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Solutions and Characterizations of Input-to-State Stabilization via State-Dependent Scaling Design[1]

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Abstract: The author presents solutions to input-to-state stabilization and integral input-to-state stabilization problems for nonlinear systems based on the concept of state-dependent scaling design. Both state-feedback and output-feedback controllers are constructed in a unified way. The method provides global solutions whenever the system is in the strict-feedback or output-feedback form. The paper also includes results of input-to-state stabilization and integral input-to-state stabilization in the presence of structured, static and dynamic uncertainties.

Key Words: Robust nonlinear stabilization; Input-to-state stability; Integral input-to-state stability; Dissipation; Robust backstepping; State-dependent scaling design; State feedback; Output feedback.

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1 Introduction

The notion of input-to-state stability(ISS) has played an important role in recent development of nonlinear control theory[11], which was originally introduced in [13]. The ISS has already found wide applicability such as nonlinear stabilization and backstepping design[11], inverse optimal control[3, 10], small-gain theorem[9].

The concept of ISS is a natural answer to the situation where boundedness of operator norms('finite linear gains' in other words) is far too strong a requirement for general nonlinear systems. The ISS replaces the finite linear gains with nonlinear gains instead of focusing only on local properties[5]. ISS is a global property which takes into account not only initial states in a manner fully compatible with Lyapunov stability, but also the effect of input perturbations. The idea of nonlinear gain was extended by the integral input-to-state stability(iISS) in which the size of inputs is measured by integral norms[14]. For linear systems, both ISS and iISS are equivalent to asymptotic stability. For general nonlinear systems, the iISS is strictly weaker than ISS although ISS implies iISS. One of necessary and sufficient conditions for iISS is that a nonlinear system is iISS if and only if there is some output function which makes the system smoothly dissipative and weakly zero-detectable[1]. This equivalence describes an important connection between the iISS concept and another popular concept 'dissipation' which has guided developments of nonlinear $\mathcal{H}_\infty$ control and related robust control techniques.

This paper address the problem of designing input-to-state and integral input-to-state stabilizing control laws. The concept of state-dependent(SD) scaling design is employed and it leads to an explicit construction of state feedback and output feedback control laws. The SD scaling design is a new technique which thoroughly utilize the state-dependent scaling and diffeomorphism to design nonlinear control systems[4, 6, 8]. This paper does not repeat the concept and details of the SD scaling design framework which has been already presented in [4, 6, 8] and references therein. In [6, 7], the SD scaling design method has succeeded in directly solving robust nonlinear global stabilization and inverse optimal control problems without resort to ISS, by contrast with other previous methods based on ISS. Since abovementioned papers bypassed the ISS, it was not clear how to solve an important class of nonlinear control problems by using the SD scaling design approach when the problems are characterized directly in terms of ISS and iISS. This paper presents new characterizations of ISS and iISS problems through the SD scaling design and explains some necessary nontrivial modifications to the scaling, Lyapunov functions and recursive design of feedback gains and observers presented in [6, 7]. Thereby, this paper enables us to solve ISS and iISS problems through the use of the SD scaling design. The stabilizing control laws are systematically generated by selecting state-dependent scaling and parameters of the coordinate change recursively.

The paper presents both state-feedback and output-feedback global stabilization of nonlinear systems in the strict-feedback form. Input-to-state and integral input-to-state stabilization is also considered for uncertain systems, which is called robust input-to-state and robust integral input-to-state stabilization. The uncertainties are allowed to be either static or dynamic. The existence of solutions to problems are proved and the controller designs of all problems are done within a single unified framework.

2 State Feedback Stabilization

Consider the nonlinear system Σ described by

$$\Sigma : \dot{x} = A(x)x + B(x)w + G(x)u \ . \tag{1}$$

where dimensions of signals are $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^p$ and $u(t) \in \mathbf{R}^1$. Functions $A(x)$, $B(x)$ and $G(x)$ are $\mathcal{C}^0$ functions.

We use a global diffeomorphism

$$\chi = S(x)x \tag{2}$$

between $x \in \mathbf{R}^n$ and $\chi \in \mathbf{R}^n$. The time-derivative of χ is given by

$$\dot{\chi} = \left[\frac{\partial S}{\partial x_1}x, \frac{\partial S}{\partial x_2}x, \cdots, \frac{\partial S}{\partial x_n}x\right]\dot{x} + S(x)\dot{x} = T(x)\dot{x} \ ,$$

where $T(x)$ is a matrix-valued $\mathcal{C}^0$ function. Let the state-feedback be represented by

$$u = K(x)x \tag{3}$$

where K is a $\mathcal{C}^0$ function. The closed-loop system consisting of (1) and (3) becomes

$$\Sigma_{cl} : \dot{\chi} = T\left(\hat{A}\hat{S}\chi + Bw\right) \tag{4}$$

$$\hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \quad \hat{A} = [A \ G] \, .$$

The following provides new characterization of the ISS property in the state-feedback case.

Theorem 1 *If there exist a positive definite matrix P and positive real numbers ν and ξ such that*

$$N^{sf}(x) = \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB \\ B^T T^T P & -\xi I \end{bmatrix} < 0 \tag{5}$$

is satisfied for all $x \in \mathbf{R}^n$, the state-feedback law (3) renders the nonlinear system Σ input-to-state stable.

Proof : Define a positive definite function $V(x) : \mathbf{R}^n \to [0, \infty)$ by

$$V(x) = \chi^T P \chi \ . \tag{6}$$

which is a radially unbounded function of x since S defines a global diffeomorphism. The time-derivative of V along the trajectory of the closed-loop system (4) satisfies

$$\frac{d}{dt}V(x) = 2\chi^T PT\left(\hat{A}\hat{S}\chi + Bw\right)$$

We have

$$\begin{aligned} \frac{d}{dt}V(x) + \nu\chi^T P\chi - \xi w^T w &= \begin{bmatrix} \chi \\ w \end{bmatrix}^T \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} & PTB \\ B^T T^T P & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \end{bmatrix} + \begin{bmatrix} \chi \\ w \end{bmatrix}^T \begin{bmatrix} \nu P & 0 \\ 0 & -\xi I \end{bmatrix} \begin{bmatrix} \chi \\ w \end{bmatrix} \\ &= \begin{bmatrix} \chi \\ w \end{bmatrix}^T N^{sf} \begin{bmatrix} \chi \\ w \end{bmatrix} \ . \end{aligned} \tag{7}$$

From $N^{sf}(x) < 0$ it follows that

$$\frac{d}{dt}V(x) \leq -\nu\chi^T P\chi + \xi w^T w, \quad \forall x \in \mathbf{R}^n \tag{8}$$

Since $S(x)$ defines a global diffeomorphism, using the characterization of the ISS Lyapunov function in [15], the closed-loop system is proved to be input-to-state stable ∎

For linear systems, it is verified that the condition in Theorem 1 is satisfied if and only if there exist $\nu > 0$, $\xi > 0$, $\epsilon > 0$ and $P > 0$ such that

$$\left(A + GK + \frac{\nu}{2}I\right)^T P + P\left(A + GK + \frac{\nu}{2}I\right) + \xi^{-1}PBB^TP + \epsilon I = 0$$

By virtue of the theory of Riccati equations, the existence of the parameters (ν, ξ, ϵ, P) and K is guaranteed if and only if the pair (A, G) is stabilizable. This property is precisely the same as the fact that a linear closed-loop system is ISS if and only if $(A + GK)$ is a Hurwitz matrix[14].

Consider an uncertain nonlinear system Σ_U described by

$$\Sigma_U : \begin{cases} \dot{x} = A(x)x + B(x)w + B_\delta(x)w_\delta + G(x)u \\ z_\delta = C_\delta(x)x + D_\delta(x)w_\delta + H_\delta(x)u \end{cases} . \tag{9}$$

where $x(t)$ is the state, $w(t) \in \mathbf{R}^p$ is the disturbance input, and $w_\delta(t), z_\delta(t) \in \mathbf{R}^q$ are channels through which the uncertain components affects the system. Functions $B_\delta(x)$, $C_\delta(x)$, $H_\delta(x)$ and $D_\delta(x)$ are $\mathcal{C}^0$. The two signals z_δ and w_δ

$$z_\delta = \begin{bmatrix} z_{\delta_1} \\ z_{\delta_2} \\ \vdots \\ z_{\delta_m} \end{bmatrix}, \quad w_\delta = \begin{bmatrix} w_{\delta_1} \\ w_{\delta_2} \\ \vdots \\ w_{\delta_m} \end{bmatrix}, \quad \begin{matrix} w_{\delta_i}(t) \in \mathbf{R}^{q_i} \\ z_{\delta_i}(t) \in \mathbf{R}^{q_i} \\ q_i \geq 0, \quad q = \sum_{i=1}^m q_i \end{matrix}$$

are connected by an uncertain system Σ_Δ which is represented by a causal nonlinear mapping $\Delta : z_\delta \mapsto w_\delta$.

$$\Sigma_\Delta : \ \Delta = \text{block-diag}[\Delta_1, \Delta_2, \cdots, \Delta_m], \tag{10}$$

Some of the mappings $\Delta_i : z_i \mapsto w_i$, $i = 1, 2, \ldots, m$ can be zero in vector size q_i. Each uncertain mapping Δ_i is defined as

$$\Delta_i : w_{\delta_i} = h_{\delta_i}(z_{\delta_i}, t), \tag{11}$$

where h_{δ_i} is a vector-valued function satisfying $h_{\Delta_{\delta_i}}(0, t) = 0$ for all $t \geq 0$. For notational simplicity, we assume that Δ_i are square in size of input and output vectors, which does not cause any loss of generality. The uncertainty Σ_Δ defined by (11) is said to be admissible if Δ_i satisfies

$$\|z_{\delta_i}(t)\| \geq \|w_{\delta_i}(t)\|, \qquad \forall t \in [0, \infty) \ . \tag{12}$$

Note that uncertainty components having super-linear growth in x can be included by a judicious choice of $B_\delta(x)$, $C_\delta(x)$, $D_\delta(x)$ and $H_\delta(x)$. Indeed, the matrices $\{B_\delta, C_\delta, D_\delta, H_\delta\}$ specify the "nonlinear size"(including magnitude, nonlinearity, location and structure) of uncertainties. The closed-loop system consisting of (9) and the state-feedback law (3) is obtained as

$$\Sigma_{clU} : \begin{cases} \dot{\chi} = T\left(\hat{A}\hat{S}\chi + Bw + B_\delta w_\delta\right) \\ z_\delta = \hat{C}_\delta \hat{S}\chi + D_\delta w_\delta \end{cases} \tag{13}$$
$$\hat{C}_\delta = [C_\delta \ H_\delta] .$$

This paper employs the idea of state-dependent scaling to achieve input-to-output stabilization of the uncertain nonlinear system. Define the following set of scaling matrices

$$\mathbf{L} = \left\{ \Lambda = \underset{i=1}{\overset{m}{\text{block-diag}}}\, \Lambda_i : \Lambda_i = \lambda_i(x) I_i, \ \lambda_i(x) > 0 \ \forall x \in \mathbf{R}^n \right\} \tag{14}$$

In the above definition, I_i denotes an identity matrix which is compatible in size with z_{δ_i}. The scaling matrices are functions of the state variable. The state-dependent scaling is useful for estimating the worst case value of the time-derivative of Lyapunov functions[4]. As in [6], another type of SD scaling matrices for repeated uncertainties can be incorporated in the set of scaling matrices straightforwardly. For brevity, they are not included in the following theorem and all results of this paper.

