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ALBANESE KERNEL OF THE PRODUCT OF CURVES OVER A p-ADIC FIELD

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Abstract

In this short note, we investigate the image of the Kummer map associated to an abelian variety over a p-adic field. As a byproduct, we give the structure of the Albanese kernel of the product of curves over a p-adic field under some assumptions. The result has already known by E. Gazaki [1], but the proof is completely different.

1. Introduction

In this note, we compute the Albanese kernel for the product of curves over k as a generalization of [10]. Precisely, we use the following notation: For $i = 1, 2$,

- X_i: a smooth projective curve over k with k-rational point $X_i(k) \neq \emptyset$, and
- $J_i := \mathrm{Jac}(X_i)$: the Jacobian variety associated to X_i of dimension g_i.

The kernel of the degree map $\deg : \mathrm{CH}_0(X_1 \times X_2) \to \mathbf{Z}$ is denoted by $A_0(X_1 \times X_2)$. The kernel $T(X_1 \times X_2)$ of the Albanese map

$$\mathrm{alb} : A_0(X_1 \times X_2) \to \mathrm{Alb}_{X_1 \times X_2}(k) = J_1(k) \oplus J_2(k)$$

which is called the **Albanese kernel** is also written by the Somekawa K-group associated to J_1 and J_2 as

$$T(X_1 \times X_2) \simeq K(k; J_1, J_2)$$

([7]). From the same computation as in [10], Theorem 4.1, we recover the following theorem which is proved in [1], Corollary 8.9:

Theorem 1.1. *Assume that the Jacobian varieties J_1 and J_2 satisfy*
(**Ord**) *J_i has good ordinary reduction, and*
(**Rat**) *$J_i[p^n] \subset J_i(k)$.*
Then, we have

$$T(X_1 \times X_2)/p^n \simeq (\mathbf{Z}/p^n)^{\oplus g_1 g_2}.$$

Note that the condition (**Rat**) implies that $\mu_p \subset k$ and hence the ramification index of k is $\geq p - 1$. On the contrary, even in the case where X_1 and X_2 are elliptic curves, it is known that $T(X_1 \times X_2)/p^n = 0$ for all n when k is unramified over $\mathbf{Q}_p$ ([3]).

Notation

Throughout this note, we use the following notation:

- k: a finite extension of $\mathbf{Q}_p$.

For a finite extension K/k, we define

- O_K: the valuation ring of K with maximal ideal $\mathfrak{m}_K$,
- $\mathbf{F}_K = O_K/\mathfrak{m}_K$: the residue field of K, and
- $U_K = O_K^{\times}$: the unit group.

For an abelian group G and $m \in \mathbf{Z}_{\geq 1}$, we write $G[m]$ and G/m for the kernel and cokernel of the multiplication by m on G respectively.

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2. Mackey functors

DEFINITION 2.1 (*cf.* [7], Sect. 3). A **Mackey functor** $\mathscr{M}$ (over k) (or a G_k**-modulation** in the sense of [6], Def. 1.5.10) is a contravariant functor from the category of étale schemes over k to the category of abelian groups equipped with a covariant structure for finite morphisms such that

$$\mathscr{M}(X_1 \sqcup X_2) = \mathscr{M}(X_1) \oplus \mathscr{M}(X_2)$$

and if

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram, then the induced diagram

$$\begin{array}{ccc} \mathscr{M}(X') & \xrightarrow{g'_*} & \mathscr{M}(X) \\ {\scriptstyle f'^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathscr{M}(Y') & \xrightarrow{g_*} & \mathscr{M}(Y) \end{array}$$

commutes.

For a Mackey functor $\mathscr{M}$, we denote by $\mathscr{M}(K)$ its value $\mathscr{M}(\mathrm{Spec}(K))$ for a field extension K of k. For any finite extensions $k \subset K \subset L$, the induced homomorphisms from the canonical map $j : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ are denoted by

$$N_{L/K} := j_* : \mathscr{M}(L) \to \mathscr{M}(K), \qquad \text{and} \qquad \mathrm{Res}_{L/K} := j^* : \mathscr{M}(K) \to \mathscr{M}(L).$$

The category of Mackey functors over k forms an abelian category with the following tensor product:

