1/h)^{1/p} \ge 1 \tag{52}$$

For the case of $\nu < 0$, we can trace the above argument again if replacing τ with $\tau - \nu$. Moreover, in the $\tau < 0$ case, accordingly to Lemma 9, the signal u achieving (52) is constructed by replacing $\mathcal{S}^\tau$ with $\mathcal{S}^{T+\tau}$, where $0 < T + \tau < T$ holds. ∎

Proof of Lemma 16

We assume first that $\nu \ge 0$ and $0 \le \tau < T$. Suppose that p is an integer belonging to $[1, \infty]$. For $u \in \mathcal{L}^p$, define the function $\bar{y}(t)$ by $\bar{y} := (D_{-\nu} - I)Fu$. Let $f(t)$ denote the impulse response of F. Then, the sequence $\{y(k)\}_{k=0}^\infty$ defined by $y := \mathcal{S}^\tau(D_{-\nu} - I)Fu$ is expressed as

$$\begin{aligned} y(k) &= \int_0^{kT+\tau+\nu} f(kT + \tau + \nu - \xi)u(\xi)d\xi - \int_0^{kT+\tau} f(kT + \tau - \xi)u(\xi)d\xi && (53) \\ &= Y_1(k) + Y_2(k), \quad k \in \mathcal{Z}_+ \ . && (54) \end{aligned}$$

where $Y_1(k)$ and $Y_2(k)$ are

$$Y_1(k) := \int_{kT+\tau}^{kT+\tau+\nu} f(kT + \tau + \nu - \xi)u(\xi)d\xi, \quad k \in \mathcal{Z}_+ \tag{55}$$

$$Y_2(k) := \int_0^{kT+\tau} (f(kT + \tau + \nu - \xi) - f(kT + \tau - \xi))u(\xi)d\xi, \quad k \in \mathcal{Z}_+ \ . \tag{56}$$

We also define

$$Y_{21}(k) := \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} (f(kT + \tau + \nu - \xi) - f(kT + \tau - \xi))u(\xi)d\xi, \quad k \in \mathcal{Z}_+ \tag{57}$$

$$Y_{22}(k) := \int_{kT}^{kT+\tau} (f(kT + \tau + \nu - \xi) - f(kT + \tau - \xi))u(\xi)d\xi, \quad k \in \mathcal{Z}_+ \ . \tag{58}$$

which are consistent with

$$Y_2(k) = Y_{21}(k) + Y_{22}(k) \ . \tag{59}$$

(i) *The $p = 1$ case :* Since F is FDLTI, strictly proper and stable, the system admits a state space representation $(A_f, B_f, C_f, 0)$ and its impulse response is $f(t) = C_f e^{A_f t} B_f$, $t \geq 0$. It is clear that $|f(t)| \leq \|C_f\| \|B_f\| \|e^{A_f t}\| \leq be^{at}, \quad t \geq 0$ holds with $a < 0$ and $b > 0$. Then

$$|Y_1(k)| \quad \leq \quad \int_0^\nu be^{a(\nu-\eta)} |u(kT + \tau + \eta)| d\eta \leq b \int_0^\nu |u(kT + \tau + \eta)| d\eta, \quad k \in \mathcal{Z}_+ \ . \tag{60}$$

We have $\|Y_1\|_{\ell^1} = \sum_{k=0}^\infty |Y_1(k)| \leq b\|u\|_{\mathcal{L}^1}$. On the other hand, by defining

$$\bar{f}(k) := \sup_{t \in [kT,(k+1)T)} |f(t + \tau + \nu) - f(t + \tau)|, \quad k \in \mathcal{Z}_+ \ ,$$

the sequence $Y_{21}(k)$ satisfies

$$|Y_{21}(k)| \leq \sum_{i=0}^{k-1} \bar{f}(k - i - 1) v(i) \ , \quad v(i) := \int_{iT}^{(i+1)T} |u(\xi)| d\xi \ , \quad i \in \mathcal{Z}_+ \ . \tag{61}$$

Note that $\|v\|_{\ell^1} = \|u\|_{\mathcal{L}^1}$ holds and that v belongs to ℓ^1. From

$$f(t + \tau + \nu) - f(t + \tau) = C_f e^{A_f(t+\tau)} (e^{A_f \nu} - I) B_f$$

it follows that there exists a real number c such that $c < a < 0$ and

$$|f(t + \tau + \nu) - f(t + \nu)| \leq be^{a(t+\tau)} (1 - e^{c\nu}) \ .$$

Hence, $\bar{f}(k) \leq be^{a(kT+\tau)}(1 - e^{c\nu})$. Here, $\bar{f}_1$ is in ℓ^1 since

$$\|\bar{f}\|_{\ell^1} \leq \frac{be^{a\tau}(1 - e^{c\nu})}{1 - e^{aT}} < \infty \quad , \nu > 0 \ .$$

