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Reduced-Order Dynamic and Static Partial-State Feedback Control for Robust Global Stabilization via State-Dependent Scaling Design[1][2]

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Abstract: This paper focuses on global robust stabilization of a class of nonlinear systems by partial-state feedback. Both reduced-order dynamic and static feedback are investigated. These control problems are solved by using the pair of state-dependent(SD) scaling and diffeomorphism with which dynamic and static partial-state feedback for nominal and robust global stabilization against memoryless and dynamic uncertainty of various structures can be handled in a unified way. The solutions are obtained as backstepping procedures which are amenable to numerical computation and optimization. Analytical formulas of solutions are also available.

Key Words: robust backstepping; state-dependent scaling; global robust stability; partial-state feedback; input-to-state stability; matrix inequality; computation-oriented design

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1 Introduction

This paper considers global stabilization of uncertain nonlinear systems in the so-called strict-feedback form(Krstić *et al.*, 1995; Freeman and Kokotović, 1996). New design procedures for reduced-order dynamic and static partial-state feedback laws are developed by using the concept of the state-dependent(SD) scaling and diffeomorphism. Recently it has been shown that the SD scaling provides a systematic and unified method of robust backstepping design for state-feedback control(Ito and Freeman, 1999*b*; Ito and Freeman, 1999*a*). The methodology of SD scaling design is applicable not only to strict feedback systems, but also to other unspecific classes of nonlinear systems(Ito, 1998*b*). The diffeomorphism between two state coordinates is a way to represent a non-quadratic Lyapunov function $V(x) = x^T P(x)x$ with $P(x) > 0$. Indeed, Choleskey factorization of P^{-1} indicates that V is quadratic in another coordinate(Ito, 1999). Compared with the parameter-dependent Lyapunov representation(Wu *et al.*, 1996; Apkarian and Adams, 1998) and the NLMI approach(Lu and Doyle, 1995), the diffeomorphism is more useful for guaranteeing global stabilization of nonlinear systems(Freeman and Kokotović, 1996; Ito and Freeman, 1999*b*).

There are many papers dealing with semi-global stabilization of nonlinear systems whose state variables are not completely available for feedback. The technique of input saturation and high-gain observer (Esfandiari and Khalil, 1992; Khalil and Esfandiari, 1993; Lin and Saberi, 1995). has been successful for semi-global stabilization. The studies (Teel and Praly, 1995; Teel and Praly, 1994) provide a useful semi-global backstepping lemma for high-gain partial-state feedback and high-gain observers with saturating control for dynamic output feedback. However, given an uncertain system, semi-global stabilization using high-gain and saturation would be meaningful only if the system cannot be globally stabilized. From this viewpoint, this paper seeks global stabilization in stead of settling for semi-global stabilization. As for global results of partial-state feedback stabilization of nonlinear systems in the strict-feedback form, typical results are applicable only to nonlinear systems whose nonlinearities depend only on the measured state, e.g. (Krstić *et al.*, 1995). In (Freeman and Kokotović, 1996) it is shown that global stability can be achievable by partial-state feedback if nonlinearities in the unmeasured part depend linearly on the unmeasured states and if the unmeasured part of the system is stable. Explicit discussion about robust stabilization via these types of control is absent. No result of reduced-order dynamic controllers is available for *partial-state* feedback *robust* control design of nonlinear systems although reduced-order controllers are well investigated for robust linear systems control. The only result relevant to reduced-order observers is found in (Kanellakopoulos, 1991) for output feedback in adaptive control.

The standpoint of this paper is similar to a common one in the linear robust control literature. This paper first proposes a method of solving or tackling design problems regardless of the existence of solutions. Since the problem may inherently have no solutions, a condition for the existence is derived. For robust stabilization, the condition is nothing but the allowable size of uncertainty. The paper also shows the class of systems for which solutions are always exist. This latter position is rather common in the nonlinear control literature.

This paper successfully extends the author's state-dependent scaling design for state-feedback backstepping to the partial-state feedback case. This paper proposes partial-state feedback controllers whose dynamics is introduced only for unmeasured states. This contrasts with output feedback control with full-order observers, e.g.(Krstić *et al.*, 1995; Ito and Krstić, 1999). This paper presents the state coordinates on which the diffeomorphism and SD scaling should be defined for the partial-state feedback problem. Such an appropriate pair of diffeomorphism and SD scaling makes the recursive procedure feasible and guarantees the existence of solutions. The paper derives a condition of "nonlinear size " of uncertainty under which global robust stabilization can be achieved. The difference between abilities of full-order observer and reduced-order observer feedback control is clearly described. The difference does

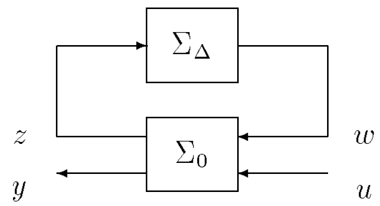

Figure 1: Uncertain nonlinear plant Σ_P

not appear in nominal stabilization. This paper also investigate a static partial-state feedback problem. It is shown that an uncertain systems can be globally robustly stabilized by static partial-state feedback unless backstepping is required for unmeasured part of the system. The robustness considered throughout this paper is practically desirable and useful in that the size, nonlinearity and location of uncertainties are prescribed *a priori*, which is completely different from the inverse optimal type of robustness(Freeman and Kokotović, 1996; Sepulchre *et al.*, 1997; Krstić and Li, 1998).

2 Uncertain plants with partial-state measurement

Consider the uncertain nonlinear system Σ_P shown in Fig.1. Here, Σ_0 denotes a nominal plant and Σ_Δ represents the uncertain part of the control system. We assume that the nominal part Σ_0 is described by

$$\Sigma_0 : \begin{cases} \dot{x} = A(y)x + B(y)w + G(y)u & x(t)\in\mathcal{R}^n, u(t)\in\mathcal{R} \\ z = C(y)x & , w(t)\in\mathcal{R}^p, z(t)\in\mathcal{R}^p \\ y = C_y x & y(t)\in\mathcal{R}^r \end{cases} \tag{1}$$

The matrix-valued functions A, B, C and G are assumed to be $\mathbf{C}^0$ functions. The vectors w and z are defined as

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad \begin{array}{c} w_i(t) \in \mathcal{R}^{p_i} \\ z_i(t) \in \mathcal{R}^{p_i} \\ p_i \geq 0, \quad p = \sum_{i=1}^n p_i \\ i = 1, 2, \cdots, n \end{array} \tag{2}$$

It is assumed that A and G are written in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots\cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots \ \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots\cdots & & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots\cdots & & a_{n,n} \end{bmatrix}, G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix}. \tag{3}$$

with $\mathbf{C}^0$ scalar functions a_{ij} satisfying

$$\begin{array}{l} a_{ij}(x) = a_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \leq i \leq n, \ 1 \leq j \leq i+1 \\ a_{i,i+1}(x_1, x_2, \cdots, x_i) \neq 0, \quad 1 \leq i \leq n, \quad \forall x \in \mathcal{R}^n \end{array} \tag{4}$$