Theorem 2 *If there exist a positive definite matrix P, positive real numbers ν, ξ and a scaling function matrix $\Lambda \in \mathbf{L}$ such that*

$$M^{sf}(x) = \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta & \hat{S}^T\hat{C}_\delta^T\Lambda \\ B^T T^T P & -\xi I & 0 & 0 \\ B_\delta^T T^T P & 0 & -\Lambda & D_\delta^T\Lambda \\ \Lambda\hat{C}_\delta\hat{S} & 0 & \Lambda_\delta D & -\Lambda \end{bmatrix} < 0 \tag{15}$$

is satisfied for all $x \in \mathbf{R}^n$, the state-feedback law (3) renders the nonlinear system Σ_U input-to-state stable for all admissible uncertainty Σ_Δ.

Proof : Define a positive definite function $V(x) : \mathbf{R}^n \to [0, \infty)$ by

$$V(x) = \chi^T P \chi \tag{16}$$

which is a radially unbounded function of x. The time-derivative of V along the trajectory of the closed-loop system (13) satisfies

$$\frac{d}{dt}V(x) = 2\chi^T PT\left(\hat{A}\hat{S}\chi + Bw + B_\delta w_\delta\right)$$

Using this equation, we obtain

$$\begin{aligned} \frac{d}{dt}V(x) + \nu\chi^T P\chi - \xi w^T w = & \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} & PTB & PTB_\delta \\ B^T T^T P & 0 & 0 \\ B_\delta^T T^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} \\ & + \begin{bmatrix} \chi \\ w \end{bmatrix}^T \begin{bmatrix} \nu P & 0 \\ 0 & -\xi I \end{bmatrix} \begin{bmatrix} \chi \\ w \end{bmatrix} \end{aligned} \tag{17}$$

Since the admissible uncertainty Δ_i satisfies

$$\begin{bmatrix} w_{\delta_i} \\ z_{\delta_i} \end{bmatrix}^T \begin{bmatrix} -\Lambda_i(x) & 0 \\ 0 & \Lambda_i(x) \end{bmatrix} \begin{bmatrix} w_{\delta_i} \\ z_{\delta_i} \end{bmatrix} \geq 0 \quad \forall x \in \mathbf{R}^n \tag{18}$$

Equation (17) becomes

$$\begin{aligned} & \frac{d}{dt}V(x) + \nu\chi^T P\chi - \xi w^T w \\ \leq & \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta \\ B^T T^T P & -\xi I & 0 \\ B_\delta^T T^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} + \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^T \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix} \quad (19) \\ = & \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \left(\begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta \\ B^T T^T P & -\xi I & 0 \\ B_\delta^T T^T P & 0 & 0 \end{bmatrix} + \right. \\ & \left. \begin{bmatrix} 0 & \hat{S}^T\hat{C}_\delta^T \\ 0 & 0 \\ I & D_\delta^T \end{bmatrix} \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ \hat{C}_\delta\hat{S} & 0 & D_\delta \end{bmatrix} \right) \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} \\ = & \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T Q \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} \end{aligned} \tag{20}$$

where the matrix $Q(x)$ is

$$Q = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta \\ B^T T^T P & -\xi I & 0 \\ B_\delta^T T^T P & 0 & -\Lambda \end{bmatrix} + \begin{bmatrix} \hat{S}^T \hat{C}_\delta^T \Lambda \\ 0 \\ D_\delta^T \Lambda \end{bmatrix} \Lambda^{-1} \begin{bmatrix} \Lambda \hat{C}_\delta \hat{S} & 0 & \Lambda D_\delta \end{bmatrix}$$

According to the Schur complement formula, the inequality (15) is equivalent to $Q < 0$. Thus, we arrive at

$$\frac{d}{dt} V(x) \leq -\nu \chi^T P \chi + \xi w^T w, \quad \forall x \in \mathbf{R}^n \tag{21}$$

Hence, the closed-loop system is input-to-state stable. ∎

The characterizations of stabilization presented in the above theorems are addressed by strict inequalities(negative definiteness of matrices). It can be verified that they can be replaced with non-strict inequalities(negative semi-definiteness). A control law satisfying the non-strict inequality is also a solution to the stabilization problem. The following corollaries state the fact precisely.

Corollary 1 *Suppose that matrices S and P are given. The following two statements are equivalent.*
*(***i***) There exist $\nu, \xi > 0$ such that $N^{sf}(x) < 0$ is satisfied for all $x \in \mathbf{R}^n$.*
*(***ii***) There exist $\tilde{\nu}, \tilde{\xi} > 0$ such that $N^{sf}(x) \leq 0$ is satisfied for all $x \in \mathbf{R}^n$, where ν, ξ are replaced by $\tilde{\nu}, \tilde{\xi}$ in the definition (5) of $N^{sf}(x)$.*
Proof : The direction (i)⇒(ii) is trivial. The converse direction is proved for any $\xi > \tilde{\xi}$ by letting $\epsilon_\nu > 0$ be small enough to satisfy $\nu = \tilde{\nu} - \epsilon_\nu > 0$. ∎

Corollary 2 *Suppose that matrices S and P are given. Consider the following two statements.*
*(***i***) There exist $\nu, \xi > 0$ and $\Lambda \in \mathbf{L}$ such that $M^{sf}(x) < 0$ is satisfied for all $x \in \mathbf{R}^n$.*
*(***ii***) There exist $\tilde{\nu}, \tilde{\xi} > 0$ and $\tilde{\Lambda} \in \mathbf{L}$ such that $M^{sf}(x) \leq 0$ is satisfied for all $x \in \mathbf{R}^n$, where ν, ξ, Λ are replaced by $\tilde{\nu}, \tilde{\xi}, \tilde{\Lambda}$ in the definition (15) of $M^{sf}(x)$.*
Then, the statement (i) always implies (ii). Furthermore, if the condition

$$\begin{bmatrix} -\tilde{\Lambda} & D_\delta^T \tilde{\Lambda} \\ \tilde{\Lambda} D_\delta & -\tilde{\Lambda} \end{bmatrix} < 0, \qquad \forall x \in \mathbf{R}^n \tag{22}$$

is satisfied, the statement (ii) implies (i).
Proof : The direction (i)⇒(ii) is trivial. In order to prove the converse, let $\xi > \tilde{\xi}$ and choose $\epsilon_\nu > 0$ to be any small number satisfying $\nu = \tilde{\nu} - \epsilon_\nu > 0$. Define $\Lambda(x) = (1 + \alpha(x))\tilde{\Lambda}(x)$ with a scalar-valued function $\alpha(x) > 0$ to be determined later. The matrix Λ obviously belongs to the set of the SD scaling $\mathbf{L}$. For these new parameters, we obtain

$$M^{sf}(\tilde{\nu}, \tilde{\xi}, \tilde{\Lambda}, x) = M^{sf}(\nu, \xi, \Lambda, x) - M_\epsilon(x) \leq 0$$

$$M_\epsilon(x) = \begin{bmatrix} -\epsilon_\nu P & 0 & 0 & \hat{S}^T \hat{C}_\delta^T \alpha \Lambda \\ 0 & -(\xi - \tilde{\xi}) I & 0 & 0 \\ 0 & 0 & -\alpha \Lambda & D_\delta^T \alpha \Lambda \\ \alpha \Lambda \hat{C}_\delta \hat{S} & 0 & \alpha \Lambda D_\delta & -\alpha \Lambda \end{bmatrix}$$

The matrix $M_\epsilon(x)$ is negative definite if and only if

$$\begin{bmatrix} -\epsilon_\nu P & 0 \\ 0 & -(\xi - \tilde{\xi}) I \end{bmatrix} - \alpha \begin{bmatrix} 0 & \hat{S}^T \hat{C}_\delta^T \tilde{\Lambda} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\tilde{\Lambda} & D_\delta^T \tilde{\Lambda} \\ \tilde{\Lambda} D_\delta & -\tilde{\Lambda} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ \tilde{\Lambda} \hat{C}_\delta \hat{S} & 0 \end{bmatrix} < 0 \tag{23}$$

holds for all x. Recall the assumption (22) and that the constant numbers ϵ_ν and $\xi - \tilde{\xi}$ are positive. Since functions $\hat{S}$, $\hat{C}$ are continuous, there always exists a function $\alpha(x)$ such that (23) and $\alpha(x) > 0$ are satisfied for all $x \in R^n$. Hence, $M^{sf}(\nu, \xi, \Lambda, x) < 0$ is proved. ∎

It should be noted that the condition (22) is satisfied for any $\tilde{\Lambda} \in \mathbf{L}$ whenever $D_\delta = 0$. The inequality (22) is a reasonable assumption for input-to-state stabilizable systems. Actually, the assumption is a condition which ensures the wellposedness of the uncertain system for all admissible uncertainties. For instance, if D_δ is block-diagonal, the inequality (22) is equivalent to

$$I - D_{\delta,i}(x) D_{\delta,i}^T(x) > 0, \quad \forall x \in \mathbf{R}^n \tag{24}$$

which is necessary for the wellposedness of the uncertain system.

Now, we focus on the existence of the state-feedback law and the construction of the controller solving the conditions in Theorem 1 and 2. We shall prove the existence for the nonlinear system Σ satisfying the following structural assumptions.

$$A(x) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots\cdots & 0 \\ a_{21} & a_{22} & a_{23} \; 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots\ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, G(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \tag{25}$$

$$B(x) = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{n1} & \cdots & B_{n,n-1} & B_{nn} \end{bmatrix} \tag{26}$$

$$a_{ij}(x) = a_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \le i \le n, \; 1 \le j \le i+1 \tag{27}$$

$$a_{i,i+1}(x_1, x_2, \cdots, x_i) \ne 0, \quad 1 \le i \le n, \quad \forall x \in \mathbf{R}^n \tag{28}$$

$$B_{ij}(x) = B_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \le i \le n, \; 1 \le j \le i \tag{29}$$