Definition 2.2 (*cf.* [4]). For Mackey functors $\mathscr{M}$ and $\mathscr{N}$, their **Mackey product** $\mathscr{M} \otimes \mathscr{N}$ is defined as follows: For any finite field extension k'/k,

$$(\mathscr{M} \otimes \mathscr{N})(k') := \left(\bigoplus_{K/k':\text{finite}} \mathscr{M}(K) \otimes_{\mathbf{Z}} \mathscr{N}(K) \right) \Big/ (\mathbf{PF}), \tag{1}$$

where $(\mathbf{PF})$ stands for the subgroup generated by elements of the following form:

$(\mathbf{PF})$ For finite field extensions $k' \subset K \subset L$,

$$N_{L/K}(x) \otimes y - x \otimes \mathrm{Res}_{L/K}(y) \qquad \text{for } x \in \mathscr{M}(L),\ y \in \mathscr{N}(K), \qquad \text{and}$$

$$x \otimes N_{L/K}(y) - \mathrm{Res}_{L/K}(x) \otimes y \qquad \text{for } x \in \mathscr{M}(K),\ y \in \mathscr{N}(L).$$

For the Mackey product $\mathscr{M} \otimes \mathscr{N}$, we denote by $\{x, y\}_{K/k'}$ the image of $x \otimes y \in \mathscr{M}(K) \otimes_{\mathbf{Z}} \mathscr{N}(K)$ in the product $(\mathscr{M} \otimes \mathscr{N})(k')$. For any finite field extension k'/k, the push-forward

$$N_{k'/k} = j_* : (\mathscr{M} \otimes \mathscr{N})(k') \to (\mathscr{M} \otimes \mathscr{N})(k) \tag{2}$$

is given by $N_{k'/k}(\{x, y\}_{K/k'}) = \{x, y\}_{K/k}$. For each $m \in \mathbf{Z}_{\geq 1}$, we define a Mackey functor $\mathscr{M}/m$ by

$$(\mathscr{M}/m)(K) := \mathscr{M}(K)/m \tag{3}$$

for any finite extension K/k. We have

$$(\mathscr{M}/m \otimes \mathscr{N}/m)(k) \simeq (\mathscr{M} \otimes \mathscr{N})(k)/m = ((\mathscr{M} \otimes \mathscr{N})/m)(k) \qquad (cf.\ (3)).$$

Every G_k-module M defines a Mackey functor defined by the fixed sub module $M(K) := M^{\mathrm{Gal}(\bar{k}/K)}$ which is also denoted by M. Conversely, assume a Mackey functor $\mathscr{M}$ satisfies **Galois descent**, meaning that, for every finite Galois extension L/K, the restriction

$$\mathrm{Res}_{L/K} : \mathscr{M}(K) \xrightarrow{\simeq} \mathscr{M}(L)^{\mathrm{Gal}(L/K)}$$

is an isomorphism. For any $m \in \mathbf{Z}_{\geq 1}$, the connecting homomorphism associated to the short exact sequence $0 \to \mathscr{M}[m] \to \mathscr{M} \xrightarrow{m} \mathscr{M} \to 0$ as G_k-modules gives

$$\delta_{\mathscr{M}} : \mathscr{M}(K)/m \hookrightarrow H^1(K, \mathscr{M}[m]) \tag{4}$$

which is often called the **Kummer map**.

Definition 2.3 (*cf.* [9], Prop. 1.5). For Mackey functors $\mathscr{M}$ and $\mathscr{N}$ with Galois descent, the **Galois symbol map**

$$s_m^M : (\mathscr{M} \otimes \mathscr{N})(k)/m \to H^2(k, \mathscr{M}[m] \otimes \mathscr{N}[m]) \tag{5}$$

is defined by the cup product and the corestriction as follows:

$$s_m^M(\{x, y\}_{K/k}) = \mathrm{Cor}_{K/k}(\delta_{\mathscr{M}}(x) \cup \delta_{\mathscr{N}}(y)).$$

3. Galois symbol map

Let A be an abelian variety of dimension g over k. We assume that
(**Ord**) A has good ordinary reduction, and
(**Rat**) $A[p^n] \subset A(k)$.
We denote by $\hat{A}$ the formal group over O_k of A. Let k^{ur} be the completion of the maximal unramified extension of k. It is known that we have $\hat{A} \times_{O_k} \mathrm{Spf}(O_{k^{\mathrm{ur}}}) \simeq (\hat{\mathbf{G}}_m)^{\oplus g}$, where $\hat{\mathbf{G}}_m$ is the multiplicative group ([5], Lem. 4.26, Lem. 4.27). Since we have $A[p^n] \subset A(k)$, $\hat{A}[p^n] \subset \hat{A}(k) =: \hat{A}(\mathfrak{m}_k)$ and hence we obtain isomorphisms

$$\hat{A}[p^n] = \hat{A}(k^{\mathrm{ur}})[p^n] \simeq ((\hat{\mathbf{G}}_m)(k^{\mathrm{ur}})[p^n])^{\oplus g} \simeq (\mu_{p^n})^{\oplus g}. \tag{6}$$