As for functions B and C, we assume

$$B = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_{n,n} \end{bmatrix}, \qquad C = \begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ C_{n,1} & \cdots & C_{n,n-1} & C_{n,n} \end{bmatrix} \tag{5}$$

where $B_{ii} \in \mathcal{R}^{1\times p_i}$ and $C_{ij} \in \mathcal{R}^{p_i\times 1}$ satisfy

$$B_{ii}(x)=B_{ij}(x_1,x_2,\cdots,x_i),\ C_{ij}(x)=C_{ij}(x_1,x_2,\cdots,x_i) \tag{6}$$

for $1 \leq i \leq n$ and $1 \leq j \leq i$. In regard to the uncertain part of the system Σ_P, we suppose that Σ_Δ has the following structure of nonlinear mappings $\Delta : z \mapsto w$.

$$\Delta = \text{block-diag}[\Delta_1, \Delta_2, \cdots, \Delta_n], \tag{7}$$

where some of the mappings $\Delta_i : z_i \mapsto w_i$, $i = 1, 2, \ldots, n$ can be zero in vector size. Each uncertainty Δ_i is allowed to have three types of components:

$$\Delta_i : z_i = \begin{bmatrix} z_{id} \\ z_{is} \\ z_{ir} \end{bmatrix} \mapsto w_i = \begin{bmatrix} w_{id} \\ w_{is} \\ w_{ir} \end{bmatrix}, \quad w_i = \begin{bmatrix} \Delta_{id} & 0 & 0 \\ 0 & \Delta_{is} & 0 \\ 0 & 0 & \Delta_{ir} \end{bmatrix} z_i\ . \tag{8}$$

Here, Δ_{id} represents a dynamic system. Δ_{is} and Δ_{ir} denote full static and repeated static scalar systems, respectively. It is unnecessary for Δ_i to have all types of uncertainty. The dynamic uncertainty Δ_{id} is defined by

$$\Delta_{id}\ : \begin{cases} \dot{x}_{\Delta_i} = f_{\Delta_{id}}(x_{\Delta_i}, z_{id}, t) \\ w_{id} = h_{\Delta_{id}}(x_{\Delta_i}, z_{id}, t) \end{cases}, \tag{9}$$

where $f_{\Delta_{id}}(0,0,t) = 0$ and $h_{\Delta_{id}}(0,0,t) = 0$ are satisfied for all $t \geq 0$ and $f_{\Delta_{id}}$ and $h_{\Delta_{id}}$ are vector-valued $\mathbf{C}^0$ functions. The full static part Δ_{is} is described by

$$\Delta_{is}\ : w_{is} = h_{\Delta_{is}}(z_{is}, t), \tag{10}$$

where $h_{\Delta_{is}}$ is a vector-valued $\mathbf{C^0}$ function and $h_{\Delta_{is}}(0,t) = 0$ for all $t \geq 0$. The repeated static part Δ_{ir} is defined with $r_i > 1$ copies of a static scalar system δ_{ir} :

$$\Delta_{ir} = \operatorname*{diag}_{j=1}^{r_i} \delta_{ir} = \delta_{ir} I_{r_i}, \qquad \delta_{ir} :\ w_{\delta_{ir}} = h_{\delta_{ir}}(t) z_{\delta_{ir}} \tag{11}$$

where $h_{\delta_{ir}}$ is a scalar-valued $\mathbf{C}^0$ function. For notational simplicity, we assume that Δ_{id} and Δ_{is} are square in size of input and output vectors. We consider the following class of uncertainty Σ_Δ.

Definition 1 *The uncertainty Σ_Δ is said to be admissible if* **(i)**-**(iii)** *are satisfied for $i = 1, 2, \ldots, n$:* **(i)** *Δ_{id} has $\mathcal{L}_2$-gain less than or equal to 1 with a radially unbounded $\mathbf{C}^1$ storage function $V_{\Delta i}(x_{\Delta_i})$ satisfying $V_{\Delta i}(0) = 0$.* **(ii)** *Δ_{is} satisfies $\|z_{\Delta_{is}}\|^2 \geq \|w_{\Delta_{is}}\|^2$ for all $t \in [0, \infty)$.* **(iii)** *δ_{ir} satisfies $\|z_{\delta_{ir}}\|^2 \geq \|w_{\delta_{ir}}\|^2$ for all $t \in [0, \infty)$.*

The uncertain system Σ_P has an equilibrium point at the origin when $u \equiv 0$. The uncertainty affects the system as

$$B(x)w = \begin{bmatrix} B_{11}\Delta_1 C_{11} & 0 & 0 & \cdots \\ B_{22}\Delta_2 C_{21} & B_{22}\Delta_2 C_{22} & 0 & \ddots \\ \vdots & \vdots & \ddots & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \tag{12}$$

The operator Δ_i does not represent matrix multiplication but nonlinear mappings which can have dynamics with initial conditions. Note that B_{ii} and C_{ij} are a row vector and a column vector, respectively. This implies that the two nonlinear uncertainties $B_{22}\Delta_2 C_{21}$ and $B_{22}\Delta_2 C_{22}$ can be completely independent of each other. It is possible to extend the materials of this paper easily to the uncertain system which has Δ blocks in a more general manner as in Ito and Freeman (1999*b*). The system Σ_0 not only describes a nominal plant, but also can include information about nonlinearities of uncertainty. Indeed, Σ_0 specifies how the uncertainty affects the nominal plant such as geometrical locations, structures of uncertainties where uncertain parameters are present. The matrices $B(x)$ and $C(x)$ specify the "nonlinear size"(including size, nonlinearity, location and structure) of uncertainties.

This paper considers feedback control with two types of partial-state measurement for the uncertain systems Σ_P shown in Fig.1.

Uncertain plant Σ_{P1} : The state x of the nominal part Σ_0 is decomposed into

$$x = \begin{bmatrix} x_M \\ x_N \end{bmatrix} \tag{13}$$

It is assumed that only the upper part $x_M \in \mathcal{R}^m$ $(1 \leq m < n)$ is measured. The matrices A, B, C and G are independent of x_N. The system Σ_0 in (1) is described as

$$\Sigma_{01}: \begin{cases} \begin{bmatrix} \dot{x}_M \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A_M(x_M) & A_{MN} \\ A_{NM}(x_M) & A_N \end{bmatrix} \begin{bmatrix} x_M \\ x_N \end{bmatrix} + \begin{bmatrix} B_M(x_M) & 0 \\ 0 & B_N(x_M) \end{bmatrix} \begin{bmatrix} w_M \\ w_N \end{bmatrix} + \begin{bmatrix} 0 \\ G_N(x_M) \end{bmatrix} u \\ \begin{bmatrix} z_M \\ z_N \end{bmatrix} = \begin{bmatrix} C_M(x_M) & 0 \\ C_{NM}(x_M) & C_N(x_M) \end{bmatrix} \begin{bmatrix} x_M \\ x_N \end{bmatrix} \\ y = [\, I_m \ \ 0 \,] \begin{bmatrix} x_M \\ x_N \end{bmatrix}, \quad w_N, z_N \in \mathcal{R}^{q_m}, \quad q_m = \sum_{i=1}^m p_i \end{cases} \tag{14}$$

where A_{MN} and A_N are assumed to be constant.