This structure of Σ is called the strict-feedback form in the literature[11, 3]. In addition, the uncertain system Σ_U is supposed to satisfy $m = 2n$ and

$$B_\delta(x) = \begin{bmatrix} B_{\delta,11} & U_{L1} & 0 & 0 & \cdots & 0 & 0 \\ B_{\delta,21} & 0 & B_{\delta,22} & U_{L2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ B_{\delta,n1} & 0 & B_{\delta,n2} & 0 & \cdots & B_{\delta,nn} & U_{Ln} \end{bmatrix} \tag{30}$$

$$C_\delta(x) = \begin{bmatrix} C_{\delta,11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_{R1} & 0 & \cdots & 0 & 0 \\ C_{\delta,21} & C_{\delta,22} & 0 & \ddots & 0 & 0 \\ 0 & 0 & U_{R2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{\delta,n-1,1} & C_{\delta,n-1,2} & \cdots & \cdots & C_{\delta,n-1,n-1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & U_{R,n-1} \\ C_{\delta,n1} & C_{\delta,n2} & \cdots & \cdots & C_{\delta,n,n-1} & C_{\delta,nn} \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \tag{31}$$

$$D_\delta(x)=\begin{bmatrix} D_{\delta,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_{\delta,2} & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & D_{\delta,n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, H_\delta(x)=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ U_{Rn} \end{bmatrix} \tag{32}$$

where $B_{\delta,ij}(x) \in \mathbf{R}^{1\times q_{(2i-1)}}$, $C_{\delta,ij}(x) \in \mathbf{R}^{q_{(2i-1)}\times 1}$, $D_{\delta,i}(x) \in \mathbf{R}^{q_{(2i-1)}\times q_{(2i-1)}}$, $U_{L,i}(x) \in \mathbf{R}^{1\times q_{2i}}$ and $U_{R,i}(x) \in \mathbf{R}^{q_{2i}\times 1}$ satisfies

$$B_{\delta,ij}(x)=B_{\delta,ij}(x_1,x_2,\cdots,x_i),\ C_{\delta,ij}(x)=C_{\delta,ij}(x_1,x_2,\cdots,x_i) \tag{33}$$

$$U_{Li}(x)=U_{Li}(x_1,x_2,\cdots,x_i),\ U_{Ri}(x)=U_{Ri}(x_1,x_2,\cdots,x_i) \tag{34}$$

$$D_{\delta,i}(x)=D_{\delta,i}(x_1,x_2,\cdots,x_i),\ I-D_{\delta,i}(x)D_{\delta,i}^T(x)>0, \quad \forall x\in\mathbf{R}^n \tag{35}$$

for $1\le i\le n$ and $1\le j\le i$. The structure of Σ_U is called the robust strict-feedback form[6]. Let $x_{[k]}$ denote the first k components of the state:

$$x_{[k]}=[x_1,x_2,\cdots,x_k]^T.$$

For the diffeomorphism between x and χ, we take

$$S(x)=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -s_1 & 1 & 0 & \cdots & 0 \\ s_1s_2 & -s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (-1)^{n-1}s_1\cdots s_{n-1} & \cdots & s_{n-2}s_{n-1} & -s_{n-1} & 1 \end{bmatrix} \tag{36}$$

Let the state-feedback be in the following form.

$$u=s_n(x)\chi_n \tag{37}$$

The smooth scalar functions $s_1(x_{[1]})$, $s_2(x_{[2]})$, $\cdots$, $s_{n-1}(x_{[n-1]})$ are to be designed from s_1 through s_n in a recursive manner. The matrix $\hat{S}$ is obtained as

$$\hat{S}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 1 \\ 0 & \cdots & 0 & 0 & s_n \end{bmatrix}$$

Finally, the state-dependent scaling is chosen as

$$\mathbf{L}=\left\{\Lambda=\operatorname*{block-diag}_{i=1}^{2n}\Lambda_i : \begin{array}{ll} \Lambda_i=\lambda_i(x_{[(i+1)/2]})I_i & \text{for odd } i \\ \Lambda_i=\lambda_i(x_{[i/2]})I_i & \text{for even } i \end{array}, \quad \lambda_i(x)>0\ \forall x\in\mathbf{R}^n\right\} \tag{38}$$

The following theorems demonstrate that the solutions $\{s_1,\cdots,s_n\}$, $\{\lambda_1,\cdots,\lambda_{2n}\}$ and P of (5) and (15) always exist for any $\nu,\xi>0$.

Theorem 3 *The system Σ in the strict-feedback form can be input-to-state stabilized by the state-feedback law (37).*

Theorem 4 *The system Σ_U in the robust strict-feedback form can be input-to-state stabilized by the state-feedback law (37) for all admissible uncertainty Σ_Δ.*

In the rest of this section, the two theorems are proved. From $\xi > 0$ it follows that $N^{sf} < 0$ and $M^{sf} < 0$ are identical with

$$\bar{N}^{sf}(x) = \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P + \frac{1}{\xi}PTBB^T T^T P < 0$$

$$\bar{M}^{sf}(x) = \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P + \frac{1}{\xi}PTBB^T T^T P & PTB_\delta & \hat{S}^T\hat{C}_\delta^T\Lambda \\ B_\delta^T T^T P & -\Lambda & D_\delta^T\Lambda \\ \Lambda\hat{C}_\delta\hat{S} & \Lambda D_\delta & -\Lambda \end{bmatrix} < 0$$

respectively. The matrices $\bar{N}^{sf}(x)$ and $\bar{M}^{sf}(x)$ are the same as those appearing in [6] except for an extra term $\nu P + \frac{1}{\xi}PTBB^T T^T P$. Let P be any diagonal matrix

$$P = \operatorname*{diag}_{i=1}^{n} P_i, \qquad P_i > 0$$

We introduce a notation $[k]$ which denotes the submatrix at the upper left corner of a matrix as follows:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1,n} \\ Q_{21} & Q_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n,1} & \cdots & \cdots & Q_{n,n} \end{bmatrix}, \quad Q_{[k]} = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1,k} \\ Q_{21} & Q_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k,1} & \cdots & \cdots & Q_{k,k} \end{bmatrix}, \quad Q_{[1]} = Q_{11}, \quad Q_{[n]} = Q$$

For example, we use

$$P_{[k]} = \left[\begin{array}{c|c} P_{[k-1]} & 0 \\ \hline 0 & P_k \end{array}\right]$$

$$T_{[k]}(x_{[k-1]}) = \left[\begin{array}{c|c} T_{[k-1]}(x_{[k-2]}) & 0 \\ \hline \star_{k-1,k-1} & 1 \end{array}\right]$$

$$B_{[k]}(x_{[k]}) = \left[\begin{array}{c|cc} B_{[k-1]}(x_{[k-1]}) & 0 & 0 \\ \hline \star_{k,0} & B_{kk} & U_{Lk} \end{array}\right]$$

Here, the entry $\star_{i,j}$ depends only on the states $x_{[i]}$, and the functions s_1 through s_j and their partial derivatives. The strict-feedback form of Σ yields the following structure.

$$P_{[k]}T_{[k]}B_{[k]}B_{[k]}^T T_{[k]}^T P_{[k]} = \left[\begin{array}{c|c} P_{[k-1]}T_{[k-1]}B_{[k-1]}B_{[k-1]}^T T_{[k-1]}^T P_{[k-1]} & \star_{k,k-1} \\ \hline \star_{k,k-1} & \star_{k,k-1} \end{array}\right]$$

Obviously, the matrix $PTBB^T T^T P$ is independent of Λ. In addition, the matrix $P_{[k]}T_{[k]}B_{[k]}B_{[k]}^T T_{[k]}^T P_{[k]}$ do not include of s_k. Due to this property of the extra term, the recursive construction of $\{s_k, \lambda_{2k-1}, \lambda_{2k}\}$ from $k = 1$ through $k = n$ is always feasible by following the procedure which is almost similar to [6]. For detailed formulas, see [6].

3 Output Feedback Stabilization

Consider the nonlinear system Σ described by

$$\Sigma : \begin{cases} \dot{x} = A(y)x + B(y)w + G(y)u \\ y = C_y x \end{cases} . \tag{39}$$

where C_y is a constant row vector, and $y(t) \in \mathbf{R}^1$ is the measurement output. Suppose that the state variable x cannot be measured. We employ the following observer to estimate the state.

$$\begin{cases} \dot{\hat{x}} = A(y)\hat{x} + Y(y,\hat{x})(y-\hat{y}) + G(y)u \\ \hat{y} = C_y \hat{x} \end{cases} \tag{40}$$

This section seeks the output feedback control consisting of (40) and

$$u = K(y,\hat{x})\hat{x} \ . \tag{41}$$

Functions Y and K are $\mathbf{C}^0$ functions which have yet to be determined. The closed-loop system is written as

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & GK \\ YC_y & A - YC_y + GK \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{42}$$

Consider a global diffeomorphism between $[\hat{x}^T, \hat{x}^T - x^T]^T \in \mathbf{R}^{2n}$ and $[\hat{\chi}^T, \eta]^T \in \mathbf{R}^{2n}$ as follows:

$$\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} = \begin{bmatrix} S(y,\hat{x}) & 0 \\ 0 & W \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix} \tag{43}$$

The time-derivative of $\hat{\chi}$ is obtained as

$$\dot{\hat{\chi}} = \left[\frac{\partial S}{\partial y_1}\hat{x}, \frac{\partial S}{\partial y_2}\hat{x}, \cdots, \frac{\partial S}{\partial y_n}\hat{x}\right] C_y \dot{x} + \left[\frac{\partial S}{\partial \hat{x}_1}\hat{x}, \frac{\partial S}{\partial \hat{x}_2}\hat{x}, \cdots, \frac{\partial S}{\partial \hat{x}_n}\hat{x}\right]\dot{\hat{x}} + S(y,\hat{x})\dot{\hat{x}} = X(y,\hat{x})\dot{x} + T(y,\hat{x})\dot{\hat{x}} \ .$$

The square matrix W is constant and non-singular. The closed-loop system on the new coordinate $(\hat{\chi}, \eta)$ is

$$\begin{bmatrix} \dot{\hat{\chi}} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} (X+T)\hat{A}\hat{S} & -(XA + TYC_y)W^{-1} \\ 0 & \hat{W}^T \bar{A} W^{-1} \end{bmatrix}\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} + \begin{bmatrix} XB \\ -WB \end{bmatrix} w \tag{44}$$

$$\bar{A} = \begin{bmatrix} C_y^T & A^T \end{bmatrix}, \quad \hat{W} = \begin{bmatrix} -Y^T W^T \\ W^T \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A & G \end{bmatrix}.$$

Theorem 5 *If there exist positive definite matrices P and $\tilde{P}$, and positive real numbers ν, ξ and $\tilde{\nu}$ such that*

$$N^{of}(y,\hat{x}) = \begin{bmatrix} \hat{S}^T\hat{A}^T(X+T)^T P + P(X+T)\hat{A}\hat{S} + \nu P & PXB & -P(XA+TYC_y)W^{-1} \\ B^T X^T P & -\xi I & -B^T W^T \tilde{P} \\ -W^{-T}(XA+TYC_y)^T P & -\tilde{P}WB & W^{-T}\bar{A}\hat{W}\tilde{P} + \tilde{P}\hat{W}^T\bar{A}^T W^{-1} + \tilde{\nu}\tilde{P} \end{bmatrix} < 0 \tag{45}$$

is satisfied for all $(y,\hat{x}) \in \mathbf{R}^{n+1}$, the output-feedback law (40-41) renders the nonlinear system Σ input-to-state stable.