Thus, v and $\bar{f}$ are in ℓ^1. Their convolution also lies in ℓ^1. Moreover, $\|Y_{21}\|_{\ell^1} \leq \|\bar{f}\|_{\ell^1} \|v\|_{\ell^1} = \|\bar{f}\|_{\ell^1} \|u\|_{\mathcal{L}^1}$. As for $Y_{22}(k)$,

$$|Y_{22}(k)| \leq \sup_{t \in [0,\tau)} |f(t + \nu) - f(t)| \int_{kT}^{kT+\tau} |u(\xi)| d\xi = b(1 - e^{c\nu}) \int_{kT}^{kT+\tau} |u(\xi)| d\xi \ , \quad k \in \mathcal{Z}_+ \tag{62}$$

The last inequality implies $\|Y_{22}\|_{\ell^1} \leq b(1 - e^{c\nu})\|u\|_{\mathcal{L}^1}$. Recalling (54) and (59), we obtain

$$\begin{aligned} &\|y\|_{\ell^1} \leq M_1(\nu) \|u\|_{\mathcal{L}^1} \\ &M_1(\nu) := b \left\{ 1 + (1 - e^{c\nu}) \left(\frac{1}{1 - e^{aT}} + 1 \right) \right\} < \infty, \quad \nu > 0 \ . \end{aligned} \tag{63}$$

Here, $\lim_{\nu \to +0} M_1(\nu) = b$. Furthermore, $M_1(\nu) > 0$ is a monotonically increasing function of ν and it is also uniformly continuous.
(ii) *The $p = \infty$ case :* From (60), we obtain $|Y_1(k)| \leq \nu b \|u\|_{\mathcal{L}^\infty}$ and $\|Y_1\|_{\ell^\infty} \leq \nu b \|u\|_{\mathcal{L}^\infty}$ for $\nu > 0$. By Equation (61), it is easy to see that $\|Y_{21}\|_{\ell^\infty} \leq T \|\bar{f}\|_{\ell^1} \|u\|_{\mathcal{L}^\infty}$ for $\nu > 0$ since $\|v\|_{\ell^\infty} \leq T\|u\|_{\mathcal{L}^\infty}$. Moreover, Equation (62) implies $\|Y_{22}\|_{\ell^\infty} \leq b(1 - e^{c\nu})\tau \|u\|_{\mathcal{L}^\infty}$ for $\nu > 0$. Hence,

$$\begin{aligned} &\|y\|_{\ell^\infty} \leq M_\infty(\nu) \|u\|_{\mathcal{L}^\infty} \\ &M_\infty(\nu) := b \left\{ \nu + (1 - e^{c\nu}) \left(\frac{T}{1 - e^{aT}} + T \right) \right\} < \infty \quad , \nu > 0 \ . \end{aligned} \tag{64}$$

Thus, $\lim_{\nu \to +0} M_\infty(\nu) = 0$ and $M_\infty(\nu) > 0$ is a monotonically increasing and uniformly continuous function of ν.
(iii) *The $1 < p < \infty$ case :* For any $u \in \ell^p$, the hold operator $\mathcal{H}^\tau$ has the property that $\|\mathcal{H}^\tau u\|_{\mathcal{L}^p} = T^{1/p} \|u\|_{\ell^p}$. Then, $\|\mathcal{H}^\tau \mathcal{S}^\tau (D_{-\nu} - I) F\|_{\mathcal{L}^p/\mathcal{L}^p} = T^{1/p} \|\mathcal{S}^\tau (D_{-\nu} - I) F\|_{\ell^p/\mathcal{L}^p}$. From inequalities (63) and (64), $\mathcal{S}^\tau (D_{-\nu} - I) F$

is a linear operator mapping $\mathcal{L}^1$ to $\mathcal{L}^1$ and $\mathcal{L}^\infty$ to $\mathcal{L}^\infty$. By using the Hölder's inequality for linear operators on $\mathcal{L}^p$, we obtain

$$\|\mathcal{H}^\tau \mathcal{S}^\tau (D_{-\nu} - I)F\|_{\mathcal{L}^p/\mathcal{L}^p} \le (TM_1(\nu))^{1/p}(M_\infty(\nu))^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1 \ .$$