Uncertain plant Σ_{P2} : The state x of the nominal part Σ_0 is decomposed into (13). Only the lower part $x_N \in \mathcal{R}^{n-m}$ is measured. The matrices A, B, G and C are independent of x_M. The system Σ_0 in (1) is described as

$$\Sigma_{02}: \begin{cases} \begin{bmatrix} \dot{x}_M \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A_M(x_{m+1}) & A_{MN}(x_{m+1}) \\ A_{NM}(x_N) & A_N(x_N) \end{bmatrix} \begin{bmatrix} x_M \\ x_N \end{bmatrix} + \begin{bmatrix} B_M(x_{m+1}) & 0 \\ 0 & B_N(x_N) \end{bmatrix} \begin{bmatrix} w_M \\ w_N \end{bmatrix} + \begin{bmatrix} 0 \\ G_N(x_N) \end{bmatrix} u \\ \begin{bmatrix} z_M \\ z_N \end{bmatrix} = \begin{bmatrix} C_M(x_{m+1}) & 0 \\ C_{NM}(x_N) & C_N(x_N) \end{bmatrix} \begin{bmatrix} x_M \\ x_N \end{bmatrix} \\ y = [\, 0 \ \ I_{n-m} \,] \begin{bmatrix} x_M \\ x_N \end{bmatrix}, \quad w_N, z_N \in \mathcal{R}^{q_m}, \quad q_m = \sum_{i=1}^m p_i \end{cases} \tag{15}$$

where A_M, A_{MN}, B_M and C_M are allowed to depend only on x_{m+1}.

The case of $x_M = x_1$ in Σ_{P1} is sometime called the output feedback problem in the literature(Krstić *et al.*, 1995). Note that A_M, A_{MN}, B_M and C_M of Σ_{02} do not satisfy (4) and (6). But, other parts of Σ_{02} are assumed to satisfy (4) and (6).

3 SD scaling and diffeomorphism for reduced-order dynamic feedback

Sections from 3 through 5 deal with the uncertain system Σ_{P1}. Consider the following dynamic feedback for Σ_{P1}.

$$\dot{\xi} = (A_{NM}(x_M) + A_N Y - H A_{MN} Y - Y A_M(x_M)) x_M + (A_N - Y A_{MN})\xi + G_N(x_M) u \tag{16}$$

$$u = [\, K_M(x_M) + K_N(x_M, \hat{x}_N) Y \ \ K_N(x_M, \hat{x}_N) \,] \begin{bmatrix} x_M \\ \xi \end{bmatrix} \tag{17}$$

The matrix Y is constant and is in the form of

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & Y_m \\ 0 & \cdots & 0 & \end{array}\right], \quad Y \in \mathcal{R}^{(n-m)\times m}, \quad Y_m \in \mathcal{R}^{(n-m)\times 1} \tag{18}$$

which is consistent with the zero entries of A_{MN}. The order of the dynamic controller is $n-m$ which is less than n of dynamic controllers based on full-order state observers(Krstić *et al.*, 1995; Ito and Krstić, 1999). The dynamics (16) reduces to the reduced-order observer for linear systems when matrices are independent of x_M. The closed-loop system with the reduced-order dynamic feedback is described by

$$\begin{bmatrix} \dot{x}_M \\ \dot{x}_N \\ \dot{\xi} \end{bmatrix} = A_{cl} \begin{bmatrix} x_M \\ x_N \\ \xi \end{bmatrix} + \begin{bmatrix} B_M & 0 \\ 0 & B_N \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_M \\ w_N \end{bmatrix} \tag{19}$$

Now, we choose the diffeomorphism between $(x_M, x_N, \xi) \in \mathcal{R}^n$ and the new coordinate $(\chi_M, \hat{\chi}_N, \hat{\xi}) \in \mathcal{R}^n$ as

$$\begin{bmatrix} \chi_M \\ \hat{\chi}_N \\ \hat{\xi} \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{S} & 0 \\ & & 0 \\ \hline 0 & 0 & W \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ Y & 0 & I \\ Y & -I & I \end{bmatrix} \begin{bmatrix} x_M \\ x_N \\ \xi \end{bmatrix} \tag{20}$$

where W is a non-singular constant matrix. The matrix-valued function $S(x_M, \hat{x}_N)$ defines a diffeomorphism between $(x_M, \hat{x}_N) \in \mathcal{R}^n$ and $(\chi_M, \hat{\chi}_N) \in \mathcal{R}^n$ as follows:

$$\begin{bmatrix} \chi_M \\ \hat{\chi}_N \end{bmatrix} = S(x_M, \hat{x}_N) \begin{bmatrix} x_M \\ \hat{x}_N \end{bmatrix}, \quad S(x_M, \hat{x}_N) = \begin{bmatrix} S_M(x_M) & 0 \\ S_{NM}(x_M, \hat{x}_N) & S_N(x_M, \hat{x}_N) \end{bmatrix} \tag{21}$$

where $\hat{x}_N = \xi + Y x_M$. The time-derivative of $(\chi_M, \hat{\chi}_N)$ is

$$\begin{aligned} \dot{\hat{\chi}} &= \left[\frac{\partial S}{\partial x_1} \begin{bmatrix} x_M \\ \hat{x}_N \end{bmatrix}, \ldots, \frac{\partial S}{\partial x_m} \begin{bmatrix} x_M \\ \hat{x}_N \end{bmatrix}, \frac{\partial S}{\partial \hat{x}_{m+1}} \begin{bmatrix} x_M \\ \hat{x}_N \end{bmatrix}, \ldots, \frac{\partial S}{\partial \hat{x}_n} \begin{bmatrix} x_M \\ \hat{x}_N \end{bmatrix} \right] \begin{bmatrix} \dot{x}_M \\ \dot{\hat{x}}_N \end{bmatrix} + S(x_M, \hat{x}_N) \begin{bmatrix} \dot{x}_M \\ \dot{\hat{x}}_N \end{bmatrix} \\ &= T(x_M, \hat{x}_N) \begin{bmatrix} \dot{x}_M \\ \dot{\hat{x}}_N \end{bmatrix} \\ T(x_M, \hat{x}_N) &= \begin{bmatrix} T_M(x_M) & 0 \\ T_{NM}(x_M, \hat{x}_N) & T_N(x_M, \hat{x}_N) \end{bmatrix} \end{aligned} \tag{22}$$

Hence, we obtain

$$\begin{bmatrix} \dot{\chi}_M \\ \dot{\hat{\chi}}_N \\ \dot{\hat{\xi}} \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{T} & 0 \\ & & 0 \\ \hline 0 & 0 & W \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ Y & 0 & I \\ Y & -I & I \end{bmatrix} \begin{bmatrix} \dot{x}_M \\ \dot{x}_N \\ \dot{\xi} \end{bmatrix}, \quad \begin{bmatrix} x_M \\ x_N \\ \xi \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -I \\ -Y & I & 0 \end{bmatrix} \left[\begin{array}{cc|c} \multicolumn{2}{c|}{S^{-1}} & 0 \\ & & 0 \\ \hline 0 & 0 & W^{-1} \end{array}\right] \begin{bmatrix} \chi_M \\ \hat{\chi}_N \\ \hat{\xi} \end{bmatrix} \tag{23}$$