Proof : Define a positive definite function $V(x,\hat{x}) : \mathbf{R}^{2n} \to [0,\infty)$ by

$$V(x,\hat{x}) = \hat{\chi}^T P \hat{\chi} + \eta^T \tilde{P} \eta \ . \tag{46}$$

which is a radially unbounded function of $(x,\hat{x})$ since S and W define a global diffeomorphism. The time-derivative of V along the trajectory of the closed-loop system (44) satisfies

$$\frac{d}{dt}V = 2\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & \tilde{P} \end{bmatrix}\left(\begin{bmatrix} (X+T)\hat{A}\hat{S} & -(XA+TYC_y)W^{-1} \\ 0 & \hat{W}^T\bar{A}W^{-1} \end{bmatrix}\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} + \begin{bmatrix} XB \\ -WB \end{bmatrix} w\right)$$

We have

$$\begin{aligned}
&\frac{d}{dt}V + \begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix}^T \begin{bmatrix}\nu P & 0\\ 0 & \tilde{\nu}\tilde{P}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} - \xi w^T w\\
&= \begin{bmatrix}\hat{\chi}\\ w\\ \eta\end{bmatrix}^T \begin{bmatrix}\hat{S}^T\hat{A}^T(X+T)^TP + P(X+T)\hat{A}\hat{S} & PXB & -P(XA+TYC_y)W^{-1}\\ B^TX^TP & 0 & -B^TW^T\tilde{P}\\ -W^{-T}(XA+TYC_y)^TP & -\tilde{P}WB & W^{-T}\bar{A}\hat{W}\tilde{P} + \tilde{P}\hat{W}^T\bar{A}^TW^{-1}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ w\\ \eta\end{bmatrix}\\
&\qquad + \begin{bmatrix}\hat{\chi}\\ w\\ \eta\end{bmatrix}^T \begin{bmatrix}\nu P & 0 & 0\\ 0 & -\xi I & 0\\ 0 & 0 & \tilde{\nu}\tilde{P}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ w\\ \eta\end{bmatrix}\\
&= \begin{bmatrix}\hat{\chi}\\ w\\ \eta\end{bmatrix}^T N^{of}\begin{bmatrix}\hat{\chi}\\ w\\ \eta\end{bmatrix}. \qquad (47)
\end{aligned}$$

From $N^{of}(y,\hat{x}) < 0$ it follows that

$$\frac{d}{dt}V \le -\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix}^T \begin{bmatrix}\nu P & 0\\ 0 & \tilde{\nu}\tilde{P}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} + \xi w^T w \qquad (48)$$

Since (43) is a global diffeomorphism, the characterization of the ISS Lyapunov function in [15] proves that the closed-loop system is input-to-state stable. ∎

Consider an uncertain nonlinear system Σ_U described by

$$\Sigma_U : \begin{cases}\dot{x} = A(y)x + B(y)w + B_\delta(y)w_\delta + G(y)u\\ z_\delta = C_\delta(y)x\\ y = C_y x\end{cases}. \qquad (49)$$

The uncertain system Σ_Δ is defined by (10) and (11). The uncertainty Σ_Δ is said to be admissible if (12) is satisfied for all $i = 1, \ldots, m$. For the output-feedback case, state-dependent scaling matrices are chosen as functions of output and state estimate.

$$\mathbf{L} = \left\{\Lambda = \operatorname*{block-diag}_{i=1}^{m} \Lambda_i : \Lambda_i = \lambda_i(y,\hat{x})I_i,\ \lambda_i(y,\hat{x}) > 0\ \forall (y,\hat{x}) \in \mathbf{R}^{n+1}\right\} \qquad (50)$$

The closed-loop system consisting of (49) and the output-feedback law (40-41) is represented by

$$\begin{bmatrix}\dot{\hat{\chi}}\\ \dot{\eta}\end{bmatrix} = \begin{bmatrix}(X+T)\hat{A}\hat{S} & -(XA+TYC_y)W^{-1}\\ 0 & \hat{W}^T\bar{A}W^{-1}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} + \begin{bmatrix}XB\\ -WB\end{bmatrix}w + \begin{bmatrix}XB_\delta\\ -WB_\delta\end{bmatrix}w_\delta \qquad (51)$$

$$z_\delta = C_\delta\begin{bmatrix}S^{-1} & -W^{-1}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} \qquad (52)$$

Theorem 6 *If there exist positive definite matrices P and $\tilde{P}$, positive real numbers ν, ξ, $\tilde{\nu}$ and a scaling function matrix $\Lambda \in \mathbf{L}$ such that*

$$M^{of}(y,\hat{x}) = \begin{bmatrix}\begin{pmatrix}\hat{S}^T\hat{A}^T(X+T)^TP+\\ P(X+T)\hat{A}\hat{S}+\nu P\end{pmatrix} & PXB & PXB_\delta & S^{-T}C_\delta^T\Lambda & -P(XA+TYC_y)W^{-1}\\ B^TX^TP & -\xi I & 0 & 0 & -B^TW^T\tilde{P}\\ B_\delta^TX^TP & 0 & -\Lambda & 0 & -B_\delta^TW^T\tilde{P}\\ \Lambda C_\delta S^{-1} & 0 & 0 & -\Lambda & -\Lambda C_\delta W^{-1}\\ -W^{-T}(XA+TYC_y)^TP & -\tilde{P}WB & -\tilde{P}WB_\delta & -W^{-T}C_\delta^T\Lambda & \begin{pmatrix}W^{-T}\bar{A}\hat{W}\tilde{P}+\\ \tilde{P}\hat{W}^T\bar{A}^TW^{-1}+\tilde{\nu}\tilde{P}\end{pmatrix}\end{bmatrix} < 0 \qquad (53)$$

is satisfied for all $(y,\hat{x}) \in \mathbf{R}^{n+1}$*, the output-feedback law (40-41) renders the nonlinear system* Σ_U *input-to-state stable for all admissible uncertainty* Σ_Δ*.*

Proof : Define a positive definite function $V(x,\hat{x}) : \mathbf{R}^{2n} \to [0,\infty)$ by

$$V(x,\hat{x}) = \hat{\chi}^T P \hat{\chi} + \eta^T \tilde{P} \eta \ . \tag{54}$$

which is a radially unbounded function of $(x,\hat{x})$. The time-derivative of V along the trajectory of the closed-loop system (51) satisfies

$$\frac{d}{dt}V = 2\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & \tilde{P}\end{bmatrix}\left(\begin{bmatrix}(X+T)\hat{A}\hat{S} & -(XA+TYC_y)W^{-1} \\ 0 & \hat{W}^T\bar{A}W^{-1}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} + \begin{bmatrix} XB \\ -WB\end{bmatrix} w + \begin{bmatrix} XB_\delta \\ -WB_\delta\end{bmatrix} w_\delta\right)$$

We arrange the time-derivative as follows:

$$\begin{aligned}
&\frac{d}{dt}V + \begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix}^T \begin{bmatrix}\nu P & 0 \\ 0 & \tilde{\nu}\tilde{P}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} - \xi w^T w \\
&= \begin{bmatrix}\hat{\chi}\\ w \\ w_\delta \\ \eta\end{bmatrix}^T \begin{bmatrix}
\begin{pmatrix}\hat{S}^T\hat{A}^T(X+T)^TP + \\ P(X+T)\hat{A}\hat{S}\end{pmatrix} & PXB & PXB_\delta & -P(XA+TYC_y)W^{-1} \\
B^TX^TP & 0 & 0 & -B^TW^T\tilde{P} \\
B_\delta^TX^TP & 0 & 0 & -B_\delta^TW^T\tilde{P} \\
-W^{-T}(XA+TYC_y)^TP & -\tilde{P}WB & -\tilde{P}WB_\delta & \begin{pmatrix} W^{-T}\bar{A}\hat{W}\tilde{P} + \\ \tilde{P}\hat{W}^T\bar{A}^TW^{-1}\end{pmatrix}
\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ w \\ w_\delta \\ \eta\end{bmatrix} \\
&+ \begin{bmatrix}\hat{\chi}\\ w \\ \eta\end{bmatrix}^T \begin{bmatrix}\nu P & 0 & 0 \\ 0 & -\xi I & 0 \\ 0 & 0 & \tilde{\nu}\tilde{P}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ w \\ \eta\end{bmatrix}
\end{aligned} \tag{55}$$

Since the admissible uncertainty Δ_i satisfies

$$\begin{bmatrix} w_{\delta_i} \\ z_{\delta_i}\end{bmatrix}^T \begin{bmatrix} -\Lambda_i(y,\hat{x}) & 0 \\ 0 & \Lambda_i(y,\hat{x})\end{bmatrix}\begin{bmatrix} w_{\delta_i} \\ z_{\delta_i}\end{bmatrix} \geq 0 \quad \forall (y,\hat{x}) \in \mathbf{R}^{n+1} \tag{56}$$

Equation (55) becomes

$$\begin{aligned}
&\frac{d}{dt}V + \begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix}^T \begin{bmatrix}\nu P & 0 \\ 0 & \tilde{\nu}\tilde{P}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} - \xi w^T w \\
&\leq \begin{bmatrix}\hat{\chi}\\ w \\ w_\delta \\ \eta\end{bmatrix}^T \begin{bmatrix}
\begin{pmatrix}\hat{S}^T\hat{A}^T(X+T)^TP + \\ P(X+T)\hat{A}\hat{S} + \nu P\end{pmatrix} & PXB & PXB_\delta & -P(XA+TYC_y)W^{-1} \\
B^TX^TP & -\xi I & 0 & -B^TW^T\tilde{P} \\
B_\delta^TX^TP & 0 & 0 & -B_\delta^TW^T\tilde{P} \\
-W^{-T}(XA+TYC_y)^TP & -\tilde{P}WB & -\tilde{P}WB_\delta & W^{-T}\bar{A}\hat{W}\tilde{P} + \tilde{P}\hat{W}^T\bar{A}^TW^{-1} + \tilde{\nu}\tilde{P}
\end{bmatrix} \\
&\quad + \begin{bmatrix} w_\delta \\ z_\delta\end{bmatrix}^T \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda\end{bmatrix}\begin{bmatrix} w_\delta \\ z_\delta\end{bmatrix} \\
&= \begin{bmatrix}\hat{\chi}\\ w \\ w_\delta \\ \eta\end{bmatrix}^T \left(\begin{bmatrix}
\hat{S}^T\hat{A}^T(X+T)^TP + P(X+T)\hat{A}\hat{S} + \nu P & PXB & PXB_\delta & -P(XA+TYC_y)W^{-1} \\
B^TX^TP & -\xi I & 0 & -B^TW^T\tilde{P} \\
B_\delta^TX^TP & 0 & 0 & -B_\delta^TW^T\tilde{P} \\
-W^{-T}(XA+TYC_y)^TP & -\tilde{P}WB & -\tilde{P}WB_\delta & W^{-T}\bar{A}\hat{W}\tilde{P} + \tilde{P}\hat{W}^T\bar{A}^TW^{-1} + \tilde{\nu}\tilde{P}
\end{bmatrix}\right. \\
&\quad \left. + \begin{bmatrix} 0 & S^{-T}C_\delta^T \\ 0 & 0 \\ I & 0 \\ 0 & -W^{-T}C_\delta^T\end{bmatrix}\begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda\end{bmatrix}\begin{bmatrix} 0 & 0 & I & 0 \\ C_\delta S^{-1} & 0 & 0 & -C_\delta W^{-1}\end{bmatrix}\right)\begin{bmatrix}\hat{\chi}\\ w \\ w_\delta \\ \eta\end{bmatrix}
\end{aligned} \tag{57}$$

$$= \begin{bmatrix} \hat{\chi} \\ w \\ w_\delta \\ \eta \end{bmatrix}^T Q \begin{bmatrix} \hat{\chi} \\ w \\ w_\delta \\ \eta \end{bmatrix} \tag{58}$$