Hence,

$$\|\mathcal{S}^\tau (D_{-\nu} - I)F\|_{\ell^p/\mathcal{L}^p} \le M_p(\nu) \ ,$$

where $M_p(\nu)$ is obtained as

$$M_p(\nu) := (M_1(\nu))^{1/p}(M_\infty(\nu))^{1/q} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 \ .$$

Obviously, for $1 < p < \infty$, $M_p(\nu)$ is a uniformly continuous function of $\nu > 0$ and $\lim_{\nu \to +0} M_p(\nu) = 0$. Also, $M_p(\nu) > 0$ is a monotonically increasing function of ν.

Hence, the proof is completed for the case of $\tau \ge 0$ and $\nu \ge 0$. Now, since every $M_p(\nu)$ obtained for $p \in [1, \infty]$ is a function independent of τ, Lemma 9 allows us to use the same $M_p(\nu)$ for $\tau < 0$. Furthermore, replacing τ with $\tau - \nu$ in the above argument, the independence of τ guarantees continuity and monotonicity properties of $M_p(|\nu|)$ to be satisfied for $\nu < 0$. This completes the proof. ∎

Proof of Lemma 17

Suppose $\nu > 0$ and $\tau \ge 0$. Consider the function $u(t)$, $t \ge 0$ defined by

$$u(t) = \begin{cases} 1 & \tau \le t < \tau + \nu \\ 0 & \text{otherwise} \end{cases} .$$

From Equation (53) and $f(t) = be^{at}$, we obtain

$$\begin{aligned} y(0) &= \int_0^\nu f(\nu - \eta) d\eta = \frac{b}{a}(e^{a\nu} - 1) \\ y(k) &= \int_0^\nu f(kT + \nu - \eta) d\eta - \int_0^\nu f(kT - \eta) d\eta = \frac{b}{a}(e^{a\nu} - 1)e^{kaT}(1 - e^{-a\nu}), \quad k = 1, 2, 3, \ldots \ . \end{aligned}$$

Now, it is straightforward to verify that

$$\|y\|_{\ell^1} = L_1(\nu)\|u\|_{\mathcal{L}^1} \tag{65}$$

where $L_1(\nu)$ is defined by

$$L_1(\nu) := \frac{|b|(1 + e^{a(T-\nu)} - 2e^{aT})}{1 - e^{aT}} \cdot \frac{(e^{a\nu} - 1)}{a\nu}, \quad \nu > 0 \ .$$

Now, note that $(e^{a\nu} - 1)/a\nu \ge e^{a\nu/2}$ for $\nu \in \mathcal{R}$. Then,

$$\infty > L_1(\nu) \ge |b|(1 + e^{a(T-\nu)} - 2e^{aT})\frac{e^{a\nu/2}}{1 - e^{aT}} > 0, \quad \nu \in (0, \infty) \ .$$

Thus, for all $0 < \nu < T$, $L_1(\nu)$ satisfies $L_1(\nu) > |b|e^{aT/2} > 0$. Taking $\epsilon = |b|e^{aT/2}$ proves the inequality claimed. If $\nu < 0$, replacing τ with $\tau - \nu$, we can trace the above argument. Moreover, in the $\tau < 0$ case, accordingly to Lemma 9, the signal u achieving (65) is constructed by replacing $\mathcal{S}^\tau$ with $\mathcal{S}^{T+\tau}$, where $0 < T + \tau < T$. ∎

Proof of Lemma 18

We assume that $T > \tau \ge 0$ and $\nu \ge 0$ hold. Let $F_1(s)$ denote the transfer function of the system F_1. Then, by assumption, it is possible to decompose F_1 as $F_1(s) = \frac{b}{s-a}Q(s)$ with $a < 0$. Here, Q is a stable FDLTI system and its transfer function $Q(s)$ is strictly proper. For $n \in \mathcal{L}^1$, define the function $u(t)$ by $u := Qn$. Then, u is a continuous function in $\mathcal{L}^1$ [7]. The impulse response of $F(s) = b/(s-a)$ is $f(t) = be^{at}, t \ge 0$. The sequence