By using the new coordinate, the closed-loop system is expressed as

$$\begin{aligned} \begin{bmatrix} \dot{\chi}_M \\ \dot{\hat{\chi}}_N \\ \dot{\hat{\xi}} \end{bmatrix} &= \left[\begin{array}{cc|c} \multicolumn{2}{c|}{T} & 0 \\ & & 0 \\ \hline 0 & 0 & W \end{array}\right] \left(\begin{bmatrix} A_M & A_{MN} & -A_{MN} \\ A_{NM} + G_N K_M & A_N + G_N K_N & -Y A_{MN} \\ 0 & 0 & A_N - Y A_{MN} \end{bmatrix} \left[\begin{array}{cc|c} \multicolumn{2}{c|}{S^{-1}} & 0 \\ & & 0 \\ \hline 0 & 0 & W^{-1} \end{array}\right] \begin{bmatrix} \chi_M \\ \hat{\chi}_N \\ \hat{\xi} \end{bmatrix} \right. \\ &\qquad \left. + \begin{bmatrix} B_M & 0 \\ Y B_M & 0 \\ Y B_M & -B_N \end{bmatrix} \begin{bmatrix} w_M \\ w_N \end{bmatrix} \right) \\ &= \begin{bmatrix} T(A+GK)S^{-1} & \begin{matrix} -T_M A_{MN} W^{-1} \\ -(T_{NM} + T_N Y) A_{MN} W^{-1} \end{matrix} \\ 0 & W(A_N - Y A_{MN}) W^{-1} \end{bmatrix} \begin{bmatrix} \chi_M \\ \hat{\chi}_N \\ \hat{\xi} \end{bmatrix} + \begin{bmatrix} T_M B_M & 0 \\ (T_{NM} + T_N Y) B_M & 0 \\ W Y B_M & -W B_N \end{bmatrix} \begin{bmatrix} w_M \\ w_N \end{bmatrix} \end{aligned} \tag{24}$$

$$\begin{bmatrix} z_M \\ z_N \end{bmatrix} = \begin{bmatrix} C_M & 0 \\ C_{NM} & C_N \end{bmatrix} \left[\begin{array}{c|c} S^{-1} & \begin{matrix} 0 \\ -W^{-1} \end{matrix} \end{array}\right] \begin{bmatrix} \chi_M \\ \hat{\chi}_N \\ \hat{\xi} \end{bmatrix} = \left[\begin{array}{c|c} C S^{-1} & \begin{matrix} 0 \\ -C_N W^{-1} \end{matrix} \end{array}\right] \begin{bmatrix} \chi_M \\ \hat{\chi}_N \\ \hat{\xi} \end{bmatrix} \tag{25}$$

Let K denote the feedback gain for the transformed states $(\chi_M, \hat{\chi}_N)$ as

$$K = [\, K_M \;\; K_N \,] \tag{26}$$

Next, sets of SD scaling matrices are defined for the uncertainty Σ_Δ according to (Ito, 1998*a*; Ito and Freeman, 1999*b*). For the dynamic uncertainty Δ_{id}, we define

$$\mathbf{L}_{id} := \{L_{id} = \lambda_{id} I_{id} : \lambda_{id} > 0\}. \tag{27}$$

Here, I_{id} denotes an identity matrix which is compatible in size with the vector z_{id}. For the full static uncertainty Δ_{is},

$$\mathbf{L}_{is} := \{L_{is} = \lambda_{is}(x_M, \hat{x}_N) I_{is} : \lambda_{is}(x_M, \hat{x}_N) > 0 \ \forall (x_M, \hat{x}_N) \in \mathcal{R}^m \times \mathcal{R}^{n-m}\}. \tag{28}$$

is defined. In the case of the repeated static uncertainty Δ_{ir}, we define

$$\mathbf{L}_{ir} := \{L_{ir} : L_{ir}^T(x_M, \hat{x}_N) = L_{ir}(x_M, \hat{x}_N), \ L_{ir}(x_M, \hat{x}_N) > 0 \ \forall (x_M, \hat{x}_N) \in \mathcal{R}^m \times \mathcal{R}^{n-m}\}. \tag{29}$$

$$\mathbf{R}_{ir} := \{R_{ir} : R_{ir}^T(x_M, \hat{x}_N) = -R_{ir}(x_M, \hat{x}_N) \ \forall (x_M, \hat{x}_N) \in \mathcal{R}^m \times \mathcal{R}^{n-m}\}. \tag{30}$$

Here, both L_{ir} and R_{ir} are square matrices whose size is the same as the dimension of z_{ir}. Scaling matrices for the whole Σ_Δ are given by

$$\mathbf{L} := \left\{ L = \operatorname*{block-diag}_{i=1}^{n} L_i(x_M, \hat{x}_N), \quad L_i \in \mathbf{L}_i \right\} \tag{31}$$

$$\mathbf{R} := \left\{ R = \operatorname*{block-diag}_{i=1}^{n} R_i(x_M, \hat{x}_N), \quad R_i \in \mathbf{R}_i \right\} \tag{32}$$

$$\mathbf{L}_i := \left\{ L_i(x_M, \hat{x}_N) = \begin{bmatrix} L_{id} & 0 & 0 \\ 0 & L_{is}(x_M, , \hat{x}_N) & 0 \\ 0 & 0 & L_{ir}(x_M, , \hat{x}_N) \end{bmatrix} : \begin{matrix} L_{id} \in \mathbf{L}_{id} \\ L_{is} \in \mathbf{L}_{is} \\ L_{ir} \in \mathbf{L}_{ir} \end{matrix} \right\} \tag{33}$$

$$\mathbf{R}_i := \left\{ R_i(x_M, , \hat{x}_N) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_{ir}(x_M, \hat{x}_N) \end{bmatrix} : R_{ir} \in \mathbf{R}_{ir} \right\}, \tag{34}$$

Note that a constant $\lambda > 0$ satisfies $\lambda I \in \mathbf{L}_i$ and $0 \in \mathbf{R}_i$. Because $B_{ii}\Delta_j C_{ji}$ is scalar, a repeated static uncertainty can always be represented by a scalar full static uncertainty. However, we include the repeated representation here because it allows more degrees of freedom in the scaling design.

By using the diffeomorphism (20) and the SD scaling (31-32), we can prove the following.

Theorem 1 *(i) Suppose that there exist constant matrices $P = P^T > 0$ and $\tilde{P} = \tilde{P}^T > 0$ such that*

$$N(x_M, \hat{x}_N) = \begin{bmatrix} S^{-T}(A+GK)^T T^T P + PT(A+GK)S^{-1} & -P \begin{bmatrix} T_M \\ T_{NM} + T_N Y \end{bmatrix} A_{MN} W^{-1} \\ -W^{-T} A_{MN}^T \begin{bmatrix} T_M^T & T_{NM}^T + Y^T T_N^T \end{bmatrix} P & W^{-T}(A_N - YA_{MN})^T W^T \tilde{P} + \tilde{P} W (A_N - YA_{MN}) W^{-1} \end{bmatrix} < 0 \tag{35}$$

is satisfied for all $(x_M, \hat{x}_N)$ in $\mathcal{R}^n$, then the nominal nonlinear system Σ_{01} is globally uniformly asymptotically stabilized by the reduced-order dynamic feedback (16-17). Furthermore, a Lyapunov function is given by

$$V(x, \xi) = \begin{bmatrix} \chi_M \\ \hat{\chi}_N \end{bmatrix}^T P \begin{bmatrix} \chi_M \\ \hat{\chi}_N \end{bmatrix} + \hat{\xi}^T \tilde{P} \hat{\xi}$$