The matrix $Q(y, \hat{x})$ is obtained as

$$Q = \begin{bmatrix} \hat{S}^T \hat{A}^T (X+T)^T P + P(X+T)\hat{A}\hat{S} + \nu P & PXB & PXB_\delta & -P(XA+TYC_y)W^{-1} \\ B^T X^T P & -\xi I & 0 & -B^T W^T \tilde{P} \\ B_\delta^T X^T P & 0 & -\Lambda & -B_\delta^T W^T \tilde{P} \\ -W^{-T}(XA+TYC_y)^T P & -\tilde{P}WB & -\tilde{P}WB_\delta & W^{-T}\bar{A}\hat{W}\tilde{P} + \tilde{P}\hat{W}^T\bar{A}^T W^{-1} + \tilde{\nu}\tilde{P} \end{bmatrix}$$
$$+ \begin{bmatrix} S^{-T} C_\delta^T \Lambda \\ 0 \\ 0 \\ -W^{-T} C_\delta^T \Lambda \end{bmatrix} \Lambda^{-1} \left[\Lambda C_\delta S^{-1} \;\; 0 \;\; 0 \;\; -\Lambda C_\delta W^{-1} \right]$$

Using the Schur complement formula, we see that the inequality (53) is equivalent to $Q < 0$. Thus, we obtain

$$\frac{d}{dt} V \le - \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix}^T \begin{bmatrix} \nu P & 0 \\ 0 & \tilde{\nu}\tilde{P} \end{bmatrix} \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} + \xi w^T w \tag{59}$$

and the closed-loop system is input-to-state stable. ∎

It can be verified that the strict inequality characterizations in Theorem 5 and 6 can be rewritten by non-strict inequalities $N^{of} \le 0$ and $M^{of} \le 0$. The equivalence demonstrated in Corollary 1 and Corollary 2 is also true for the output-feedback case.

Now we suppose that the system Σ and Σ_U satisfy the following triangular structure.

$$A(y) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, G(y) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \tag{60}$$

$$a_{i,i+1}(y) \neq 0, \quad 1 \le i \le n, \quad \forall y \in \mathbf{R} \tag{61}$$

$$B(y) = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{n1} & \cdots & B_{n,n-1} & B_{nn} \end{bmatrix} \tag{62}$$

$$B_\delta(y) = \begin{bmatrix} B_{\delta,11} & 0 & \cdots & 0 \\ B_{\delta,21} & B_{\delta,22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{\delta,n1} & \cdots & B_{\delta,n,n-1} & B_{\delta,nn} \end{bmatrix}, \; C_\delta(y) = \begin{bmatrix} C_{\delta,11} & 0 & \cdots & 0 \\ C_{\delta,21} & C_{\delta,22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ C_{\delta,n1} & \cdots & C_{\delta,n,n-1} & C_{\delta,nn} \end{bmatrix} \tag{63}$$

where $B_{\delta,ij}(y) \in \mathbf{R}^{1 \times q_{(2i-1)}}$, $C_{\delta,ij}(y) \in \mathbf{R}^{q_{(2i-1)} \times 1}$ and $m = n$. The above matrices are dependent only on the output y so that this paper call the structure of Σ and Σ_U the output-feedback form and the robust output-feedback form, respectively. Note that the class is more general than a standard output-feedback form[11] in which the nonlinearity is restricted to $A(y)x = A_0 x + \psi(y)$. We assume that the output equation of Σ and Σ_U is given by

$$y = x_1$$

or equivalently $C_y = [1\ 0 \cdots 0]$. This case is sometimes called output feedback in the nonlinear control literature[11]. We define $S(x_1, \hat{x})$ and the feedback gain as follows:

$$S^{-1}(x_1, \hat{x}_{[n-2]}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 1 \end{bmatrix} \tag{64}$$

$$u = s_n(x_1, \hat{x}_{[n-1]})\hat{\chi}_n \tag{65}$$

The parameters $s_1(x_1)$, $s_2(x_1, \hat{x}_1)$, $\cdots$, $s_n(x_1, \hat{x}_{[n-1]})$ are smooth scalar-valued functions which are to be determined in a recursive manner from s_1 through s_n. Let the matrix W be

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ w_2 & 1 & 0 & \cdots & 0 \\ 0 & w_3 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & w_n & 1 \end{bmatrix} \tag{66}$$

whose entries w_i for $2 \le i \le n$ are constant. Define the observer gain by

$$Y(x_1) = -W^{-1} \begin{bmatrix} w_1(x_1) \\ 0 \end{bmatrix} = - \begin{bmatrix} w_1 \\ -w_1 w_2 \\ \vdots \\ (-1)^{n-1} w_1 w_2 \cdots w_n \end{bmatrix} \tag{67}$$

where w_1 is a $\mathbf{C}^0$ function of x_1. The parameters $w_1, \cdots, w_n$ have yet to be determined recursively from $k = n$ through $k = 1$. The state-dependent scaling for the output-feedback problem is chosen as

$$\mathbf{L} = \left\{ \Lambda = \operatorname*{block-diag}_{i=1}^{n} \Lambda_i : \Lambda_i = \lambda_i(y, \hat{x}_{[i-2]}) I_i > 0 \ \forall (y, \hat{x}_{[i-2]}) \in \mathbf{R} \times \mathbf{R}^{i-2} \right\} \tag{68}$$

We restrict our attention to the following class of systems.

Assumption 1 *The function $A(x_1)x$ satisfies*

$$A(x_1)x = A_0 x + \psi(x_1) + \phi(x_1) x_2 \tag{69}$$

with a constant matrix A_0 and $\mathbf{C}^0$ functions ψ and ϕ. There exist constants $\alpha_i > 0$ such that

$$\left| a_{i2}^2(x_1) / a_{12}(x_1) \right| \le \alpha_i, \quad i = 2, 3, \ldots, n \tag{70}$$

hold for all $x_1 \in \mathbf{R}$. The matrix B satisfies

$$B(x_1) = \begin{bmatrix} B_{11}(x_1) & 0 & \cdots & 0 \\ 0 & B_{22}(x_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_{nn}(x_1) \end{bmatrix}, \qquad B_{ii}(x_1) \in \mathbf{R}^{1 \times p_i} \tag{71}$$

and there exist constants $\beta_i > 0$ such that

$$\frac{B_{11}(x_1) B_{11}^T(x_1)}{\sqrt{a_{12}^2(x_1)}} \le \beta_0, \quad \frac{B_{22}(x_1) B_{22}^T(x_1)}{\sqrt{a_{12}^2(x_1)}} \le \beta_1, \quad B_{ii}(x_1) B_{ii}^T(x_1) \le \beta_i, \ i = 2, 3, \ldots, n \tag{72}$$

It should be noted that the diagonal restriction (71) imposed on B does not cause any loss of generality. Indeed, an ISS problem with a triangular B can be recasted as another ISS problem with a diagonal B by using the following idea.

$$\underbrace{\left[\begin{array}{c|c|c} b_{11} & 0 & 0 \\ \hline b_{21} & b_{22} & 0 \\ \hline b_{31} & b_{32} & b_{33} \end{array}\right]}_{B_t} \left[\begin{array}{c} w_1 \\ \hline w_2 \\ \hline w_3 \end{array}\right] = \underbrace{\left[\begin{array}{ccc|ccc|ccc} b_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_{21} & b_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{array}\right]}_{B_d} \left[\begin{array}{c} w_1 \\ w_2 \\ w_3 \\ \hline w_1 \\ w_2 \\ w_3 \\ \hline w_1 \\ w_2 \\ w_3 \end{array}\right]$$

A system defined by a state-space equation with a disturbance input term $B_t w$ is ISS if it is ISS with $B_d w$.

We are now in a position to state the following theorem.

Theorem 7 *Under the Assumption 1, the system Σ in the output-feedback form can be input-to-state stabilized by the output-feedback law (40-41) with (65) and (67).*

For robust input-to-state stabilization, we need the following assumption.

Assumption 2 *The matrices B_δ and C_δ satisfy*

$$B_\delta(x_1) = \begin{bmatrix} B_{\delta,11}(x_1) \\ B_{\delta,21}(x_1) \\ \vdots \\ B_{\delta,n1}(x_1) \end{bmatrix}, \quad C_\delta(x_1) = \begin{bmatrix} C_{\delta,11}(x_1) & 0 & \cdots & 0 \end{bmatrix} \tag{73}$$

where $B_{\delta,i1}(x_1) \in \mathbf{R}^{1\times q_1}$, $C_{\delta,11}(x_1) \in \mathbf{R}^{q_1\times 1}$ and $q_1 = q$.

This assumption is the same as that in [8].

Theorem 8 *Under the Assumption 1 and 2, the system Σ_U in the output-feedback form can be input-to-state stabilized for all admissible uncertainty Σ_Δ by the output-feedback law (40-41) with (65) and (67).*

The rest of this section describes the proof of the above theorems and how to design the output-feedback control law. Using the Schur complement formula, $N^{of} < 0$ and $M^{of} < 0$ are equivalently transformed into

$$\bar{N}^{of}(y,\hat{x}) = \begin{bmatrix} N + \xi^{-1}PXBB^TX^TP & -P(XA+TYC_y)W^{-1} - \xi^{-1}PXBB^TW^T\tilde{P} \\ * & H + \xi^{-1}\tilde{P}WBB^TW^T\tilde{P} \end{bmatrix} < 0$$

$$\bar{M}^{of}(y,\hat{x}) = \begin{bmatrix} N + \xi^{-1}PXBB^TX^TP & PXB_\delta & S^{-T}C_\delta^T\Lambda & \begin{pmatrix} -P(XA+TYC_y)W^{-1} - \\ \xi^{-1}PXBB^TW^T\tilde{P} \end{pmatrix} \\ * & -\Lambda & 0 & -B_\delta^TW^T\tilde{P} \\ * & * & -\Lambda & -\Lambda C_\delta W^{-1} \\ * & * & * & H + \xi^{-1}\tilde{P}WBB^TW^T\tilde{P} \end{bmatrix} < 0$$

respectively, where

$$N = \hat{S}^T\hat{A}^T(X+T)^TP + P(X+T)\hat{A}\hat{S} + \nu P$$
$$H = W^{-T}\bar{A}\hat{W}\tilde{P} + \tilde{P}\hat{W}^T\bar{A}^TW^{-1} + \tilde{\nu}\tilde{P}$$

The matrices $\bar{N}^{of}$ and $\bar{M}^{of}$ are symmetric. The parts filled with $*$ are defined accordingly. The matrices $\bar{N}^{of}$ and $\bar{M}^{of}$ are the same as those appeared in [8] except for the terms

$$\xi^{-1}PXBB^TX^TP,\ \xi^{-1}\tilde{P}WBB^TW^T\tilde{P},\ \xi^{-1}PXBB^TW^T\tilde{P},\ \nu P,\ \tilde{\nu}\tilde{P}$$

In order to solve $\bar{M}^{of} < 0$ and $\bar{M}^{of} < 0$, we divide the design procedure into two phases. The first phase is to design the observer (40). The second phase is the design of the feedback gain (41). The observer problem is to find W and Y which solve $H + \xi^{-1}\tilde{P}WBB^TW^T\tilde{P} < 0$. The feedback gain design takes place after W and Y are determined. In the second phase, we compute S and K meeting the condition $\bar{M}^{of} < 0$ or $\bar{M}^{of} < 0$. Both the designs will be done in a recursive manner. The feedback gain design determines s_k from $k = 1$ up to $k = n$. The observer gain design determines w_k from $k = n$ down to $k = 1$. The idea of recursive construction of observers originates from [8]. It, however, needs nontrivial modifications for dealing with problems of this section.