$\{y(k)\}_{k=0}^{\infty}$ defined by $y := \mathcal{S}^{\tau}(D_{-\nu} - I)Fu$ is represented by $y(k) = Y_1(k) + Y_{21}(k) + Y_{22}(k)$ for $k \in \mathcal{Z}_+$, where $Y_1(k)$, $Y_{21}(k)$, $Y_{22}(k)$ are defined as in (55), (57) and (58). Now, $Y_1(k)$ is rewritten as

$$Y_1(k) := \int_0^{\nu} be^{a(\nu-\eta)}u(kT + \tau + \eta)d\eta, \quad k \in \mathcal{Z}_+ \ . \tag{66}$$

Since u is a continuous function of t, applying the first mean value theorem to (66), there exists a real number $\alpha_k \in (0, \nu)$ such that $Y_1(k) = \nu b e^{a(\nu-\alpha_k)}u(kT + \tau + \alpha_k)$ for every $k \in \mathcal{Z}_+$. Then, we have $|Y_1(k)| \leq \nu|b||u(kT + \tau + \alpha_k)|$, $k \in \mathcal{Z}_+$. By defining the sequence $\bar{u}(k) = u(kT + \tau + \alpha_k)$, $k \in \mathcal{Z}_+$,

$$\|Y_1\|_{\ell^1} \leq \nu|b|\|\bar{u}\|_{\ell^1} \ . \tag{67}$$

Let $q(t)$ denote the impulse response of Q. Obviously, q is $\in \mathcal{L}^1$. Then, the signal $\bar{u}$ is

$$\bar{u}(k) = \sum_{i=0}^{k-1}\int_{iT}^{(i+1)T} q(kT + \alpha_k - \xi)n(\xi)d\xi + \int_{kT}^{kT+\alpha_k} q(kT + \alpha_k - \xi)n(\xi)d\xi \ .$$

for $k \in \mathcal{Z}_+$. It is clear that there exist $a_q < 0$ and $b_q > 0$ such that $|q(t)| \leq b_q e^{a_q t}$ for all $t \geq 0$. Hence, we obtain.

$$|\bar{u}(k)| \leq \sum_{i=0}^{k-1}\bar{q}(k-i-1)|\int_{iT}^{(i+1)T}|n(\xi)|d\xi + \bar{q}(0)\int_{kT}^{kT+\alpha_k}|n(\xi)|d\xi \ , k \in \mathcal{Z}_+,$$

where $\bar{q}(k) := b_q e^{k a_q T}$ and $\bar{q} \in \ell^1$. Thus,

$$\|\bar{u}\|_{\ell^1} \leq \|\bar{q}\|_{\ell^1}\|\bar{n}\|_{\ell^1} + b_q\|n\|_{\mathcal{L}^1} \ , \quad \bar{n}(i) := \int_{iT}^{(i+1)T}|n(\xi)|d\xi, \quad i \in \mathcal{Z}_+ \ .$$

It follows from $\|\bar{n}\|_{\ell^1} = \|n\|_{\mathcal{L}^1}$ that $\|\bar{u}\|_{\ell^1} \leq (\|\bar{q}\|_{\ell^1} + b_q)\|n\|_{\mathcal{L}^1}$. Apply this inequality to (67) to get

$$\|Y_1\|_{\ell^1} \leq \nu|b|(\|\bar{q}\|_{\ell^1} + b_q)\|n\|_{\mathcal{L}^1} \ . \tag{68}$$

Recalling $\|u\|_{\mathcal{L}^1} \leq \|q\|_{\mathcal{L}^1}\|n\|_{\mathcal{L}^1}$ and tracing the proof of Lemma 16, $Y_{21}(k)$ is bounded as

$$\|Y_{21}\|_{\ell^1} \leq \frac{|b|e^{a\tau}(1 - e^{a\nu})}{1 - e^{aT}}\|q\|_{\mathcal{L}^1}\|n\|_{\mathcal{L}^1} \tag{69}$$

and $Y_{22}(k)$ satisfies

$$\|Y_{22}\|_{\ell^1} \leq |b|(1 - e^{a\nu})\|q\|_{\mathcal{L}^1}\|n\|_{\mathcal{L}^1} \ . \tag{70}$$

Combining (68), (69) and (70), we obtain

$$\begin{aligned} &\|y\|_{\ell^1} \leq N_1(\nu)\|n\|_{\mathcal{L}^1} \\ &N_1(\nu) := |b|\left\{\nu(\|\bar{q}\|_{\ell^1} + b_q) + (1 - e^{a\nu})\left(\frac{1}{1 - e^{aT}} + 1\right)\|q\|_{\mathcal{L}^1}\right\} < \infty, \quad \nu > 0. \end{aligned} \tag{71}$$

Here, $\lim_{\nu \to +0} N_1(\nu) = 0$ and $N_1(\nu)$ is a monotonically increasing function of ν. The function is also uniformly continuous so that there exists a positive number δ which achieves the required property in the case of $\tau \geq 0$ and $\nu \geq 0$.

