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CONVERGING DECISION PROCESSES WITH MULTIPLICATIVE REWARD SYSTEM

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Abstract

Converging decision process is a decision process model with converging transition, which is one of the nonserial branch systems proposed by Nemhauser. This paper deals with multiplicative reward system on a finite-stage deterministic converging decision process. The purpose of this work is to give a recursive method to solve our model by bidecision approach.

1. Introduction

Nonserial dynamic programming was introduced by Nemhauser [15] and has been widely discussed [1, 2, 3]. Nonserial dynamic systems are classified into the four structures: diverging branch systems, converging branch systems, feedback loop systems, and feedforward loop systems. We have also studied nonserial dynamic programming, especially diverging branch systems and converging branch systems. Nondeterministic dynamic programming [5, 11] and mutually dependent decision processes [4, 6, 9] are a type of dynamic programming model with diverging branch systems. Recently, we discussed some converging decision process models [7, 8]. In this paper, we introduce a converging decision process model with multiplicative reward system [10, 14]. We consider our model in a framework of bidecision processes [12, 13] and derive bicursive formula which consists of two interrelated recursive equations. These equations enable to solve our model recursively.

2. Notation and formulation

We introduce a finite-stage converging decision process model with a multiplicative reward system.

1. X, a nonempty finite set, is the state space. The states in the process are expressed by $x_1, x_2, \ldots, x_N \in X$. The set of indexes that indicate the initial states is denoted by $I_{\mathrm{Init}} \subset \{1, 2, \ldots, N-1\}$ and x_i $(i \in I_{\mathrm{Init}})$ are specified at the beginning of the process. Moreover we define two index sets as follows:

$$\overline{I_{\mathrm{Init}}} = \{1, 2, \ldots, N\} \backslash I_{\mathrm{Init}}$$

and

$$\overline{I_{\mathrm{Init}}^*} = \{1, 2, \ldots, N-1\} \backslash I_{\mathrm{Init}}.$$

The process progresses through states x_i $(i \in \overline{I^*_{\mathrm{Init}}})$ according to converging branch system and is terminated at state x_N.

2. U, a nonempty finite set, is the decision space. Furthermore, we denote by U_n a point-to-set valued mapping from X to $2^U \backslash \{\phi\}$, where we denote the power set of U by 2^U. $U_n(x_n)$, called the feasible decision space, represents the set of all feasible decisions in state x_n. The selected decision for state x_n $(n = 1, 2, \dots, N-1)$ is represented by $u_n \in U_n(x_n)$ $(n = 1, 2, \dots, N-1)$.
3. The transition matrix $E = (e_{ij}) \in \{0, 1\}^{N \times N}$ is defined by

$$e_{ij} = \begin{cases} 1 & (\text{if } x_j \text{ is the next state to } x_i) \\ 0 & (\text{otherwise}), \end{cases}$$

and let $I_j = \{i \mid e_{ij} = 1\}$ $(j = L+1, L+2, \dots, N)$. We assume that the directed graph which represents the state transition does not contain a loop and that each state (node) has a unique path to the terminal state.

Let $Gr(U_n)$ be the graph of $U_n(\cdot)$:

$$Gr(U_n) = \{(x_n, u_n) \mid u_n \in U_n(x_n),\ x_n \in X\}.$$

When an index set $I = \{m_1, m_2, \dots, m_M\}$ $(m_1 < m_2 < \cdots < m_M)$ is given, the corresponding sequence

$$x_{m_1}, u_{m_1}, x_{m_2}, u_{m_2}, \dots, x_{m_M}, u_{m_M}$$

is denoted by $\langle x_m, u_m \rangle_{m \in I}$. Similarly

$$x_{m_1}, x_{m_2}, \dots, x_{m_M} \qquad \text{and} \qquad u_{m_1}, u_{m_2}, \dots, u_{m_M}$$

are denoted by $\langle x_m \rangle_{m \in I}$ and $\langle u_m \rangle_{m \in I}$, respectively.

4. $r_n : G_r(U_n) \to \mathbf{R}$ $(n = 1, 2, \dots, N-1)$ are the reward functions, where $\mathbf{R} = (-\infty, \infty)$. A decision u_n selected in state x_n confers a reward $r_n(x_n, u_n)$. The function $k : X \to \mathbf{R}$ is the terminal reward function.
5. The converging transition laws are given by

$$f_n : Gr(U_{m_1}) \times Gr(U_{m_2}) \times \cdots \times Gr(U_{m_M}) \to X \qquad (n \in \overline{I_{\mathrm{Init}}}),$$

where $I_n = \{m_1, m_2, \dots, m_M\}$. If a process in states $\langle x_m \rangle_{m \in I_n}$ selects actions $\langle u_m \rangle_{m \in I_n}$, it proceeds deterministically to the next state $f_n(\langle x_m, u_m \rangle_{m \in I_n})$.