(ii) Suppose that there exist constant matrices $P = P^T > 0$, $\tilde{P} = \tilde{P}^T > 0$ and scaling functions $L \in \mathbf{L}$ and $R \in \mathbf{R}$ such that

$$
\begin{aligned}
&M(x_M, \hat{x}_N) = \\
&\begin{bmatrix}
\left\{ \begin{array}{c} S^{-T}(A+GK)^T T^T P + \\ PT(A+GK)S^{-1} \end{array} \right\} & P \begin{bmatrix} T_M B_M & 0 \\ (T_{NM}+T_N Y)B_M & 0 \end{bmatrix} + S^{-T}C^T R^T & S^{-T}C^T L & -P \begin{bmatrix} T_M \\ T_{NM}+T_N Y \end{bmatrix} A_{MN} W^{-1} \\
* & -L & 0 & \begin{bmatrix} B_M^T Y^T \\ -B_N^T \end{bmatrix} W^T \tilde{P} - R \begin{bmatrix} 0 \\ C_N \end{bmatrix} W^{-1} \\
* & * & -L & -L \begin{bmatrix} 0 \\ C_N \end{bmatrix} W^{-1} \\
* & * & * & \left\{ \begin{array}{c} W^{-T}(A_N - YA_{MN})^T W^T \tilde{P} + \\ \tilde{P} W (A_N - YA_{MN}) W^{-1} \end{array} \right\}
\end{bmatrix} \\
&< 0
\end{aligned}
\tag{36}
$$

is satisfied for all $(x_M, \hat{x}_N)$ in $\mathcal{R}^n$, then the uncertain nonlinear system Σ_{P1} is globally uniformly asymptotically stabilized by reduced-order dynamic feedback (16-17) for any admissible uncertainty Σ_Δ. Furthermore, a Lyapunov function is given by

$$V(x, \xi) = \begin{bmatrix} \chi_M \\ \hat{\chi}_N \end{bmatrix}^T P \begin{bmatrix} \chi_M \\ \hat{\chi}_N \end{bmatrix} + \hat{\xi}^T \tilde{P} \hat{\xi} + \sum_{i=1}^{n} \lambda_{id} V_{\Delta i}(x_{\Delta_i})$$

The robust (nominal) stabilization problem is reduced into the existence of scaling matrices and a diffeomorphism which make the matrix M (N, respectively) negative. This is the fundamental of SD scaling design. Section 4 and 5 investigate how to solve the negativity problems.

Since A_N and A_{MN} are constant and the pair is observable by the assumption (4), there exist constant matrices $\hat{P} > 0$ and Y such that

$$(A_N - YA_{MN})^T \hat{P} + \hat{P}(A_N - YA_{MN}) < 0 \tag{37}$$

is satisfied. The observer gain Y can be constructed by using the standard linear control theory. By Cholesky factorization of $\hat{P}^{-1}$, there exists a non-singular lower triangular matrix W such that $\hat{P} = W^T \tilde{P} W$ with a diagonal matrix $\tilde{P} > 0$. Hence, the inequality (37) is equivalent to

$$
\begin{aligned}
H = W^{-T}(A_N - YA_{MN})^T W^T \tilde{P} + \\
\tilde{P} W (A_N - YA_{MN}) W^{-1} < 0
\end{aligned}
\tag{38}
$$

Although $\tilde{P}$ can be always an identity matrix in Cholesky factorization, the choice of $\tilde{P} \neq I$ may be exploited to obtain different solutions in design. Note that if A_N and A_{MN} satisfy

$$(A_N(x_M) - YA_{MN}(x_M))^T \hat{P} + \hat{P}(A_N(x_M) - YA_{MN}(x_M)) < 0 \tag{39}$$

for all $x_M \in \mathcal{R}^m$ with constant matrices $\hat{P} > 0$ and Y, the matrices A_N and A_{MN} are allowed to depend on x_M in Section 3-5.

4 Recursive selection of SD scaling and diffeomorphism

This section demonstrates that the structures of diffeomorphism and SD scaling newly proposed in (20-21) and (31-32) lead us to a recursive procedure of SD scaling design for reduced-order partial-state feedback. Let $\bar{x}$ denote

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_m \\ \bar{x}_{m+1} \\ \vdots \\ \bar{x}_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ \hat{x}_{m+1} \\ \vdots \\ \hat{x}_n \end{bmatrix} = \begin{bmatrix} x_M \\ \hat{x}_N \end{bmatrix} \tag{40}$$

We choose the diffeomorphism $S(\bar{x}_{[n-1]})$ in the form of

$$S^{-1}(x_M, \hat{x}_N) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 1 \end{bmatrix} \tag{41}$$

Let smooth scalar functions s_1 , $s_2, \cdots, s_n$ be

$$s_i(\bar{x}_{[i]}) \quad \text{for} \quad 1 \le i \le m \tag{42}$$
$$s_i(\bar{x}_{[i-1]}) \quad \text{for} \quad m+1 \le i \le n \tag{43}$$

The smooth function matrix $T(\bar{x}_{[n-1]})$ becomes

$$T(x_M, \hat{x}_N) = \left[\begin{array}{cccc|c|ccc} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \star_{1,1} & 1 & 0 & \ddots & \vdots & 0 & \cdots & 0 \\ \star_{2,2} & \star_{2,2} & 1 & \ddots & \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 & 0 & \cdots & 0 \\ \hline \star_{m,m} & \star_{m,m} & \cdots & \star_{m,m} & 1 & 0 & \cdots & 0 \\ \hline \star_{m,m+1} & \star_{m,m+1} & \cdots & \star_{m,m+1} & \star_{m,m+1} & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ \star_{n-2,n-1} & \star_{n-2,n-1} & \cdots & \star_{n-2,n-1} & \star_{n-2,n-1} & \cdots & \star_{n-2,n-1} & 1 \end{array}\right], \tag{44}$$

where $\star_{i,j}$ denotes any function depending only on $\bar{x}_{[i]}$, and the functions s_1 through s_j and their partial derivatives. Note that the only source of $\hat{x}$-dependence in N and M is T and that $T_{[m+1]}$ does not contain $\hat{x}_{m+1}$. That is why the function s_{m+1} is chosen to be independent of $\hat{x}_{m+1}$. Let W be represented by

$$W = \begin{bmatrix} W_{11} & 0 & \cdots & 0 \\ W_{21} & W_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ W_{n-m,1} & \cdots & W_{n-m,n-m-1} & W_{n-m,n-m} \end{bmatrix} \tag{45}$$

Due to the structure of T, we have

$$\begin{bmatrix} T_M \\ T_{NM} + T_N Y \end{bmatrix} A_{MN} W^{-1} = \left[\begin{array}{c} 0_{(m-1)\times(n-m)} \\ \hline a_{m,m+1} W_{11}^{-1} \;\; 0 \cdots 0 \\ \hline \begin{array}{cc} \star_{m,m} & 0 \cdots 0 \\ \star_{m,m+1} & 0 \cdots 0 \\ \vdots & \vdots \cdots 0 \\ \star_{n-2,n-1} & 0 \cdots 0 \end{array} \end{array}\right] \tag{46}$$

We consider a feedback gain (26) in the form of

$$K = \left[(-1)^{n-1} s_1 \cdots s_n \ \cdots \ -s_{n-1}s_n \ \ s_n \right] \ . \tag{47}$$

The following definitions are needed.

$$\hat{A} := [A \ G] \,, \quad \hat{S} := \left[\begin{array}{c|c} \multicolumn{2}{c}{S^{-1}} \\ \hline 0 \cdots 0 & s_n \end{array} \right] \tag{48}$$

We choose P as a diagonal matrix:

$$P = \operatorname*{diag}_{i=1}^{n} P_i, \qquad P_i > 0, \quad P_{[k]} = \operatorname*{diag}_{i=1}^{k} P_i \tag{49}$$