Noting that $N_1(\nu)$ obtained is independent of τ, by Lemma 9, $N_1(\nu)$ is valid for $\tau < 0$ in (71). Furthermore, the independence of τ allow us to replace τ with $\tau - \nu$ in the above argument to obtain the same function $N_1(|\nu|)$ for $\nu < 0$. ∎

Proof of Lemma 19

Define

$$H := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \\ 0 & 0 & F \end{bmatrix}\begin{bmatrix} 0 & 0 & I \\ 0 & E_{11} & E_{12} \\ I & E_{22} & E_{22} \end{bmatrix}\begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & I \end{bmatrix} \ .$$

Then, the system $\Sigma[E, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}F]$ is equivalent to $\Sigma[H, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ in Fig.11. Since F is stable, $\Sigma[H, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ is internally stable by the assumption. Now apply Theorem 5 to $\Sigma[H, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$. By Theorem 4.1 in [15], it is easy to see that $\Sigma[H, \mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ is $\mathcal{L}^p$ hybrid stable, which proves the claim. ∎

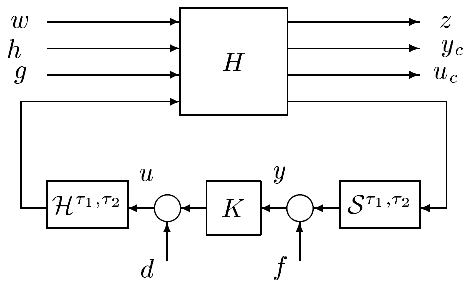

Figure 11: Equivalent system

Proof of Theorem 13

The sampled-data controllers with anti-aliasing filters are given by

$$\begin{aligned}\mathcal{H}^{\tau_1+\nu_1}K_1\mathcal{S}^{\tau_1+\nu_1}F_1 &= (\mathcal{H}^{\tau_1}+\Lambda_{O,1})K(\mathcal{S}^{\tau_1}F_1+\Lambda_{I,1})\\ \mathcal{H}^{\tau_2+\nu_2}K_2\mathcal{S}^{\tau_2+\nu_2}F_2 &= (\mathcal{H}^{\tau_2}+\Lambda_{O,2})K(\mathcal{S}^{\tau_2}F_2+\Lambda_{I,2}),\end{aligned}$$

where $\Lambda_{O,i}$ and $\Lambda_{I,i}, i=1,2$ are phase perturbation operators defined by

$$\begin{aligned}\Lambda_{I,i} &:= (\mathcal{S}^{\tau_i+\nu_i}-\mathcal{S}^{\tau_i})F_i, \quad -T_i<\tau_i+\nu_i<T_i, \quad i=1,2\\ \Lambda_{O,i} &:= \mathcal{H}^{\tau_i+\nu_i}-\mathcal{H}^{\tau_i} \quad -T_i<\tau_i+\nu_i<T_i, \quad i=1,2\ .\end{aligned}$$

Let

$$M := \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{F}_\ell(E,\mathcal{H}^\tau K\mathcal{S}^\tau F) & E_{12}(I-\mathcal{H}^\tau K\mathcal{S}^\tau FE_{22})^{-1}[I \quad \mathcal{H}^\tau K]\\ \begin{bmatrix} K\mathcal{S}^\tau F\\ I\end{bmatrix}(I-E_{22}\mathcal{H}^\tau K\mathcal{S}^\tau F)^{-1}E_{21} & \begin{bmatrix} 0 & K\\ 0 & 0\end{bmatrix}+\begin{bmatrix} K\mathcal{S}^\tau F\\ I\end{bmatrix}E_{22}(I-\mathcal{H}^\tau K\mathcal{S}^\tau FE_{22})^{-1}[I \quad \mathcal{H}^\tau K]\end{bmatrix}.$$