Now, we choose an isomorphism

$$A[p^n] \xrightarrow{\simeq} (\mu_{p^n})^{\oplus 2g} \tag{7}$$

of (trivial) Galois modules which makes the following diagram commutative:

$$\begin{tikzcd} \hat{A}[p^n] \arrow[r, hook] \arrow[d, "\simeq"] & A[p^n] \arrow[d, "\simeq"] \\ (\mu_{p^n})^{\oplus g} \arrow[r, hook, "(\mathrm{id}, 1)"] & (\mu_{p^n})^{\oplus g} \oplus (\mu_{p^n})^{\oplus g}, \end{tikzcd}$$

where the left vertical map is given in (6), and the bottom horizontal map is defined by

$$(\mu_{p^n})^{\oplus g} \to (\mu_{p^n})^{\oplus 2g}; \qquad (x_1, \dots, x_g) \mapsto (x_1, \dots, x_g, 1, \dots, 1).$$

By the same proof of [2], Prop. 3.1, one can determine the image of the Kummer map as follows:

Proposition 3.1. *For any finite extension K/k, the image of the Kummer map*

$$\delta_A : A(K)/p^n \to H^1(K, A[p^n]) \simeq H^1(K, \mu_{p^n}^{\oplus 2g}) \simeq (K^\times/p^n)^{\oplus 2g}$$

coincides with

$$(\overline{U}_K)^{\oplus g} \oplus \mathrm{Ker}(j : K^\times/p^n \to (K^{\mathrm{ur}})^\times/p^n)^{\oplus g},$$

where $\overline{U}_K$ is the image of $U_K = O_K^\times$ in $K^\times/p$, and j is the map induced from the inclusion $K^\times \hookrightarrow (K^{\mathrm{ur}})^\times$.

The above isomorphism is extended to the isomorphism

$$A/p^n \simeq \mathscr{U}^{\oplus g} \oplus \mathscr{V}^{\oplus g} \tag{8}$$

of Mackey functors, where $\mathscr{U}$ and $\mathscr{V}$ are the sub Mackey functors of $\mathbf{G}_m/p^n$ defined by

$$\mathscr{U}(K) := \mathrm{Im}(U_K \to K^\times/p^n) = \overline{U}_K, \qquad \text{and} \qquad \mathscr{V}(K) := \mathrm{Ker}(j : K^\times/p^n \to (K^{\mathrm{ur}})^\times/p^n),$$

for any finite extension K/k (*cf.* [2], Cor. 3.4).

4. Proof of Thm. 1.1

We show Thm. 1.1. From $K(k; J_1, J_2) \simeq T(X_1 \times X_2)$ (as noted in Introduction), it is enough to prove $K(k; J_1, J_2)/p^n \simeq (\mathbf{Z}/p^n)^{\oplus g_1 g_2}$. Applying (8) in the last section to J_i, we have $J_i/p^n \simeq (\mathscr{U} \oplus \mathscr{V})^{\oplus g_i}$ after fixing $J_i[p^n] \simeq (\mu_{p^n})^{\oplus 2g_i}$ as in (7). We have

$$J_1/p^n \otimes J_2/p^n \simeq ((\mathscr{U} \otimes \mathscr{U}) \oplus (\mathscr{U} \otimes \mathscr{V}) \oplus (\mathscr{V} \otimes \mathscr{U}) \oplus (\mathscr{V} \otimes \mathscr{V}))^{\oplus g_1 g_2}.$$

The Galois symbol maps give the following commutative diagram:

(9)
$$\begin{tikzcd}
(J_1/p^n \otimes J_2/p^n)(k) \arrow[r, two heads] \arrow[rr, bend left, "s^M_{p^n}"] \arrow[d, "\simeq"] & K(k; J_1, J_2)/p^n \arrow[r, hook, "s_{p^n}"] & H^2(k, J_1[p^n] \otimes J_2[p^n]) \arrow[d, "\simeq"] \\
\begin{array}{c} (\mathscr{U} \otimes \mathscr{U})(k)^{\oplus g_1 g_2} \\ \oplus \\ (\mathscr{U} \otimes \mathscr{V})(k)^{\oplus 2 g_1 g_2} \\ \oplus \\ (\mathscr{V} \otimes \mathscr{V})(k)^{\oplus g_1 g_2}. \end{array} \arrow[rr] & & H^2(k, \mu_{p^n}^{\otimes 2})^{\oplus 4 g_1 g_2}
\end{tikzcd}$$