Recursive representation is introduced to matrices as follows:

$$S_{[k]}^{-1} = \left[\begin{array}{c|c} S_{[k-1]}^{-1} & 0 \\ \hline 0 \cdots 0 \ s_{k-1} & 1 \end{array} \right], \quad \hat{S}_{[k]} = \left[\begin{array}{c} S_{[k]}^{-1} \\ \hline 0 \cdots 0 \ s_k \end{array} \right] \tag{50}$$

$$T_{[k]} = \left[\begin{array}{c|c} T_{[k-1]} & 0 \\ \hline \star_{k-1,k-1} & 1 \end{array} \right] \text{ for } 1 \leq k \leq m+1, \quad T_{[k]} = \left[\begin{array}{c|c} T_{[k-1]} & 0 \\ \hline \star_{k-2,k-1} & 1 \end{array} \right] \text{ for } m+2 \leq k \leq n \tag{51}$$

$$\hat{A}_{[k]} = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & \cdots & a_{k-1,k} & 0 \\ a_{k1} & a_{k2} & \cdots & \cdots & a_{kk} & a_{k,k+1} \end{bmatrix}, \ C_{[k]} = \begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ C_{k1} & \cdots & C_{k,k-1} & C_{kk} \end{bmatrix} \tag{52}$$

The $k \times q_k$ left upper part of

$$\begin{bmatrix} T_M B_M & 0 \\ (T_{NM} + T_N Y) B_M & 0 \end{bmatrix} \tag{53}$$

is denoted by $\bar{B}_{[k]}$, where $q_k = \sum_{i=1}^{k} p_i$. The recursive definitions of the scaling matrices are

$$\mathbf{L}_{[k]} := \left\{ L_{[k]} = \operatorname*{block-diag}_{i=1}^{k} L_i : \begin{array}{ll} L_i(\bar{x}_{[i]}) \in \mathbf{L}_i & \text{for } \ 1 \leq i \leq m \\ L_i(\bar{x}_{[i-1]}) \in \mathbf{L}_i & \text{for } \ i = m+1 \\ L_i(\bar{x}_{[i-2]}) \in \mathbf{L}_i & \text{for } \ m+2 \leq i \leq n \end{array} \right\} \tag{54}$$

$$\mathbf{R}_{[k]} := \left\{ R_{[k]} = \operatorname*{block-diag}_{i=1}^{k} R_i : \begin{array}{ll} R_i(\bar{x}_{[i]}) \in \mathbf{R}_i & \text{for } \ 1 \leq i \leq m \\ R_i(\bar{x}_{[i-1]}) \in \mathbf{R}_i & \text{for } \ i = m+1 \\ R_i(\bar{x}_{[i-2]}) \in \mathbf{R}_i & \text{for } \ m+1 \leq i \leq n \end{array} \right\} \tag{55}$$

Let $\tilde{N}_{[k]}$ be defined by

$$\tilde{N}_{[k]} = \begin{bmatrix} N_{[k]11} & Q_k^T N_{12} \\ N_{12}^T Q_k & H \end{bmatrix}, \quad \tilde{N}_{[n]} = N \tag{56}$$

$$N_{[k]11} := \hat{S}_{[k]}^T \hat{A}_{[k]}^T P_{[k]} + P_{[k]} T_{[k]} \hat{A}_{[k]} \hat{S}_{[k]} \tag{57}$$

$$N_{12}(\bar{x}_{[n-2]}) := -P \begin{bmatrix} T_M \\ T_{NM} + T_N Y \end{bmatrix} A_{MN} W^{-1}, \quad (N_{12}(\bar{x}_{[n-1]}) \text{ if } m = n-1) \tag{58}$$

$$Q_k = \begin{bmatrix} I_k \\ 0 \end{bmatrix}, \quad Q_n = I_n \tag{59}$$

where I_k denotes a $k \times k$ identity matrix. The dependence of $\tilde{N}_{[k]}$ and $N_{[k]11}$ on $\bar{x}$ is

$$\tilde{N}_{[k]}(\bar{x}_{[k]}), \ N_{[k]11}(\bar{x}_{[k]}) \ \text{ if } \ 1 \leq k \leq m \tag{60}$$

$$\tilde{N}_{[k]}(\bar{x}_{[k-1]}), \ N_{[k]11}(\bar{x}_{[k-1]}) \ \text{ if } \ m+1 \leq k \leq n \tag{61}$$

In a similar manner, $\tilde{M}_{[k]}$ is defined as

$$\tilde{M}_{[k]} = \begin{bmatrix} M_{[k]11} & \bar{Q}_k^T M_{12} \\ M_{12}^T \bar{Q}_k & H \end{bmatrix}, \quad \tilde{M}_{[n]} = M \tag{62}$$

$$M_{[k]11} := \begin{bmatrix} \hat{S}_{[k]}^T \hat{A}_{[k]}^T T_{[k]}^T P_{[k]} + P_{[k]} T_{[k]} \hat{A}_{[k]} \hat{S}_{[k]} & P_{[k]} \bar{B}_{[k]} + S_{[k]}^{-T} C_{[k]}^T R_{[k]}^T & S_{[k]}^{-T} C_{[k]}^T L_{[k]} \\ * & -L_{[k]} & 0 \\ * & * & -L_{[k]} \end{bmatrix} \tag{63}$$

$$M_{12}(\bar{x}_{[n-2]}) := \begin{bmatrix} -P \begin{bmatrix} T_M \\ T_{NM} + T_N Y \end{bmatrix} A_{MN} W^{-1} \\ \begin{bmatrix} B_M^T Y^T \\ -B_N^T \end{bmatrix} W^T \tilde{P} - R \begin{bmatrix} 0 \\ C_N \end{bmatrix} W^{-1} \\ -L \begin{bmatrix} 0 \\ C_N \end{bmatrix} W^{-1} \end{bmatrix}, \quad (M_{12}(\bar{x}_{[n-1]}) \text{ if } m = n-1) \tag{64}$$

$$\bar{Q}_k = \begin{bmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{q_k} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{q_k} \end{bmatrix}, \quad \bar{Q}_n = I_{n+2P} \tag{65}$$

The dependence of $\tilde{M}_{[k]}$ and $M_{[k]11}$ on $\bar{x}$ is the same as in (60-61). Note that $N_{[k]11} = Q_k^T N_{11} Q_k$ and $M_{[k]11} = \bar{Q}_k^T M_{11} \bar{Q}_k$. We can verify the following easily.

Theorem 2 *Suppose* $1 \le k \le n$.

(i-a) $\tilde{N}_{[k]}$ *does not include* $\{s_{k+1}, s_{k+2}, \cdots, s_n\}$.

(i-b) *Every entry of* $\tilde{N}_{[k]}$ *is affine in* s_k.

(i-c) *Every entry of* $\tilde{N}_{[k]}$ *is simultaneously affine in all the entries of* $P_{[k]}$.

(i-d) $\tilde{N}_{[k]} < 0$ *implies* $\tilde{N}_{[k-1]} < 0$ *unless* $k = 1$.

(ii-a) $\tilde{M}_{[k]}$ *does not include either* $\{s_{k+1}, s_{k+2}, \cdots, s_n\}$, $\{L_{k+1}, L_{k+2}, \cdots, L_n\}$ *or* $\{R_{k+1}, R_{k+2}, \cdots, R_n\}$.

(ii-b) *Every entry of* $\tilde{M}_{[k]}$ *is simultaneously affine in* L_k, R_k *and* s_k.