Then our model is formulated as follows:

$$\begin{aligned} \text{(P)} \quad \text{Max} \quad & r_1(x_1, u_1) r_2(x_2, u_2) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N) \\ \text{s.t.} \quad & x_n = f_n(\langle x_m, u_m \rangle_{m \in I_n}) \qquad n \in \overline{I_{\mathrm{Init}}} \\ & u_n \in U_n(x_n) \qquad n = 1, 2, \dots, N-1. \end{aligned}$$

EXAMPLE 2.1. Let $N = 7$, $I_{\mathrm{Init}} = \{1, 2, 3, 4\}$, and $e_{15} = e_{26} = e_{36} = e_{47} = e_{57} = e_{67} = 1$ ($e_{ij} = 0$ for the other pairs (i, j)). Then, for the given initial states x_1, x_2, x_3, x_4, the

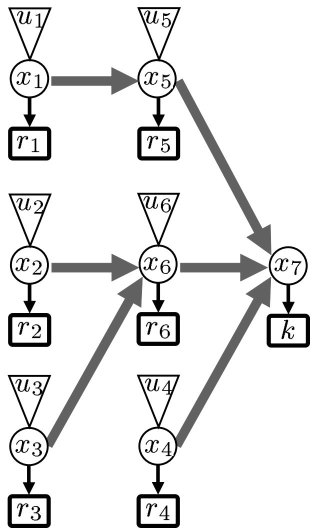

Figure 1. State transition tree for Example 2.1

other states x_5, x_6, x_7 are determined by

$$x_5 = f_5(x_1, u_1), \qquad x_1 \in X,\ u_1 \in U_1(x_1)$$

$$x_6 = f_6(x_2, u_2, x_3, u_3), \qquad x_2, x_3 \in X,\ u_2 \in U_2(x_2),\ u_3 \in U_3(x_3)$$

$$x_7 = f_7(x_4, u_4, x_5, u_5, x_6, u_6), \qquad x_4, x_5, x_6 \in X,\ u_4 \in U_4(x_4),\ u_5 \in U_5(x_5),\ u_6 \in U_6(x_6)$$

(see Figure 1). In this case,

$$I_5 = \{1\}, \qquad I_6 = \{2, 3\}, \qquad I_7 = \{3, 5, 6\}$$

and the problem is described as follows:

$$\begin{array}{rl} \text{Max} & r_1(x_1, u_1) r_2(x_2, u_2) r_3(x_3, u_3) r_4(x_4, u_4) r_5(x_5, u_5) r_6(x_6, u_6) k(x_7) \\ \text{s.t.} & x_n = f_n(\langle x_m, u_m \rangle_{m \in I_n}) \qquad n = 5, 6, 7 \\ & u_n \in U_n(x_n) \qquad n = 1, 2, 3, 4, 5, 6. \end{array}$$ □

3. Bidecision approach

3.1. Bidecision processes

On bidecision processes, both the family of maximum subproblems and the family of minimum subproblems are considered. In constructing the subproblems, we start with the initial target state sequence $Q = (x_N)$. Then, we add x_n to Q in the order that coincides one of the node order for a depth-first search for a state transition tree with the root x_N. Without loss of generality, we regard that index order as

$$N \to N-1 \to N-2 \to \cdots \to 2 \to 1,$$

by renumbering the state index.

EXAMPLE 3.1. When we consider the decision process whose state transition tree is given by Figure 2, the sequence of the target state sequences becomes

$$(x_6) \to (x_5, x_6) \to (x_4, x_5, x_6) \to \cdots \to (x_1, x_2, x_3, x_4, x_5, x_6).$$

Then, we define the corresponding maximum subproblems and minimum subproblems as follows:

$$\begin{aligned}
v^6(x_6) &= k(x_6) \\
v^5(x_5; x_2, u_2) &= \max_{u_5 \in U_5(x_5)} [r_5(x_5, u_5)k(x_6)] \\
v^4(x_4; x_2, u_2, x_3, u_3) &= \max_{u_m \in U_m(x_m)\ (4 \le m \le 5)} [r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
v^3(x_3; x_2, u_2) &= \max_{u_m \in U_m(x_m)\ (3 \le m \le 5)} [r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
v^2(x_2) &= \max_{u_m \in U_m(x_m)\ (2 \le m \le 5)} [r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
v^1(x_1) &= \max_{u_m \in U_m(x_m)\ (1 \le m \le 5)} [r_1(x_1, u_1)r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
w^6(x_6) &= k(x_6) \\
w^5(x_5; x_2, u_2) &= \min_{u_5 \in U_5(x_5)} [r_5(x_5, u_5)k(x_6)] \\
w^4(x_4; x_2, u_2, x_3, u_3) &= \min_{u_m \in U_m(x_m)\ (4 \le m \le 5)} [r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
w^3(x_3; x_2, u_2) &= \min_{u_m \in U_m(x_m)\ (3 \le m \le 5)} [r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
w^2(x_2) &= \min_{u_m \in U_m(x_m)\ (2 \le m \le 5)} [r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\
w^1(x_1) &= \min_{u_m \in U_m(x_m)\ (1 \le m \le 5)} [r_1(x_1, u_1)r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)],
\end{aligned}$$

where v^n and w^n $(n = 1, 2, \ldots, 6)$ are the optimal value functions for maximum subproblems and minimum subproblems, respectively. □

We give the general form of subproblems.