Begin by picking any diagonal matrices $P > 0$ and $\tilde{P} > 0$. The output-feedback form of Σ and Σ_U leads to the following recursive structure.

$$P_{[k]}X_{[k]}B_{[k]}B_{[k]}^TX_{[k]}^TP_{[k]} = \left[\begin{array}{c|c} P_{[k-1]}X_{[k-1]}B_{[k-1]}B_{[k-1]}^TX_{[k-1]}^TP_{[k-1]} & \square_{k-1,k-1} \\ \hline \square_{k-1,k-1} & \square_{k-1,k-1} \end{array}\right] \tag{74}$$

where $\square_{i,j}$ denotes any function depending only on $(x_1, \hat{x}_{[i]})$, and the functions s_1 through s_j and their partial derivatives. Due to the diagonal structure (71), we also obtain

$$P_{[k]}X_{[k]}B_{[k]}B_{[k]}^TW_{[k]}^T\tilde{P}_{[k]} = \left[\begin{array}{c|c} P_{[k-1]}X_{[k-1]}B_{[k-1]}B_{[k-1]}^TW_{[k-1]}^T\tilde{P}_{[k-1]} & 0 \\ \hline \square_{k-1,k-1} & 0 \end{array}\right] \tag{75}$$

Since ξ is positive, the inequality

$$H + \xi^{-1}\tilde{P}WBB^TW^T\tilde{P} < 0 \tag{76}$$

is equivalent to

$$-B^TW^T\tilde{P}H^{-1}\tilde{P}WB < \xi I, \quad H < 0 \tag{77}$$

Let Γ be a diagonal matrix satisfying

$$\Gamma = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_n \end{bmatrix} > 0 \tag{78}$$

$$-H^{-1} < \Gamma \tag{79}$$

Then, (77) is implied by

$$B^TW^T\tilde{P}\Gamma\tilde{P}WB < \xi I \tag{80}$$

Now, consider

$$B_{\langle k\rangle}^TW_{\langle k\rangle}^T\tilde{P}_{\langle k\rangle}\Gamma_{\langle k\rangle}\tilde{P}_{\langle k\rangle}W_{\langle k\rangle}B_{\langle k\rangle} < \xi_kI \tag{81}$$

Here the subscript $\langle k\rangle$ denotes the submatrix at the lower right corner as follows:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1,n} \\ Q_{21} & Q_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n,1} & \cdots & \cdots & Q_{n,n} \end{bmatrix}, \quad Q_{\langle k\rangle} = \begin{bmatrix} Q_{k,k} & Q_{k,k+1} & \cdots & Q_{k,n} \\ Q_{k+1,k} & Q_{k+1,k+1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n,k} & \cdots & \cdots & Q_{n,n} \end{bmatrix}, \quad Q_{\langle 1\rangle} = Q, \quad Q_{\langle n\rangle} = Q_{n,n}$$

Due to the structure of Σ, we have

$$B_{\langle k\rangle}^T W_{\langle k\rangle}^T \tilde{P}_{\langle k\rangle}\Gamma_{\langle k\rangle}\tilde{P}_{\langle k\rangle}W_{\langle k\rangle}B_{\langle k\rangle}$$
$$= \left[\begin{array}{c|c} (\gamma_k\tilde{P}_k^2+\gamma_{k+1}\tilde{P}_{k+1}^2 w_{k+1}^2)B_{kk}^T B_{kk} & \gamma_{k+1}\tilde{P}_{k+1}^2 w_{k+1}B_{kk}^T B_{k+1,k+1} \qquad\qquad 0 \\ \hline \Box_{0,0} & B_{\langle k+1\rangle}^T W_{\langle k+1\rangle}^T \tilde{P}_{\langle k+1\rangle}\Gamma_{\langle k+1\rangle}\tilde{P}_{\langle k+1\rangle}W_{\langle k+1\rangle}B_{\langle k+1\rangle}\end{array}\right] \tag{82}$$

Thus, there exists a sequence $\xi = \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n > 0$ achieving (81) whenever there exist $\xi > 0$ such that (80) holds. Using (82), we can pick constants $\gamma_k > 0$, $\xi_k > 0$ and $\mu_k > 0$ satisfying

$$B_{\langle k\rangle}^T W_{\langle k\rangle}^T \tilde{P}_{\langle k\rangle}\Gamma_{\langle k\rangle}\tilde{P}_{\langle k\rangle}W_{\langle k\rangle}B_{\langle k\rangle} - \xi_k I \leq -\mu_k I \tag{83}$$

recursively from $k = n$ up to $k = 3$ since $B_{\langle 3\rangle}$ is uniformly bounded. For $k = 2$, take $\gamma_2 = c/\sqrt{a_{12}^2}$ with a positive constant $c > 0$. Because of the uniform boundedness of $B_{\langle 2\rangle}$ and $B_{22}B_{22}^T/\sqrt{a_{12}^2}$, we can find $c > 0$, $\xi_2 > 0$ and $\mu_2 > 0$ such that (83) holds for $k = 2$. Again from (82) and the uniform boundedness of $B_{11}B_{11}^T/\sqrt{a_{12}^2}$ and $B_{22}B_{22}^T/\sqrt{a_{12}^2}$, it follows that we can pick up $\gamma_1(x_1) > 0$ and $\xi_1 > 0$ with which (81) is achieved for $k = 1$. Note here that under the Assumption 1, at each step k, we can find the parameter w_k satisfying

$$-H_{\langle k\rangle}^{-1} < \Gamma_{\langle k\rangle} \tag{84}$$

for the chosen γ_k by following the design procedure of robust observer in [8]. In this way, the observer parameters $Y(x_1)$ and W which meet (76) can be always constructed by solving (83) and (84) for γ_k and w_k recursively from $k = n$ down to $k = 1$. Finally, using the structure (74) and (75), it is proved that the parameters $\{s_k, \lambda_k\}$ of feedback gain design solving $\bar{N}^{of} < 0$ and $\bar{M}^{of} < 0$ can be determined recursively from $k = 1$ up to $k = n$ by following the procedure established in [8]. Thus, Theorem 7 and 8 are proved on Assumption 1 and 2.

4 Characterization for Dynamic Uncertainty

The uncertainty components defined by (11) are static systems. In this section, we consider the uncertainty Σ_Δ which is allowed to be dynamic. Now, the causal mapping Δ_i in (10) is defined by

$$\Delta_i : z_{\delta_i} = \begin{bmatrix}\zeta_{d,i}\\ \zeta_{s,i}\end{bmatrix} \mapsto w_{\delta_i} = \begin{bmatrix}\omega_{d,i}\\ \omega_{s,i}\end{bmatrix} \tag{85}$$

$$\begin{cases} \dot{x}_{\delta_i} = f_{\delta_i}(x_{\delta_i}, \zeta_{d,i}, t) \\ \omega_{d,i} = g_{\delta_i}(x_{\delta_i}, \zeta_{d,i}, t) \end{cases} \tag{86}$$

$$\omega_{s,i} = h_{\delta_i}(\zeta_{s,i}, t) \tag{87}$$

where the vector-valued functions satisfy $f_{\delta_i}(0,0,t)=0$ and $g_{\delta_i}(0,0,t)=0$ for all $t \geq 0$. Let x_δ denote the whole state variable of Σ_Δ, i.e., $x_\delta = [x_{\delta,1}^T, \ldots, x_{\delta,n}^T]^T$. In this section, the uncertainty Σ_Δ is said to be admissible if each static mapping $\zeta_{s,i} \mapsto \omega_{s,i}$ satisfies

$$\|\zeta_{s,i}(t)\| \geq \|\omega_{s,i}(t)\|, \qquad \forall t \in [0,\infty) \quad . \tag{88}$$

and if for each dynamic mapping $\zeta_{d,i} \mapsto \omega_{d,i}$, there exists a radially unbounded positive definite $\mathbf{C}^1$ function $V_{\delta_i}(x_{\delta_i})$ such that

$$V_{\delta_i}(x_{\delta_i}(0)) + \int_0^T (\zeta_{\delta_i}^T\zeta_{\delta_i} - \omega_{\delta_i}^T\omega_{\delta_i})dt \geq V_{\delta_i}(x_{\delta_i}(T)) + \int_0^T \psi(\|x_\delta(t)\|)dt \quad , \forall T \in [0,\infty), \forall \zeta_{\delta_i} \in \mathcal{L}_2[0,T]$$

holds globally for a class $\mathcal{K}_\infty$ function $\psi(\cdot)$. The following theorem describes robust input-to-state stabilization for dynamic uncertainty by state feedback control.

Theorem 9 *If there exist a positive definite matrix P, positive real numbers ν and ξ, and a constant matrix $\Lambda \in \mathbf{L}$ such that*

$$M^{sfd}(x) = \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta & \hat{S}^T\hat{C}_\delta^T\Lambda \\ B^T T^T P & -\xi I & 0 & 0 \\ B_\delta^T T^T P & 0 & -\Lambda + \zeta I & D_\delta^T \Lambda \\ \Lambda \hat{C}_\delta \hat{S} & 0 & \Lambda D_\delta & -\Lambda \end{bmatrix} < 0 \tag{89}$$

is satisfied for all $x \in \mathbf{R}^n$, the state-feedback law (3) renders the nonlinear system Σ_U input-to-state stable for all admissible dynamic uncertainty Σ_Δ.