(ii-c) *Every entry of* $\tilde{M}_{[k]}$ *is simultaneously affine in all the entries of* $L_{[k]}$, $R_{[k]}$ *and* $P_{[k]}$.

(ii-d) $\tilde{M}_{[k]} < 0$ *implies* $\tilde{M}_{[k-1]} < 0$ *unless* $k = 1$.

The problem of SD scaling is recursively linear in design parameters. On the basis of Theorem 2, this paper proposes the following procedures for reduced-order dynamic partial-state feedback design.
Nominal backstepping : Solve $\tilde{N}_{[k]} < 0$ for s_k from $k = 1$ through $k = n$.
Robust backstepping : Solve $\tilde{M}_{[k]} < 0$ for $\{s_k, L_k, R_k\}$ from $k = 1$ through $k = n$.
Both the procedures assume that P, $\tilde{P}$, W and Y are given. The recursive procedures can be carried out since the process of finding design parameters at Step k does not require any design parameters to be found at Step $k+1, k+2, \ldots, n$. The recursive procedures are justified by (i-d) and (ii-d). The problem of finding $\{L_k, R_k, s_k\}$ satisfying $\tilde{M}_{[k]} < 0$ (or $\tilde{N}_{[k]} < 0$) is a convex problem. Thus these backstepping procedures via SD scaling are amenable to numerical computation and optimization as it has been shown for state-feedback control(Ito and Freeman, 1999*b*).

5 Existence of solutions

This section investigates whether the solutions exist or not in the recursive procedures proposed in the previous section. This section also provides their analytical solutions.

Define the following two functions.

$$\bar{N}_{[k]} := N_{[k]11} - Q_k^T N_{12} H^{-1} N_{12}^T Q_k \tag{66}$$
$$\bar{M}_{[k]} := M_{[k]11} - \bar{Q}_k^T M_{12} H^{-1} M_{12}^T \bar{Q}_k \tag{67}$$

From the Schur complements formula it follows that the equivalence

$$\bar{N}_{[k]} < 0 \Leftrightarrow \tilde{N}_{[k]} < 0 \tag{68}$$
$$\bar{M}_{[k]} < 0 \Leftrightarrow \tilde{M}_{[k]} < 0 \tag{69}$$

are true on the assumption that $H < 0$ holds. The matrices $\bar{N}_{[k]}$ and $\bar{M}_{[k]}$ are represented by

$$\begin{array}{ll} \bar{N}_{[k]}(\bar{x}_{[k]}) := N_{[k]11} & \text{for } \ 1 \le k \le m-1 \\ \bar{N}_{[k]}(\bar{x}_{[k]}) := N_{[k]11} - \tilde{\Theta} & \text{for } \ k = m \\ \bar{N}_{[k]}(\bar{x}_{[k-1]}) := N_{[k]11} - Q_k^T N_{12} H^{-1} N_{12}^T Q_k & \text{for } \ m+1 \le k \le n \end{array} \tag{70}$$

$$\begin{array}{ll} \bar{M}_{[k]}(\bar{x}_{[k]}) := M_{[k]11} & \text{for } \ 1 \le k \le m-1 \\ \bar{M}_{[k]}(\bar{x}_{[k]}) := M_{[k]11} - \Theta & \text{for } \ k = m \\ \bar{M}_{[k]}(\bar{x}_{[k-1]}) := M_{[k]11} - \bar{Q}_k^T M_{12} H^{-1} M_{12}^T \bar{Q}_k & \text{for } \ m+1 \le k \le n \end{array} \tag{71}$$

The matrices $\tilde{\Theta}$ and Θ are given by

$$\tilde{\Theta}(\bar{x}_{[m]}) := \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & P_m^2 W_{11}^{-2} a_{m,m+1}^2 [H^{-1}]_{11} \end{bmatrix} \tag{72}$$

$$\Theta(\bar{x}_{[m]}) := \begin{bmatrix} \tilde{\Theta} & 0 & a_{m,m+1} W_{11}^{-1} [H^{-1}]_{1,*} Y_m B_{mm} & 0 \\ 0 & 0 & 0 & 0 \\ B_{mm}^T Y_m^T [H^{-1}]_{1,*}^T W_{11}^{-1} a_{m,m+1} 0 & 0 & B_{mm}^T Y_m^T [H^{-1}] Y_m B_{mm} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{73}$$

Here, $[H^{-1}]_{1,*}$ denotes the first row of H^{-1}.

Lemma 1 *Suppose* $2 \le k \le n$.
(i) The symmetric matrix

$$\bar{N}_{[1]}(x_1) = \tilde{\Psi}_1(x_1) \tag{74}$$

depends only on s_1. $\bar{N}_{[k]} < 0$ *is equivalent to*

$$\begin{array}{ll} \begin{bmatrix} \bar{N}_{[k-1]}(\bar{x}_{[k-1]}) & \tilde{\Phi}_k(\bar{x}_{[k]}) \\ \tilde{\Phi}_k^T(\bar{x}_{[k]}) & \tilde{\Psi}_k(\bar{x}_{[k]}) \end{bmatrix} < 0 & \textit{for} \ \ 2 \le k \le m \\ \begin{bmatrix} \bar{N}_{[k-1]}(\bar{x}_{[k-1]}) & \tilde{\Phi}_k(\bar{x}_{[k-1]}) \\ \tilde{\Phi}_k^T(\bar{x}_{[k-1]}) & \tilde{\Psi}_k(\bar{x}_{[k-1]}) \end{bmatrix} < 0 & \textit{for} \ \ k = m+1 \\ \begin{bmatrix} \bar{N}_{[k-1]}(\bar{x}_{[k-2]}) & \tilde{\Phi}_k(\bar{x}_{[k-1]}) \\ \tilde{\Phi}_k^T(\bar{x}_{[k-1]}) & \tilde{\Psi}_k(\bar{x}_{[k-1]}) \end{bmatrix} < 0 & \textit{for} \ \ m+2 \le k \le n \end{array} \tag{75}$$

where $\tilde{\Phi}_k$ depends only on $(s_1, \cdots, s_{k-1})$ and their partial derivatives. The symmetric matrix $\tilde{\Psi}_k$ depends on s_k.
(ii) Assume that $\tilde{P}$ is diagonal. Then, the symmetric matrix

$$\bar{M}_{[1]}(x_1) = \Psi_1(x_1) \tag{76}$$

depends only on (L_1, R_1) and s_1. $\bar{M}_{[k]} < 0$ is equivalent to

$$\begin{aligned} &\begin{bmatrix} \bar{M}_{[k-1]}(\bar{x}_{[k-1]}) & \Phi_k(\bar{x}_{[k]}) \\ \Phi_k^T(\bar{x}_{[k]}) & \Psi_k(\bar{x}_{[k]}) \end{bmatrix} < 0 \quad \textit{for} \;\; 2 \le k \le m \\ &\begin{bmatrix} \bar{M}_{[k-1]}(\bar{x}_{[k-1]}) & \Phi_k(\bar{x}_{[k-1]}) \\ \Phi_k^T(\bar{x}_{[k-1]}) & \Psi_k(\bar{x}_{[k-1]}) \end{bmatrix} < 0 \;\; \textit{for} \;\; k = m+1 \\ &\begin{bmatrix} \bar{M}_{[k-1]}(\bar{x}_{[k-2]}) & \Phi_k(\bar{x}_{[k-1]}) \\ \Phi_k^T(\bar{x}_{[k-1]}) & \Psi_k(\bar{x}_{[k-1]}) \end{bmatrix} < 0 \;\; \textit{for} \;\; m+2 \le k \le n \end{aligned} \tag{77}$$

where Φ_k depends only on $(L_1, R_1, \cdots, L_k, R_k)$ and $(s_1, \cdots, s_{k-1})$ and their partial derivatives. The symmetric matrix Ψ_k depends on (L_k, R_k) and s_k.