$\underline{n = N}$

Both of the bidecision subproblems corresponding to the state sequence (x_N) formed of the final state x_N are given by

$$v^N(x_N) = w^N(x_N) = k(x_N), \qquad x_N \in X.$$

$\underline{1 \leq n < N}$

The bidecision subproblems corresponding to the state sequence $(x_n, x_{n+1}, \ldots, x_N)$ with the initial state x_m $(m \in \{v \mid v \geq n,\ I_v \cap \{n, n+1, \ldots, N\} = \phi\})$ are given by

$$\begin{aligned} & v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ & \quad = \max_{u_m \in U_m(x_m)\ (m=n,n+1,\ldots,N-1)} [r_n(x_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots k(x_N)], \\ & \qquad\qquad\qquad x_n \in X,\ x_m \in X,\ u_m \in U(x_m)\ \ (m \in J_n) \end{aligned}$$

and

$$\begin{aligned} & w^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ & \quad = \min_{u_m \in U_m(x_m)\ (m=n,n+1,\ldots,N-1)} [r_n(x_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots k(x_N)], \\ & \qquad\qquad\qquad x_n \in X,\ x_m \in X,\ u_m \in U(x_m)\ \ (m \in J_n), \end{aligned}$$

where

$$J_n = \bigcup_{l=n+1}^{N} \{j \in I_l \mid j < n\}.$$

The following proposition gives a method to get the sets J_n recursively.

Proposition 3.1. *Put* $J_N = \phi$, *then,*

(i) *if* $n + 1 \in \overline{I_{\mathrm{Init}}}$,

$$J_n = J_{n+1} \cup \{j \in I_{n+1} \mid j < n\}.$$

(ii) *if* $n + 1 \in I_{\mathrm{Init}} \setminus \{1\}$,

$$J_n = J_{n+1} \setminus \{n\}.$$

For each $n = 1, 2, \ldots, N-1$, we divide each feasible decision set $U_n(x_n)$, $x_n \in X$ into two disjoint subsets:

$$U_n^+(x_n) = \{u \in U_n(x_n) \mid r_n(x_n, u) \geq 0\}, \qquad U_n^-(x_n) = \{u \in U_n(x_n) \mid r_n(x_n, u) < 0\}$$

that satisfy

$$U_n^+(x_n) \cap U_n^-(x_n) = \phi, \qquad U_n^+(x_n) \cup U_n^-(x_n) = U_n(x_n).$$

Then we have the bicursive formula (system of two recursive equations) for the both subproblems:

THEOREM 3.1.

$$(1) \qquad v^N(x_N) = k(x_N), \qquad x_N \in X$$

$$(2) \qquad v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n})$$
$$= \max_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) v^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})]$$
$$\vee \max_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) w^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})],$$
$$x_n \in X,\ n+1 \in \overline{I_{\mathrm{Init}}}$$

$$(3) \qquad v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n})$$
$$= \max_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) v^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})]$$
$$\vee \max_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) w^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})],$$
$$x_n \in X,\ n+1 \in I_{\mathrm{Init}} \backslash \{1\}.$$

$$(4) \qquad w^N(x_N) = k(x_N), \qquad x_N \in X$$

$$(5) \qquad w^n(x_n; \langle x_m, u_m \rangle_{m \in J_n})$$
$$= \min_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) w^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})]$$
$$\wedge \min_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) v^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})],$$
$$x_n \in X,\ n+1 \in \overline{I_{\mathrm{Init}}}$$

$$(6) \qquad w^n(x_n; \langle x_m, u_m \rangle_{m \in J_n})$$
$$= \min_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) w^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})]$$
$$\wedge \min_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) v^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})],$$
$$x_n \in X, n+1 \in I_{\mathrm{Init}} \backslash \{1\},$$

where $\vee(\wedge)$ *is the maximum (minimum) operator:*

$$a \vee b = \max(a, b) \;\; (a \wedge b = \min(a, b)) \qquad a, b \in \mathbf{R}.$$

Proof. We consider the nth maximum subproblem with the given current state $\bar{x}_n$ and parameters $\langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}$. We note that $m < n$ for $^\forall m \in J_n$. Here, we show the first equation (2). (This statement also means that we suppose $n + 1 \notin I_{\mathrm{Init}}$.) By definition of the subproblems,

$$\begin{aligned} & v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\ & \quad = \max_{u_m \in U_m(x_m) \ (m=n, n+1, \ldots, N-1)} [r_n(\bar{x}_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)]. \end{aligned}$$

From the existence of the maximum value, there exists an optimal decision sequence $u_n^*, u_{n+1}^*, \ldots, u_{N-1}^*$ satisfying

$$v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) = r_n(\bar{x}_n, u_n^*) r_{n+1}(x_{n+1}^*, u_{n+1}^*) \cdots r_{N-1}(x_{N-1}^*, u_{N-1}^*) k(x_N^*).$$

The corresponding state sequence $x_{n+1}^*, \ldots, x_{N-1}^*, x_N^*$ is sequentially determined by

$$x_l^* = \begin{cases} \text{(the given initial state)} & (l \in I_{\mathrm{Init}}) \\ f_l(\langle x_m^*, u_m^* \rangle_{m \in I_l}) & (l \notin I_{\mathrm{Init}}), \end{cases}$$

where

$$x_m^* = \begin{cases} \bar{x}_m & (m \le n) \\ x_m^* & (m > n) \end{cases}, \qquad u_m^* = \begin{cases} \bar{u}_m & (m < n) \\ u_m^* & (m \ge n). \end{cases}$$