Proof : Suppose that there exist real numbers $\lambda_i > 0$, $i = 1, 2, \ldots, n$ such that the inequality (89) holds with $\Lambda_i = \lambda_i I$. By definition of the admissible uncertainty, the time-derivative of $V_{\delta_i}(x_{\delta_i})$ is obtained as

$$\frac{d}{dt} V_{\delta_i} \le \frac{1}{\lambda_i} \begin{bmatrix} \omega_{d,i} \\ \zeta_{d,i} \end{bmatrix}^T \begin{bmatrix} -\Lambda_i & 0 \\ 0 & \Lambda_i \end{bmatrix} \begin{bmatrix} \omega_{d,i} \\ \zeta_{d,i} \end{bmatrix} - \psi(\|x_\delta(t)\|), \qquad \forall t \in [0, \infty)$$

for all i. Define a positive definite function

$$V(x_{cl}) = \chi^T P \chi + \sum_{i=1}^{n} \lambda_i V_{\delta_i} \tag{90}$$

which is a radially unbounded function of $x_{cl} = [x^T, x_\delta^T]^T$. The time-derivative of V along the trajectory of the closed-loop system (13) satisfies

$$\begin{aligned} &\frac{d}{dt} V(x_{cl}) + \nu \chi^T P \chi - \xi w^T w \\ &\le \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} & PTB & PTB_\delta \\ B^T T^T P & 0 & 0 \\ B_\delta^T T^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} + \begin{bmatrix} \chi \\ w \end{bmatrix}^T \begin{bmatrix} \nu P & 0 \\ 0 & -\xi I \end{bmatrix} \begin{bmatrix} \chi \\ w \end{bmatrix} + \\ &\qquad \sum_{i=1}^{n} \begin{bmatrix} \omega_{d,i} \\ \zeta_{d,i} \end{bmatrix}^T \begin{bmatrix} -\Lambda_i & 0 \\ 0 & \Lambda_i \end{bmatrix} \begin{bmatrix} \omega_{d,i} \\ \zeta_{d,i} \end{bmatrix} - \lambda_i \psi(\|x_\delta\|) && (91) \\ &\le \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta \\ B^T T^T P & -\xi I & 0 \\ B_\delta^T T^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} \\ &\qquad + \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^T \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix} - \lambda_i \psi(\|x_\delta\|) && (92) \\ &= \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \left(\begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta \\ B^T T^T P & -\xi I & 0 \\ B_\delta^T T^T P & 0 & 0 \end{bmatrix} \right. \\ &\qquad \left. + \begin{bmatrix} 0 & \hat{S}^T\hat{C}_\delta^T \\ 0 & 0 \\ I & D_\delta^T \end{bmatrix} \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ \hat{C}_\delta \hat{S} & 0 & D_\delta \end{bmatrix} \right) \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} - \lambda_i \psi(\|x_\delta\|) && (93) \\ &= \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T Q \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} - \lambda_i \psi(\|x_\delta\|) \end{aligned}$$

The matrix $Q(x)$ is

$$Q = \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB & PTB_\delta \\ B^T T^T P & -\xi I & 0 \\ B_\delta^T T^T P & 0 & -\Lambda \end{bmatrix} + \begin{bmatrix} \hat{S}^T\hat{C}_\delta^T\Lambda \\ 0 \\ D_\delta^T\Lambda \end{bmatrix} \Lambda^{-1} \begin{bmatrix} \Lambda\hat{C}_\delta\hat{S} & 0 & \Lambda D_\delta \end{bmatrix}$$

Applying the Schur complement formula to $Q < 0$, we can see that the inequality (89) is equivalent to $Q < 0$. Thus, we obtain

$$\frac{d}{dt}V(x_{cl}) \leq -\nu\chi^T P\chi - \psi(\|x_\delta\|) + \xi w^T w \tag{94}$$

This inequality proves that the closed-loop system is input-to-state stable. ∎

In the same manner, the following is proved for output feedback control.

Theorem 10 *If there exist positive definite matrices P and $\tilde{P}$, positive real numbers ν, ξ, $\tilde{\nu}$ and a constant matrix $\Lambda \in \mathbf{L}$ such that*

$$M^{ofd}(y,\hat{x}) = \begin{bmatrix} \begin{pmatrix} \hat{S}^T\hat{A}^T(X+T)^T P + \\ P(X+T)\hat{A}\hat{S} + \nu P \end{pmatrix} & PXB & PXB_\delta & S^{-T}C_\delta^T\Lambda & -P(XA+TYC_y)W^{-1} \\ B^T X^T P & -\xi I & 0 & 0 & -B^T W^T \tilde{P} \\ B_\delta^T X^T P & 0 & -\Lambda I & 0 & -B_\delta^T W^T \tilde{P} \\ \Lambda C_\delta S^{-1} & 0 & 0 & -\Lambda & -\Lambda C_\delta W^{-1} \\ -W^{-T}(XA+TYC_y)^T P & -\tilde{P}WB & -\tilde{P}WB_\delta & -W^{-T}C_\delta^T\Lambda & \begin{pmatrix} W^{-T}\bar{A}\hat{W}\tilde{P} + \\ \tilde{P}\hat{W}^T\bar{A}^T W^{-1} + \tilde{\nu}\tilde{P} \end{pmatrix} \end{bmatrix} < 0 \tag{95}$$

is satisfied for all $(y,\hat{x}) \in \mathbf{R}^{n+1}$, the output-feedback law (40-41) renders the nonlinear system Σ input-to-state stable for all admissible dynamic uncertainty Σ_Δ.

Moreover, in the conditions $M^{sfd} < 0$ and $M^{ofd} < 0$, the parameter λ_i is allowed to be a function of x and $(y,\hat{x})$, respectively if the dimension of the vector $\omega_{d,i}$ is zero for that i.

In the case of dynamic uncertainty, there might be no solution of the input-to-state stabilization problem even if the system Σ_U is in the robust strict feedback form or the robust output-feedback form as it was for the robust stabilization problem[6].

5 Integral Input-to-State Stabilization

Let $\nu = 0$ and $\tilde{\nu} = 0$ in the characterization $N^{sf} < 0$, $M^{sf} < 0$, $N^{of} < 0$ and $M^{of} < 0$ of previous sections. Then, the time-derivative of the quadratic Lyapunov functions satisfies

$$\frac{d}{dt}V(x_{all}) \leq \xi w^T w \tag{96}$$

where x_{all} denotes the state of the entire closed-loop system. The inequality (96) implies that the closed-loop system is zero-output smoothly dissipative[1]. The closed-loop system is also proved to be 0-GAS since

$$\frac{d}{dt}V(x_{all}) \leq x_{all}^T \mathcal{M}(x_{all}) x_{all} \tag{97}$$

holds for $w \equiv 0$, and $\mathcal{M}(x_{all}) < 0$ holds for all x_{all}. Owing to the result of [1, 12], Theorem 1, 2, 5 and 6 guarantee iISS of the closed-loop systems when $\nu = 0$ and $\tilde{\nu} = 0$. In the same way, Theorem 9 and 10 with $\nu = 0$ and $\tilde{\nu} = 0$ guarantee iISS. Note that every input-to-state stable system is necessarily integral input-to-state stable but the converse is not true[12]. For linear systems, it is obvious that there exists $\nu > 0$ such that $N^{sf} < 0$ is satisfied if and only if $N^{sf} < 0$ is satisfied with

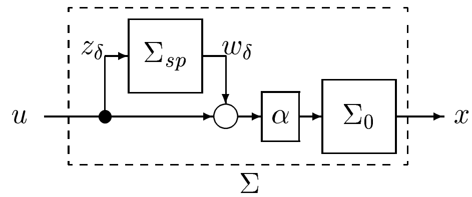

Figure 1: Nonlinear plant with input unmodeled dynamics

$\nu = 0$. Similarly, the existence of $\nu > 0$ and $\tilde{\nu} > 0$ satisfying $N^{of} < 0$ also implies and is implied by $N^{of} < 0$ of $\nu = \tilde{\nu} = 0$. The same property is true for M^{sf}, M^{of}, M^{sfd} and M^{ofd} in the case of linear systems. This fact explains exactly the equivalence between ISS and iISS property for linear systems[14]. Since we have succeeded in constructing a state-feedback controller which renders the nonlinear strict-feedback system Σ input-to-state stable using the state-dependent scaling technique for the $\nu > 0$ case, it is trivial that the strict-feedback system Σ is also integral input-to-state stabilizable (for all admissible uncertainty Σ_Δ) via state feedback by using the state-dependent scaling method. The same fact is true for output feedback control under the condition of Assumption 1 and 2.

6 Robustness for Passive Uncertainty

This section addresses the problem of designing controllers which remain input-to-state stabilizing in the presence of a certain class of dynamic uncertainties. The following is the definition of strict passivity[2].

Definition 1 *The system*

$$\Sigma_{sp} : \begin{cases} \dot{x}_\delta = f_\delta(x_\delta) + g_\delta(x_\delta) z_\delta \\ w_\delta = h_\delta(x_\delta), \qquad\qquad x_\delta(t) \in \mathbf{R}^{n_\delta} \end{cases} \tag{98}$$

is said to be strictly passive if there exist a $\mathcal{C}^1$ positive definite radially unbounded function $V_\delta(x_\delta)$ and a class $\mathcal{K}_\infty$ function $\psi(\cdot)$ such that

$$\int_0^t w_\delta^T z_\delta d\tau \geq V_\delta(x_\delta(t)) - V_\delta(x_\delta(0)) + \int_0^t \psi(\|x_\delta(\tau)\|) d\tau \tag{99}$$

for all $z_\delta \in \mathcal{C}^0$, $x_\delta(0) \in \mathbf{R}^n$ and $t \geq 0$.

Consider the uncertain system Σ shown in Fig1 in which $\Sigma_{sp} : z_\delta \mapsto w_\delta$ is a dynamic uncertainty which is assumed to be strictly passive. The system is described by

$$\Sigma : \begin{cases} \dot{x} = A(x)x + B(x)w + G(x)\alpha(w_\delta + u) \\ z_\delta = u \end{cases} . \tag{100}$$

where α is a real number and $\alpha > 0$. We consider the following state-feedback control

$$u = K(x)x \tag{101}$$

and define the following functions.

$$\hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \quad \hat{A} = [A \ \alpha G]$$

We now introduce a new class of scaling matrices as follows:

$$\mathbf{L}_d = \left\{ \Lambda(s) = \lambda(s) I : \lambda \in \mathcal{C}^0, \quad 0 < \lambda(s) \leq \bar{\lambda}, \ \forall s \in [0, \infty) \right\} \tag{102}$$

where $\bar{\lambda}$ is an arbitrary finite number. In particular, we are interested in $\Lambda(s)$ whose independent variable s is a quadratic function of χ. This new class of scaling is different from state-dependent scaling for static uncertainties in that it is uniformly bounded. This new class of scaling enables us to establish the input-to-state stabilization in the presence of the dynamic input uncertainty.

Theorem 11 *Given any $\alpha_l > 0$, the uncertain system consisting of (100) and (98) is input-to-state stabilized by a state feedback law (101) for all $\alpha \in [\alpha_l, \infty)$ if there exist a positive definite matrix P and positive real numbers ν, ξ and a scaling function $\Lambda \in \mathbf{L}_d$ such that*

$$M^{sp}(x) = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + P T \hat{A} \hat{S} + \nu P & PTB \\ B^T T^T P & -\xi I \end{bmatrix} \leq 0 \tag{103}$$

$$PTG + S^{-T} K^T \Lambda = 0 \tag{104}$$

are satisfied with $s = x^T S^T P S x$ for all $x \in \mathbf{R}^n$ and all $\alpha \in [\alpha_l, \infty)$.