This operator M represents the mapping from $[w^T,g^T,f^T]^T$ to $[z^T,u^T,v_c^T]^T$ of $\Sigma[E,\mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}F]$ in Fig.9. By the assumption of internal stability of $\Sigma[G,\mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$, Lemma 19 guarantees that M is a bounded causal operator from $\mathcal{L}^p\times\mathcal{L}^p\times\ell^p$ to $\mathcal{L}^p\times\ell^p\times\mathcal{L}^p$. Then, it is easy to see that the closed-loop operators from w to z of $\Sigma[G,\mathcal{H}^{\tau_1,\tau_2}K\mathcal{S}^{\tau_1,\tau_2}]$ are given by

$$T_{zw}(T_1,\tau_1,T_2,\tau_2) = \mathcal{F}_\ell(E,\mathcal{H}^\tau K\mathcal{S}^\tau F) = \mathcal{F}_\ell(M,0) \tag{72}$$
$$T_{zw}(T_1,\tau_1+\nu_1,T_2,\tau_2+\nu_2) = \mathcal{F}_\ell(E,\mathcal{H}^{\tau_1+\nu_1,\tau_2+\nu_2}K\mathcal{S}^{\tau_1+\nu_1,\tau_2+\nu_2}F) = \mathcal{F}_\ell(M,\Lambda) \tag{73}$$

Here, the perturbation Λ is defined by

$$\Lambda := \begin{bmatrix} \Lambda_{O,1} & 0 & 0 & 0\\ 0 & \Lambda_{O,2} & 0 & 0\\ 0 & 0 & \Lambda_{I,1} & 0\\ 0 & 0 & 0 & \Lambda_{I,2}\end{bmatrix} : \ell^p\times\ell^p\times\mathcal{L}^p\times\mathcal{L}^p \to \mathcal{L}^p\times\mathcal{L}^p\times\ell^p\times\ell^p$$

Due to Lemma 12 and Lemma 16, Λ is bounded and for any given $\alpha>0$, there exist $\delta_1,\delta_2>0$ such that $\|\Lambda\|<\alpha$ holds for all $|\nu_1|\in[0,\delta_1)$. and for all $|\nu_2|\in[0,\delta_2)$. Therefore, the boundedness of M_{22} guarantees the existence of $\delta_1,\delta_2>0$ satisfying $\|\Lambda\|<1/\|M_{22}\|$ for all $|\nu_i|\in[0,\delta_i), i=1,2$. If $\|M_{22}\Lambda\|<1$ is satisfied for $M_{22}\Lambda$ on the Banach space $\ell^p\times\ell^p\times\mathcal{L}^p\times\mathcal{L}^p$, $(I-M_{22}\Lambda)^{-1}$ exists and it is expressed in the form $(I-M_{22}\Lambda)^{-1}=\sum_{j=0}^{\infty}(M_{22}\Lambda)^j$. Thus

$$\|(I-M_{22}\Lambda)^{-1}\| \le \sum_{j=0}^{\infty}\|(M_{22}\Lambda)^j\| \le \frac{1}{1-\|M_{22}\Lambda\|}\ .$$

Since M_{11}, M_{12} and M_{12} are bounded, $\|\Lambda\|<1/\|M_{22}\|$ and Equation (73) show that $T_{zw}(T_1,\tau_1+\nu_1,T_2,\tau_2+\nu_2)$ is bounded on $\mathcal{L}^p$. On the other hand, from (72) and (73) it follows that

$$T_{zw}(T_1,\tau_1+\nu_1,T_2,\tau_2+\nu_2)-T_{zw}(T_1,\tau_1,T_2,\tau_2) = M_{12}\Lambda(I-M_{22}\Lambda)^{-1}M_{21}\ .$$

Then, we have

$$\|T_{zw}(T_1, \tau_1 + \nu_1, T_2, \tau_2 + \nu_2) - T_{zw}(T_1, \tau_1, T_2, \tau_2)\| \quad \leq \quad \|M_{12}\| \frac{\|\Lambda\|}{1 - \|M_{22}\|\|\Lambda\|} \|M_{21}\| \ .$$

Therefore, there exist $\delta_1, \delta_2 > 0$ satisfying

$$\|\Lambda\| < \min \left\{ \frac{1}{\|M_{22}\|}, \ \frac{\epsilon}{\|M_{12}\|\|M_{21}\| + \epsilon\|M_{22}\|} \right\} \ .$$

for which

$$\|T_{zw}(T_1, \tau_1 + \nu_1, T_2, \tau_2 + \nu_2) - T_{zw}(T_1, \tau_1, T_2, \tau_2)\|_{\mathcal{L}^p/\mathcal{L}^p} < \epsilon$$

holds. This completes the proof. ∎