If $r_n(\bar{x}_n, u_n^*) \ge 0$, then

$$\begin{aligned} & v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\ & \quad = r_n(\bar{x}_n, u_n^*) \times [r_{n+1}(x_{n+1}^*, u_{n+1}^*) \cdots r_{N-1}(x_{N-1}^*, u_{N-1}^*) k(x_N^*)] \\ & \quad \le r_n(\bar{x}_n, u_n^*) \max_{u_m \in U_m(x_m) \ (m=n+1, \ldots, N-1)} [r_{n+1}(x_{n+1}^*, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)] \\ & \quad = [r_n(\bar{x}_n, u_n^*) v^{n+1}(x_{n+1}^*; \langle x_m^*, u_m^* \rangle_{m \in J_{n+1}})] \\ & \quad = [r_n(\bar{x}_n, u_n^*) v^{n+1}(f_{n+1}(\langle x_m^*, u_m^* \rangle_{m \in I_{n+1}}); \langle x_m^*, u_m^* \rangle_{m \in J_{n+1}})] \\ & \quad = [r_n(\bar{x}_n, u_n^*) v^{n+1}(f_{n+1}(\bar{x}_n, u_n^*, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\ & \quad \le \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})]. \end{aligned}$$

Similarly, if $r_n(\bar{x}_n, u_n^*) < 0$, then

$$
\begin{aligned}
& v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\
& \quad \le r_n(\bar{x}_n, u_n^*) \min_{\substack{u_m \in U_m(x_m) \\ (m=n+1,\dots,N-1)}} [r_{n+1}(x_{n+1}^*, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)] \\
& \quad = [r_n(\bar{x}_n, u_n^*) w^{n+1}(x_{n+1}^*; \langle x_m^*, u_m^* \rangle_{m \in J_{n+1}})] \\
& \quad \le \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(7) \qquad & v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\
& \quad \le \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
& \qquad \vee \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})].
\end{aligned}
$$

On the other hand, there exist optimal decisions $u_n^+ \in U_n^+(\bar{x}_n)$, $u_n^- \in U_n^-(\bar{x}_n)$ satisfying

$$
\begin{aligned}
& \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
& \quad = r_n(\bar{x}_n, u_n^+) v^{n+1}(f_{n+1}(\bar{x}_n, u_n^+, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})
\end{aligned}
$$

and

$$
\begin{aligned}
& \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
& \quad = r_n(\bar{x}_n, u_n^-) w^{n+1}(f_{n+1}(\bar{x}_n, u_n^-, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}),
\end{aligned}
$$

respectively. Let

$$x_{n+1}^+ = f_{n+1}(\bar{x}_n, u_n^+, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}})$$

and

$$x_{n+1}^- = f_{n+1}(\bar{x}_n, u_n^-, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}).$$

Then there exist two optimal decision sequences $u_{n+1}^+, \dots, u_{N-1}^+$ and $u_{n+1}^-, \dots, u_{N-1}^-$ that meet the following equations:

$$
\begin{aligned}
& v^{n+1}(f_{n+1}(\bar{x}_n, u_n^+, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\
& \quad = v^{n+1}(x_{n+1}^+; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\
& \quad = r_{n+1}(x_{n+1}^+, u_{n+1}^+) r_{n+2}(x_{n+2}^+, u_{n+2}^+) \cdots r_{N-1}(x_{N-1}^+, u_{N-1}^+) k(x_N^+)
\end{aligned}
$$

and

$$\begin{aligned} & w^{n+1}(f_{n+1}(\bar{x}_n, u_n^-, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\ & \quad = w^{n+1}(x_{n+1}^-; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\ & \quad = r_{n+1}(x_{n+1}^-, u_{n+1}^-) r_{n+2}(x_{n+2}^-, u_{n+2}^-) \cdots r_{N-1}(x_{N-1}^-, u_{N-1}^-) k(x_N^-), \end{aligned}$$

respectively. The corresponding state sequences $x_{n+2}^+, \ldots, x_{N-1}^+, x_N^+$ and $x_{n+2}^-, \ldots, x_{N-1}^-$, x_N^- is sequentially determined by the following manner.

(i) For x_l^+ $(l = n+2, n+3, \ldots, N)$,

$$x_l^+ = \begin{cases} \text{(the given initial state)} & (l \in I_{\text{Init}}) \\ f_l(\langle x_m^+, u_m^+ \rangle_{m \in I_l}) & (l \notin I_{\text{Init}}), \end{cases}$$

where

$$x_m^+ = \begin{cases} \bar{x}_m & (m < n+1) \\ x_m^+ & (m \geq n+1) \end{cases}, \qquad u_m^+ = \begin{cases} \bar{u}_m & (m < n) \\ u_m^+ & (m \geq n) \end{cases}.$$

(ii) For x_l^- $(l = n+2, n+3, \ldots, N)$,

$$x_l^- = \begin{cases} \text{(the given initial state)} & (l \in I_{\text{Init}}) \\ f_l(\langle x_m^-, u_m^- \rangle_{m \in I_l}) & (l \notin I_{\text{Init}}), \end{cases}$$

where

$$x_m^- = \begin{cases} \bar{x}_m & (m < n+1) \\ x_m^- & (m \geq n+1) \end{cases}, \qquad u_m^- = \begin{cases} \bar{u}_m & (m < n) \\ u_m^- & (m \geq n) \end{cases}.$$

Hence

$$\begin{aligned} (8) \quad & \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\ & \quad \vee \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\ & = [r_n(\bar{x}_n, u_n^+) r_{n+1}(x_{n+1}^+, u_{n+1}^+) r_{n+2}(x_{n+2}^+, u_{n+2}^+) \cdots r_{N-1}(x_{N-1}^+, u_{N-1}^+) k(x_N^+)] \\ & \quad \vee [r_n(\bar{x}_n, u_n^-) r_{n+1}(x_{n+1}^-, u_{n+1}^-) r_{n+2}(x_{n+2}^-, u_{n+2}^-) \cdots r_{N-1}(x_{N-1}^-, u_{N-1}^-) k(x_N^-)] \\ & \leq \max_{\substack{u_m \in U_m(x_m) \\ (m=n, n+1, \ldots, N-1)}} [r_n(\bar{x}_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)] \\ & = v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}). \end{aligned}$$

From Eqs. (7) and (8), we have Eq. (2).