Proof : By definition, the strictly passive uncertainty has a positive definite radially unbounded function V_δ satisfying

$$\frac{d}{dt} V_\delta(x_\delta) \leq w_\delta^T z_\delta - \psi(\|x_\delta\|)$$

where $\psi(\cdot)$ is a class $\mathcal{K}_\infty$ function. Thus, we have

$$2\lambda(s) \frac{d}{dt} V_\delta(x_\delta) \leq \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^T \begin{bmatrix} 0 & \Lambda(s) \\ \Lambda(s) & 0 \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix} - 2\lambda(s)\psi(\|x_\delta\|), \qquad \forall t \in [0, \infty) \tag{105}$$

for an arbitrary function $\lambda(s) > 0$. Next, define a function $V(x_{cl})$ by

$$V(x_{cl}) = \int_0^{V_0(x)} \frac{1}{\lambda(s)} ds + 2\alpha V_\delta(x_\delta), \qquad V_0(x) = \chi^T P \chi \tag{106}$$

where $x_{cl} = [x^T, x_\delta^T]^T$ and P is a positive definite matrix. Since $\lambda(s)$ is $\mathcal{C}^0$, positive and uniformly bounded, the function V is a positive definite radially unbounded $\mathcal{C}^1$ function of x_{cl}. The time-derivative of V along the trajectory of the closed-loop system satisfies

$$\begin{aligned}
&\frac{d}{dt} V(x_{cl}) + \frac{\nu}{\lambda(V_0(x))} \chi^T P \chi - \frac{\xi}{\lambda(V_0(x))} w^T w \\
&= \frac{1}{\lambda(V_0(x))} \left[\frac{d}{dt} V_0(x) + 2\alpha \lambda(V_\delta(x_\delta)) \frac{d}{dt} V_0(x) + \nu \chi^T P \chi - \xi w^T w \right] \\
&\leq \frac{1}{\lambda(V_0(x))} \left\{ \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + P T \hat{A} \hat{S} & PTB & PTG\alpha \\ B^T T^T P & 0 & 0 \\ \alpha G^T T^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} + \begin{bmatrix} \chi \\ w \end{bmatrix}^T \begin{bmatrix} \nu P & 0 \\ 0 & -\xi I \end{bmatrix} \begin{bmatrix} \chi \\ w \end{bmatrix} \right. \\
&\qquad \left. + \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^T \begin{bmatrix} 0 & \alpha\Lambda \\ \alpha\Lambda & 0 \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix} \right\} - 2\alpha\psi(\|x_\delta\|) \qquad (107) \\
&= \frac{1}{\lambda(V_0(x))} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix}^T \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + P T \hat{A} \hat{S} + \nu P & PTB & PTG\alpha + S^{-T} K^T \Lambda \alpha \\ B^T T^T P & -\xi I & 0 \\ \alpha G^T T^T P + \alpha \Lambda K S^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi \\ w \\ w_\delta \end{bmatrix} - 2\alpha\psi(\|x_\delta\|)
\end{aligned}$$

From (103) and (104), we obtain

$$\frac{d}{dt}V(x_{cl}) \le -\frac{\nu}{\lambda(V_0(x))}\chi^T P\chi - 2\alpha\psi(\|x_\delta\|) + \frac{\xi}{\lambda(V_0(x))}w^T w$$

Since $0 < \lambda(V_0(x)) \le \bar{\lambda}$ holds for all x, there exist class $\mathcal{K}_\infty$ functions ρ_1, ρ_2 and ρ_3 such that

$$\frac{d}{dt}V(x_{cl}) \le -\frac{1}{\lambda(V_0(x))}(\rho_1(\|x\|) - \rho_2(\|w\|)) - \rho_3(\|x_\delta\|) \tag{108}$$

for any $x \in \mathbf{R}^n$, $x_\delta \in \mathbf{R}^{n_\delta}$ and any $w \in \mathbf{R}^p$. Now, pick another class $\mathcal{K}_\infty$ function $\rho_4(\|w\|) = \rho_1^{-1}(2\rho_2(\|w\|))$. Then, whenever $\|x\| \ge \rho_4(\|w\|)$, we have

$$\frac{1}{\lambda(V_0(x))}\rho_2(\|w\|) \le \frac{1}{2\lambda(V_0(x))}\rho_1(\|x\|)$$

from $\lambda > 0$. Since $0 < \bar{\lambda}^{-1} \le \lambda^{-1}$ is satisfied for all x, we arrive at

$$\|x\| \ge \rho_4(\|w\|) \;\Rightarrow\; \frac{d}{dt}V(x_{cl}) \le -\frac{1}{2\bar{\lambda}}\rho_1(\|x\|) - \rho_3(\|x_\delta\|) \le -\rho_5(\|x\|) - \rho_3(\|x_\delta\|) \tag{109}$$

where $\rho_5(\|x\|) = \rho_1(\|x\|)/2\bar{\lambda} \in \mathcal{K}_\infty$. On the other hand, from (108), we obtain

$$\begin{aligned} \frac{d}{dt}V(x_{cl}) + \rho_5(\|x\|) + \rho_3(\|x_\delta\|) &\le \left(\frac{1}{2\bar{\lambda}} - \frac{1}{\lambda(V_0(x))}\right)\rho_1(\|x\|) + \frac{1}{\lambda(V_0(x))}\rho_2(\|w\|) \\ &\le \frac{1}{\lambda(V_0(x))}\rho_2(\|w\|) \end{aligned} \tag{110}$$

for all $x \in \mathbf{R}^n$, $x_\delta \in \mathbf{R}^{n_\delta}$ and all $w \in \mathbf{R}^p$. Assume now that $\|x\| \le \rho_4(\|w\|)$ holds. The inequality (110) becomes

$$\frac{d}{dt}V(x_{cl}) + \rho_5(\|x\|) + \rho_3(\|x_\delta\|) \le \rho_2(\|w\|)\max_{\|x\|\le\rho_4(\|w\|)}\left\{\frac{1}{\lambda(V_0(x))}\right\}$$

Since λ is a continuous function satisfying $0 < \lambda \le \bar{\lambda}$, from $\rho_4 \in \mathcal{K}_\infty$ it follows that there exists a class $\mathcal{K}_\infty$ function ρ_6 such that

$$\|x\| \le \rho_4(\|w\|) \;\Rightarrow\; \frac{d}{dt}V(x_{cl}) + \rho_5(\|x\|) + \rho_3(\|x_\delta\|) \le \rho_6(\|w\|) \tag{111}$$

holds. Finally, by combining (109) and (111), we obtain

$$\frac{d}{dt}V(x_{cl}) \le -\rho_5(\|x\|) - \rho_3(\|x_\delta\|) + \rho_6(\|w\|) \tag{112}$$

for all $x \in \mathbf{R}^n$, $x_\delta \in \mathbf{R}^{n_\delta}$ and all $w \in \mathbf{R}^p$. This completes the proof. ∎

Next, we show that a controller which fulfills (103) and (104) can be always constructed if the system Σ is in the strict-feedback form. Suppose that the matrices $A(x)$, $B(x)$ and $G(x)$ are given as (25-29). Let the state-feedback law be (37) and P is a diagonal matrix. Then, the equation (104) reduces to

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_n a_{n,n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s_n\lambda \end{bmatrix} = 0$$

Thus, for the feedback gain, we pick

$$s_n = -\frac{P_n a_{n,n+1}}{\lambda} \tag{113}$$

By virtue of the development in [6], the condition $M^{sp} < 0$ is satisfied if

$$\begin{array}{ll} 2P_k a_{k,k+1} s_k - \beta_k(x_{[k]}) < 0, & \text{for } k = 1, 2, \ldots, n-1 \\ 2P_k \alpha a_{k,k+1} s_k - \beta_k(x_{[k]}) < 0, & \text{for } k = n \end{array} \tag{114}$$

are achieved by finding $s_k(x_{[k]})$ recursively from $k = 1$ through $k = n$. The function $\beta_k(x)$ is an appropriate $\mathcal{C}^0$ function which is independent of $\{s_k, \cdots, s_n\}$. Since $a_{k,k+1}(x_{[k]})$ are positive and $P_k, \alpha > 0$, there always exist $\{s_1(x_{[1]}), \cdots, s_{n-1}(x_{[n-1]})\}$ satisfying (114). As for $k = n$, substituting (113) into (114), we obtain

$$2\alpha P_n^2 a_{n,n+1}^2 > \lambda \beta_n(x), \quad \forall x \in \mathbf{R}^n \tag{115}$$

It is easy to see that there exits a $\mathcal{C}^0$ function $\lambda(\chi^T P \chi)$ such that

$$2\alpha_l P_n^2 a_{n,n+1}^2 > \lambda \beta_n(x), \quad \forall x \in \mathbf{R}^n \tag{116}$$

$$0 < \lambda(\chi^T P \chi) < \bar{\lambda}, \quad \forall x \in \mathbf{R}^n \tag{117}$$

are satisfied with a finite number $\bar{\lambda}$. It should be noted that s_n and λ do not include α. To summarize the above discussion, we state the following theorem.

Theorem 12 *Suppose that the system (100) is in the strict-feedback form. Given any $\alpha_l > 0$, the uncertain system consisting of (100) and (98) can be always input-to-state stabilized by a state feedback law (113) for all $\alpha \in [\alpha_l, \infty)$.*

An important point of the above theorem is that the robust ISS can be achieved by using the state-dependent scaling and the Schur complements formula recursively. This feature is quite different from, for example, the development[10] where the Legendre-Fenchel transform and Young's Inequality are employed to prove ISS in the presence of the passive uncertainty. It is also interesting that the state-dependent scaling approach is able to construct an inverse optimal controller without referring to the Sontag-type controller[7].

According to Theorem 12, by letting $\alpha_l \to 0$, we can make the stability margin extremely large, which means the gain margin tend to $(0, \infty)$ and the phase margin tends to $90°$. However, we should be careful that the gain of the control law can be harmfully very high. To see this point, consider $\alpha_l \to 0$ in (116). Then, the scaling factor λ should be small enough, which implies that the feedback gain in (113) becomes very large. The large stability margin characterized by Theorem 11 is achievable when the state variable is available for feedback. For output-feedback control, it is generally known that the state-feedback/observer design reduces stability margins. It is possible to characterize the reduced margins in the case of the output-feedback by restricting the set of uncertain dynamics and uncertain parameters accordingly.

The introduction of the new type of scaling (102) is crucial for establishing the input-to-state stability in the presence of input unmodeled dynamics. If the scaling is replaced by the unbounded one (14), the ISS is not guaranteed in the presence of dynamic uncertainties. If the scaling is replaced by constant scaling, in general, the condition (115) cannot be met globally for nonlinear systems. Thus, the new scaling (102) and the creation of a new type of Lyapunov functions (106) from the scaling are important ingredients in this section.

7 Concluding Remarks

In this paper, the input-to-state stabilization and the integral input-to-state stabilization have been characterized by using the state-dependent scaling and diffeomorphism exclusively. The recursive design procedure presented is based on recursive application of the Schur complements formula to the characterization. This paper use neither Young's formula nor completing the squares which are usually conservative than the Schur complements formula[8]. All developments in this paper only use the state-dependent scaling, the diffeomorphism and the Schur complements, and combination of them has been found useful in dealing with ISS and iISS problems.

The paper has solved both state-feedback and output-feedback global stabilization. The systems are allowed to have uncertain parameters and dynamics. In the case of input unmodeled uncertainty, a new class of state-dependent scaling factors has been introduced to create Lyapunov functions of a new type in the state-dependent scaling design.

Theorem 1, 3, 5 and 7 of this paper can be considered as an improved version of the input-to-state stabilization results presented in [7, 8]. The key difference is that this paper does not introduce unnecessary fictitious output functions which was used as free parameters in [7, 8]. That is why the previous papers [7, 8] need to compute scaling matrices at the fictitious channels. For instance, the characterizing matrix N^{sf} of Theorem 1 does not have any fictitious output and scaling matrices, while the previous papers use larger matrices which include the fictitious output and scaling matrices in the augmented part of the characterizing matrix. By virtue of the Schur complements formula, it can be easily seen that the design in [7, 8] may require more effort of the controller than the method of this paper to make the characterizing matrix negative definite. Thus, in general, the method of [7, 8] tends to produce higher gain controllers. Finally, it may be worth mentioning that the characterization of this paper has more flexibility to deal with advanced problems such as robust ISS problems discussed in this paper.

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