Here, $s_{p^n} : K(k; J_1, J_2)/p^n \to H^2(k, J_1[p^n] \otimes J_2[p^n])$ is injective ([7], Rem. 4.5.8 (b)), and the bottom map is the direct sum of the three kind of maps given by the composing the Galois symbol map after the natural maps $(\mathscr{U} \otimes \mathscr{U})(k) \to (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k)$, $(\mathscr{U} \otimes \mathscr{V})(k) \to (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k)$, or $(\mathscr{V} \otimes \mathscr{V})(k) \to (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k)$. Precisely,

$$s_1 : (\mathscr{U} \otimes \mathscr{U})(k) \to (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k) \xrightarrow{s^M_{p^n}} H^2(k, \mu_{p^n}^{\otimes 2}),$$

$$s_2 : (\mathscr{U} \otimes \mathscr{V})(k) \to (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k) \xrightarrow{s^M_{p^n}} H^2(k, \mu_{p^n}^{\otimes 2}), \qquad \text{and}$$

$$s_3 : (\mathscr{V} \otimes \mathscr{V})(k) \to (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k) \xrightarrow{s^M_{p^n}} H^2(k, \mu_{p^n}^{\otimes 2}).$$

The image of the maps s_i are computed as follows:

LEMMA 4.1. (i) *The map* s_1 *is surjective.*
(ii) *The image of* s_2 *and* s_3 *are trivial.*

PROOF. (i) The Galois symbol map $s_{p^n}^M : (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k) \to H^2(k, \mu_{p^n}^{\otimes 2})$ is written by the Hilbert symbol ([8], Chap. XIV, Sect. 2, Prop. 5) as the following commutative diagram indicates:

$$\begin{tikzcd}
(\mathcal{U} \otimes \mathcal{U})(k) \arrow[r] & (\mathbf{G}_m/p^n \otimes \mathbf{G}_m/p^n)(k) \arrow[r, "s_{p^n}^M"] & H^2(k, \mu_{p^n}^{\otimes 2}) \\
\bigoplus_{K/k} \mathcal{U}(K) \otimes_{\mathbf{Z}} \mathcal{U}(K) \arrow[r] \arrow[u, two heads] \arrow[rrd, dotted, bend right] & \bigoplus_{K/k} K^\times/p^n \otimes_{\mathbf{Z}} K^\times/p^n \arrow[r, "\delta\cup\delta"] \arrow[u, two heads] \arrow[rd, bend right, "{(-,-)}"] & \bigoplus_{K/k} H^2(K, \mu_{p^n}^{\otimes 2}) \arrow[u, "\mathrm{Cor}_{K/k}"'] \\
 & & \bigoplus_{K/k} \mu_{p^n}, \arrow[u, "\simeq"']
\end{tikzcd}$$

where $(-,-) : K^\times \otimes_{\mathbf{Z}} K^\times \to \mu_{p^n}$ is the Hilbert symbol. For each finite extension K/k, the Hilbert symbol from $\mathcal{U}(K) \otimes_{\mathbf{Z}} \mathcal{U}(K)$ (the dotted arrow in the above diagram) is surjective ([10], Prop. 2.5) so is s_1.

(ii) By the same reasons as in (i), the image of $\mathcal{U}(K) \otimes_{\mathbf{Z}} \mathcal{V}(K)$ and $\mathcal{V}(K) \otimes_{\mathbf{Z}} \mathcal{V}(K)$ by the Hilbert symbol is trivial and hence $\mathrm{Im}(s_1) = \mathrm{Im}(s_2) = 0$ in $H^2(k, \mu_{p^n}^{\otimes 2})$. $\square$

Recall that we have $H^2(k, \mu_{p^n}^{\otimes 2}) \simeq \mathbf{Z}/p^n$. From the above diagram (9), the above lemma implies

$$s_{p^n}^M((J_1/p^n \otimes J_2/p^n)(k)) \simeq K(k; J_1, J_2)/p^n \simeq (\mathbf{Z}/p^n)^{\oplus g_1 g_2}.$$

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