The other recursive equations can be shown in a similar way. □

In case that non-negativity of all reward functions can be assumed, the following recursive equations which are the same type as those for additive reward system hold.

Corollary 3.1.

$$v^N(x_N) = k(x_N), \qquad x_N \in X$$

$$v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n})$$
$$= \max_{u_n \in U_n(x_n)} [r_n(x_n, u_n) v^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})]$$
$$x_n \in X,\ n+1 \in \overline{I_{\mathrm{Init}}}$$

$$v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n})$$
$$= \max_{u_n \in U_n(x_n)} [r_n(x_n, u_n) v^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})]$$
$$x_n \in X,\ n+1 \in I_{\mathrm{Init}} \backslash \{1\}.$$

Proof. Since $U_n^+(x_n) = U_n(x_n)$ and $U_n^-(x_n) = \phi$, it is clear from Theorem 3.1. □

4. Example

Example 4.1. We consider the converging decision process shown in Figure 2. By Proposition 3.1, we can get J_j $(j = 6, 5, \ldots, 1)$ as follows:

$$J_6 = \phi.$$

Since, for $n = 5$, $n + 1 = 6 \in \overline{I_{\mathrm{Init}}}$,

$$J_5 = J_6 \cup \{j \in I_6 \mid j < 5\} = \phi \cup \{2\} = \{2\}.$$

Similarly,

$$n = 4 : n + 1 = 5 \notin I_{\mathrm{Init}}$$
$$\Longrightarrow \quad J_4 = J_5 \cup \{j \in I_5 \mid j < 4\} = \{2\} \cup \{3\} = \{2, 3\},$$
$$n = 3 : n + 1 = 4 \in I_{\mathrm{Init}}$$
$$\Longrightarrow \quad J_3 = J_4 \backslash \{3\} = \{2, 3\} \backslash \{3\} = \{2\},$$
$$n = 2 : n + 1 = 3 \in I_{\mathrm{Init}}$$
$$\Longrightarrow \quad J_2 = J_3 \backslash \{2\} = \{2\} \backslash \{2\} = \phi,$$

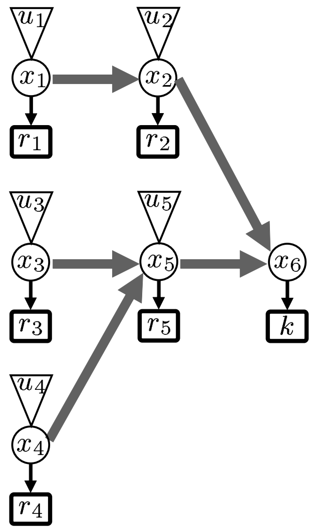

Figure 2. State transition tree for Example 4.1 and 4.2

$$n = 1 : n + 1 = 2 \notin I_{\text{Init}}$$
$$\Longrightarrow \quad J_1 = J_2 \cup \{ j \in I_2 \mid j < 1 \} = \phi \cup \phi = \phi.$$

Then, we can get the bicursive formula by Theorem 3.1. First, let $N = 6$, we have

$$v^6(x_6) = w^6(x_6) = k(x_6).$$

Next, from $6(= n + 1) \in \overline{I_{\text{Init}}}$, the system of recursive equations for $n = 5$ is

$$\begin{aligned} v^5(x_5; \langle x_m, u_m \rangle_{m \in J_5}) = & \max_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) v^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})] \\ & \vee \max_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) w^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})] \\ w^5(x_5; \langle x_m, u_m \rangle_{m \in J_5}) = & \min_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) w^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})] \\ & \wedge \min_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) v^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})]. \end{aligned}$$

Since $J_5 = \{2\}$, $J_6 = \phi$, $I_6 = \{2, 5\}$, the above equations become

$$\begin{aligned} v^5(x_5; x_2, u_2) = & \max_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) v^6(f_6(x_2, u_2, x_5, u_5))] \\ & \vee \max_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) w^6(f_6(x_2, u_2, x_5, u_5))] \end{aligned}$$

and

$$w^5(x_5; x_2, u_2) = \min_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) w^6(f_6(x_2, u_2, x_5, u_5))]$$
$$\wedge \min_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) v^6(f_6(x_2, u_2, x_5, u_5))],$$

respectively.

Similarly, we have

$$v^4(x_4; x_2, u_2, x_3, u_3) = \max_{u_4 \in U_4^+(x_4)} [r_4(x_4, u_4) v^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)]$$
$$\vee \max_{u_4 \in U_4^-(x_4)} [r_4(x_4, u_4) w^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)]$$
$$w^4(x_4; x_2, u_2, x_3, u_3) = \min_{u_4 \in U_4^+(x_4)} [r_4(x_4, u_4) w^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)]$$
$$\wedge \min_{u_4 \in U_4^-(x_4)} [r_4(x_4, u_4) v^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)]$$
$$v^3(x_3; x_2, u_2) = \max_{u_3 \in U_3^+(x_3)} [r_3(x_3, u_3) v^4(x_4; x_2, u_2, x_3, u_3)]$$
$$\vee \max_{u_3 \in U_3^-(x_3)} [r_3(x_3, u_3) w^4(x_4; x_2, u_2, x_3, u_3)]$$
$$w^3(x_3; x_2, u_2) = \min_{u_3 \in U_3^+(x_3)} [r_3(x_3, u_3) w^4(x_4; x_2, u_2, x_3, u_3)]$$
$$\wedge \min_{u_3 \in U_3^-(x_3)} [r_3(x_3, u_3) v^4(x_4; x_2, u_2, x_3, u_3)]$$
$$v^2(x_2) = \max_{u_2 \in U_2^+(x_2)} [r_2(x_2, u_2) v^3(x_3; x_2, u_2)] \vee \max_{u_2 \in U_2^-(x_2)} [r_2(x_2, u_2) w^3(x_3; x_2, u_2)]$$
$$w^2(x_2) = \min_{u_2 \in U_2^+(x_2)} [r_2(x_2, u_2) w^3(x_3; x_2, u_2)] \wedge \min_{u_2 \in U_2^-(x_2)} [r_2(x_2, u_2) v^3(x_3; x_2, u_2)]$$

and

$$v^1(x_1) = \max_{u_1 \in U_1^+(x_1)} [r_1(x_1, u_1) v^2(f_2(x_1, u_1))] \vee \max_{u_1 \in U_1^-(x_1)} [r_1(x_1, u_1) w^2(f_2(x_1, u_1))]$$
$$w^1(x_1) = \min_{u_1 \in U_1^+(x_1)} [r_1(x_1, u_1) w^2(f_2(x_1, u_1))] \wedge \min_{u_1 \in U_1^-(x_1)} [r_1(x_1, u_1) v^2(f_2(x_1, u_1))]. \qquad \square$$

Example 4.2. Let $N = 6$, $I_{\text{Init}} = \{1, 3, 4\}$. We consider the converging decision process problem shown in Figure 2 with the following data:

$$X = \{s_1, s_2, s_3\}, \qquad U_n(x) = U = \{a_1, a_2\} \quad ({}^{\forall}n \in \{1,2,3,4,5\},\ {}^{\forall}x \in X)$$

$$x_1 = s_1, \qquad x_3 = s_2, \qquad x_4 = s_1$$

$\underline{f_2(x,u)}$

$x \backslash u$	a_1	a_2
s_1	s_3	s_2

$\underline{f_5(x,u,y,v)}$

$(x,u)\backslash(y,v)$	(s_1,a_1)	(s_1,a_2)
(s_2,a_1)	s_2	s_3
(s_2,a_2)	s_1	s_2

$\underline{f_6(x,u,y,v)}$

$(x,u)\backslash(y,v)$	(s_1,a_1)	(s_1,a_2)	(s_2,a_1)	(s_2,a_2)	(s_3,a_1)	(s_3,a_2)
(s_2,a_1)	s_3	s_2	s_2	s_3	s_1	s_2
(s_2,a_2)	s_1	s_3	s_3	s_1	s_2	s_3
(s_3,a_1)	s_1	s_2	s_2	s_1	s_3	s_1
(s_3,a_2)	s_3	s_1	s_3	s_2	s_2	s_2

and

(x,u)	(s_1,a_1)	(s_1,a_2)	(s_2,a_1)	(s_2,a_2)	(s_3,a_1)	(s_3,a_2)
$r_1(x,u)$	2	-1				
$r_2(x,u)$			-1	1	2	-1
$r_3(x,u)$			3	-2		
$r_4(x,u)$	-1	2				
$r_5(x,u)$	-1	1	-2	1	2	1

x	s_1	s_2	s_3
$k(x)$	2	-2	-1

.

(In the above tables, unnecessary values for the given initial states are omitted.)

We have already obtained the bicursive formula in Example 4.1. Hereafter, the calculation will proceed based on them.

First, for the terminal state x_6, we get

$$v^6(x_6) = w^6(x_6) = k(x_6) = \begin{cases} 2 & (x_6 = s_1) \\ -2 & (x_6 = s_2). \\ -1 & (x_6 = s_3) \end{cases}$$

Then, we calculate $v^5(x_5; x_2, u_2)$, $w^5(x_5; x_2, u_2)$ and the corresponding optimal decision functions $\bar{\pi}_5^*(x_5; x_2, u_2)$, $\underline{\pi}_5^*(x_5; x_2, u_2)$. When $x_5 = s_1$ and $(x_2, u_2) = (s_2, a_1)$, we have

$$\begin{aligned} v^5(s_1; s_2, a_1) &= \max_{u_5 \in \{a_2\}} [r_5(s_1, u_5) v^6(f_6(s_2, a_1, s_1, u_5))] \\ &\quad \vee \max_{u_5 \in \{a_1\}} [r_5(s_1, u_5) w^6(f_6(s_2, a_1, s_1, u_5))] \\ &= [r_5(s_1, a_2) v^6(f_6(s_2, a_1, s_1, a_2))] \vee [r_5(s_1, a_1) w^6(f_6(s_2, a_1, s_1, a_1))] \end{aligned}$$

$$
\begin{aligned}
&= [1 \times v^6(s_2)] \vee [-1 \times w^6(s_3)] = [1 \times (-2)] \vee [-1 \times (-1)] \\
&= 1 \\
\bar{\pi}_5^*(s_1; s_2, a_1) &= a_1
\end{aligned}
$$

and

$$
\begin{aligned}
w^5(s_1; s_2, a_1) &= \max_{u_5 \in \{a_2\}} [r_5(s_1, u_5) w^6(f_6(s_2, a_1, s_1, u_5))] \\
&\quad \wedge \max_{u_5 \in \{a_1\}} [r_5(s_1, u_5) v^6(f_6(s_2, a_1, s_1, u_5))] \\
&= [r_5(s_1, a_2) w^6(f_6(s_2, a_1, s_1, a_2))] \wedge [r_5(s_1, a_1) v^6(f_6(s_2, a_1, s_1, a_1))] \\
&= [1 \times w^6(s_2)] \wedge [-1 \times v^6(s_3)] = [1 \times (-2)] \wedge [-1 \times (-1)] \\
&= -2 \\
\underline{\pi}_5^*(s_1; s_2, a_1) &= a_2.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
v^5(s_1; s_2, a_2) &= [r_5(s_1, a_2) v^6(s_3)] \vee [r_5(s_1, a_1) w^6(s_1)] = [1 \times (-1)] \vee [-1 \times 2] \\
&= -1, \\
\bar{\pi}_5^*(s_1; s_2, a_2) &= a_2 \\
w^5(s_1; s_2, a_2) &= [r_5(s_1, a_2) w^6(s_3)] \wedge [r_5(s_1, a_1) v^6(s_1)] = [1 \times (-1)] \wedge [-1 \times 2] \\
&= -2, \\
\underline{\pi}_5^*(s_1; s_2, a_2) &= a_1 \\
v^5(s_1; s_3, a_1) &= [r_5(s_1, a_2) v^6(s_2)] \vee [r_5(s_1, a_1) w^6(s_1)] = [1 \times (-2)] \vee [-1 \times 2] \\
&= -2, \\
\bar{\pi}_5^*(s_1; s_3, a_1) &= a_1, a_2 \\
w^5(s_1; s_3, a_1) &= [r_5(s_1, a_2) w^6(s_2)] \wedge [r_5(s_1, a_1) v^6(s_1)] = [1 \times (-2)] \wedge [-1 \times 2] \\
&= -2, \\
\underline{\pi}_5^*(s_1; s_3, a_1) &= a_1, a_2 \\
v^5(s_1; s_3, a_2) &= [r_5(s_1, a_2) v^6(s_1)] \vee [r_5(s_1, a_1) w^6(s_3)] = [1 \times 2] \vee [-1 \times (-1)] \\
&= 2,
\end{aligned}
$$

$$\bar{\pi}_5^*(s_1; s_3, a_2) = a_2$$

$$w^5(s_1; s_3, a_2) = [r_5(s_1, a_2)w^6(s_1)] \wedge [r_5(s_1, a_1)v^6(s_3)] = [1 \times 2] \wedge [-1 \times (-1)]$$
$$= 1,$$

$$\underline{\pi}_5^*(s_1; s_3, a_2) = a_1.$$

Other results required for $x_5 = s_2, s_3$ are

(x_2, u_2)	(s_2, a_1)	(s_2, a_2)	(s_3, a_1)	(s_3, a_2)
$v^5(s_2; x_2, u_2)$	4	2	4	2
$\bar{\pi}_5^*(s_2; x_2, u_2)$	a_1	a_1, a_2	a_1	a_1
$w^5(s_2; x_2, u_2)$	-1	2	2	-2
$\underline{\pi}_5^*(s_2; x_2, u_2)$	a_2	a_1, a_2	a_2	a_2
$v^5(s_3; x_2, u_2)$	4	-1	2	-2
$\bar{\pi}_5^*(s_3; x_2, u_2)$	a_1	a_2	a_2	a_2
$w^5(s_3; x_2, u_2)$	-2	-4	-2	-4
$\underline{\pi}_5^*(s_3; x_2, u_2)$	a_2	a_1	a_1	a_1

(The results for $x_2 = s_1$ are omitted as they are unnecessary for this example. The same applies below.)

Next, we calculate $v^4(x_4; x_2, u_2, x_3, u_3)$, $w^4(x_4; x_2, u_2, x_3, u_3)$ and the corresponding optimal decision functions $\bar{\pi}_4^*(x_4; x_2, u_2, x_3, u_3)$, $\underline{\pi}_4^*(x_4; x_2, u_2, x_3, u_3)$ for the given initial states $x_4 = s_1$ and $x_3 = s_2$. When $(x_2, u_2, u_3) = (s_2, a_1, a_1)$, we have

$$\begin{aligned} v^4(s_1; s_2, a_1, s_2, a_1) &= \max_{u_4 \in \{a_2\}} [r_4(s_1, u_4)v^5(f_5(s_2, a_1, s_1, u_4); s_2, a_1)] \\ &\quad \vee \max_{u_4 \in \{a_1\}} [r_4(s_1, u_4)w^5(f_5(s_2, a_1, s_1, u_4); s_2, a_1)] \\ &= [r_4(s_1, a_2)v^5(f_5(s_2, a_1, s_1, a_2); s_2, a_1)] \\ &\quad \vee [r_4(s_1, a_1)w^5(f_5(s_2, a_1, s_1, a_1); s_2, a_1)] \\ &= [2 \times v^5(s_3; s_2, a_1)] \vee [-1 \times w^5(s_2; s_2, a_1)] \\ &= [2 \times 4] \vee [-1 \times (-1)] = 8 \end{aligned}$$

$$\bar{\pi}_4^*(s_1; s_2, a_1, s_2, a_1) = a_2.$$

Through similar calculations, we get values of $v^4(s_1; x_2, u_2, s_2, u_3)$ and $w^4(s_1; x_2, u_2, s_2, u_3)$ as follows:

(x_2, u_2)	(s_2, a_1)	(s_2, a_2)	(s_3, a_1)	(s_3, a_2)
$v^4(s_1; x_2, u_2, s_2, a_1)$	8	-2	4	2
$\bar{\pi}_4^*(s_1; x_2, u_2, s_2, a_1)$	a_2	a_1, a_2	a_2	a_1
$w^4(s_1; x_2, u_2, s_2, a_1)$	-4	-8	-4	-8
$\underline{\pi}_4^*(s_1; x_2, u_2, s_2, a_1)$	a_1, a_2	a_2	a_1, a_2	a_2
$v^4(s_1; x_2, u_2, s_2, a_2)$	8	4	8	4
$\bar{\pi}_4^*(s_1; x_2, u_2, s_2, a_2)$	a_2	a_2	a_2	a_2
$w^4(s_1; x_2, u_2, s_2, a_2)$	-2	1	2	-4
$\underline{\pi}_4^*(s_1; x_2, u_2, s_2, a_2)$	a_2	a_1	a_1	a_2

Moreover, the remaining results are given by the following:

(x_2, u_2)	(s_2, a_1)	(s_2, a_2)	(s_3, a_1)	(s_3, a_2)
$v^3(s_2; x_2, u_2)$	24	-2	12	8
$\bar{\pi}_3^*(s_2; x_2, u_2)$	a_1	a_2	a_1	a_2
$w^3(s_2; x_2, u_2)$	-16	-24	-16	-24
$\underline{\pi}_3^*(s_2; x_2, u_2)$	a_2	a_1	a_2	a_1

$$
\begin{aligned}
v^2(s_2) &= 16, & \bar{\pi}_2^*(s_2) &= a_1 \\
w^2(s_2) &= -24, & \underline{\pi}_2^*(s_2) &= a_1, a_2 \\
v^2(s_3) &= 24, & \bar{\pi}_2^*(s_3) &= a_1, a_2 \\
w^2(s_3) &= -32, & \underline{\pi}_2^*(s_3) &= a_1 \\
v^1(s_1) &= 48, & \bar{\pi}_2^*(s_1) &= a_1 \\
(w^1(s_1) &= -64, & \underline{\pi}_2^*(s_1) &= a_1).
\end{aligned}
$$

Thus, the optimal value is $v^1(s_1) = 48$ and the optimal state-decision sequences are obtained as follows.

First, set $x_1 = s_1$ $(1 \in I_{\text{Init}})$, then

$$x_1 = s_1 \quad \longrightarrow \quad u_1^* = \bar{\pi}_2^*(s_1) = a_1$$

$$\longrightarrow \quad x_2 = f_2(s_1, a_1) = s_3, \qquad r_1(s_1, a_1) = 2 \geq 0 \quad \longrightarrow \quad u_2^* = \bar{\pi}_2^*(s_3) = a_1, a_2.$$

(i) If we take $u_2^* = a_1$,

$$\longrightarrow \quad x_3 = s_2 \ \ (3 \in I_{\text{Init}}), \qquad r_2(s_3, a_1) = 2 \geq 0 \quad \longrightarrow \quad u_3^* = \bar{\pi}_3^*(s_2; s_3, a_1) = a_1$$

$$\longrightarrow \quad x_4 = s_1 \ \ (4 \in I_{\text{Init}}), \qquad r_3(s_2, a_1) = 3 \geq 0 \quad \longrightarrow \quad u_4^* = \bar{\pi}_4^*(s_1; s_3, a_1, s_2, a_1) = a_2$$

$$\longrightarrow \quad x_5 = f_5(s_2, a_1, s_1, a_2) = s_3, \qquad r_4(s_1, a_2) = 2 \geq 0 \quad \longrightarrow \quad u_5^* = \bar{\pi}_5^*(s_3; s_3, a_1) = a_2$$

$$\longrightarrow \quad x_6 = f_6(s_3, a_1, s_3, a_2) = s_1.$$

(ii) If we take $u_2^* = a_2$,

$$\longrightarrow \quad x_3 = s_2 \ \ (3 \in I_{\text{Init}}), \qquad r_2(s_3, a_2) = -1 < 0 \quad \longrightarrow \quad u_3^* = \underline{\pi}_3^*(s_2; s_3, a_2) = a_1$$

$$\longrightarrow \quad x_4 = s_1 \ \ (4 \in I_{\text{Init}}), \qquad r_3(s_2, a_1) = 3 \geq 0 \quad \longrightarrow \quad u_4^* = \underline{\pi}_4^*(s_1; s_3, a_2, s_2, a_1) = a_2$$

$$\longrightarrow \quad x_5 = f_5(s_2, a_1, s_1, a_2) = s_3, \qquad r_4(s_1, a_2) = 2 \geq 0 \quad \longrightarrow \quad u_5^* = \underline{\pi}_5^*(s_3; s_3, a_2) = a_1$$

$$\longrightarrow \quad x_6 = f_6(s_3, a_2, s_3, a_1) = s_2.$$

Finally, we have two optimal decision sequences:

$$(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (a_1, a_1, a_1, a_2, a_2), (a_1, a_2, a_1, a_2, a_1),$$

which attain the optimal value. □

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