The proof is straightforward and is similar to (Ito and Krstić, 1999). The matrices $\tilde{\Phi}$ and $\tilde{\Psi}$ for $\bar{N}_{[k]}$ are obtained as

$$\tilde{\Phi}_k = \star_{k,k-1}, \quad \tilde{\Psi}_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \star_{k-1,k-1}) \text{ for } 1 \le k \le m-1 \tag{78}$$

$$\tilde{\Phi}_k = \star_{k,k-1}, \quad \tilde{\Psi}_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \star_{k,k-1}) \text{ for } k = m \tag{79}$$

$$\tilde{\Phi}_k = \bullet_{m,k-1,k-1}, \quad \tilde{\Psi}_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \bullet_{m,k-1,k-1}) \text{ for } m+1 \le k \le n \tag{80}$$

where $\bullet_{m,j,k}$ denotes any function depending only on $x_{[m]}$, $\hat{x}_{[j-m]}$, and $(s_1, \cdots, s_k)$ and their partial derivatives. As for $\bar{M}_{[k]}$, matrices Φ_k and Ψ_k are obtained as follows:
for $1 \le k \le m-1$

$$\Phi_k = \begin{bmatrix} \star_{k,k-1} & \star_{k,k-1}R_k^T & \star_{k-1,k-1}C_{k,-}^T L_k \\ \star_{k,k-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{81}$$

$$\Psi_k = \begin{bmatrix} 2P_k(a_{kk} + a_{k,k+1}s_k + \star_{k-1,k-1}) & P_k B_{kk} + C_{kk}^T R_k^T & C_{kk}^T L_k \\ * & -L_k & 0 \\ * & * & -L_k \end{bmatrix}. \tag{82}$$

for $k = m$

$$\Phi_k = \begin{bmatrix} \star_{k,k-1} & \star_{k,k-1}R_k^T & \star_{k-1,k-1}C_{k,-}^T L_k \\ \star_{k,k-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{83}$$

$$\Psi_k = \begin{bmatrix} 2P_k(a_{kk} + a_{k,k+1}s_k + \star_{k,k-1}) & P_k B_{kk} - a_{m,m+1}W_{11}^{-1}[H^{-1}]_{1,*}Y_m B_{mm} + C_{kk}^T R_k^T & C_{kk}^T L_k \\ * & -L_k - B_{mm}^T Y_m^T [H^{-1}] Y_m B_{mm} & 0 \\ * & * & -L_k \end{bmatrix}. \tag{84}$$

for $m+1 \le k \le n$

$$\Phi_k = \begin{bmatrix} \bullet_{m,k-1,k-1} & \bullet_{m,k-2,k-1}R_k^T & \bullet_{m,k-2,k-1}C_{k,-}^T L_k \\ \diamondsuit_{m,k-1,k-1} & U_{*,k} & \diamondsuit_{m,k-3,k-1}C_{N,k,-}^T L_k \\ \diamondsuit_{m,k-1,k-1} & \diamondsuit_{1,k-3,k-1}R_k^T & \diamondsuit_{m,k-3,k-1}C_{N,k,-}^T L_k \end{bmatrix} \tag{85}$$

$$\Psi_k =$$

$$\begin{bmatrix} 2P_k(a_{kk} + a_{k,k+1}s_k + \bullet_{m,k-1,k-1}) & \bullet_{m,k-1,k-1} + (\bullet_{m,k-1,k-1} + C_{kk}^T)R_k^T & (\bullet_{m,k-1,k-1} + C_{kk}^T)L_k \\ * & -L_k + U_{kk} & (R_k \star_{m,0} + \star_{m,0})C_{N,k,-}^T L_k \\ * & * & -L_k - L_k C_{N,k,-} W_{[k-m]}^{-1} [H^{-1}]_{[k-m]} W_{[k-m]}^{-T} C_{N,k,-}^T L_k \end{bmatrix} \tag{86}$$

where $\diamondsuit_{m,j,k}$ denotes any function depending only on $(x_{[m]}, \hat{x}_{[j-m]})$ and $(L_{[k]}, R_{[k]})$, and $(s_1, \cdots, s_k)$ and their partial derivatives. For $m+1 \le k \le n$,

$$[H^{-1}]_{[k]} = \left[\begin{array}{c|c} [H^{-1}]_{[k-1]} & \star_{m,0} \\ \hline \star_{m,0} & [H^{-1}]_{k,k} \end{array}\right], \quad [H^{-1}]_{[n-m]} = H^{-1}, \quad [H^{-1}]_{[1]} = [H^{-1}]_{11} \tag{87}$$

$$W_{[k]}^{-1} = \left[\begin{array}{c|c} W_{[k-1]}^{-1} & 0 \\ \hline \star_{0,0} & W_{k,k}^{-1} \end{array}\right], \quad W_{[n-m]}^{-1} = W^{-1}, \quad W_{[1]}^{-1} = W_{11}^{-1} \tag{88}$$

$$C_{[k]} = \left[\begin{array}{c|c} C_{[k-1]} & 0 \\ \hline \star_{m,0} & C_{kk} \end{array}\right] = \left[\begin{array}{c} C_{[k-1]} | 0 \\ \hline C_{k,-} \end{array}\right], \quad C_{k,-} = [\, C_{k,1} \cdots C_{k,m} | C_{N,k,-} \,] \tag{89}$$

are used. The definition of

$$U_{m+1,m+1}(x_{[m]}) \in \mathcal{R}^{p_{m+1} \times p_{m+1}}, \quad U_{*,m+1}(x_{[m]}) \in \mathcal{R}^{q_m \times p_{m+1}} \quad \text{for } k = m+1 \tag{90}$$

$$U_{kk}(x_{[m]}, \hat{x}_{[k-2]}) \in \mathcal{R}^{p_k \times p_k}, \quad U_{*,k}(x_{[m]}, \hat{x}_{[k-2]}) \in \mathcal{R}^{q_{k-1} \times p_k} \quad \text{for } m+2 \le k \le n \tag{91}$$

are given by

$$U_{kk} = -[B_N^T W^T \tilde{P} + R_N C_N W^{-1}]_k H^{-1} [B_N^T W^T \tilde{P} + R_N C_N W^{-1}]_k^T \tag{92}$$

$$U_{*,k} = -\tilde{U}_{[k-1]} H^{-1} [B_N^T W^T \tilde{P} + R_N C_N W^{-1}]_k^T \tag{93}$$

where

$$R_N = \operatorname*{block\text{-}diag}_{i=m}^{n} R_i \tag{94}$$

$$B_N^T W^T \tilde{P} + R_N C_N W^{-1} = \begin{bmatrix} [B_N^T W^T \tilde{P} + R_N C_N W^{-1}]_m \\ [B_N^T W^T \tilde{P} + R_N C_N W^{-1}]_{m+1} \\ \cdots \\ [B_N^T W^T \tilde{P} + R_N C_N W^{-1}]_n \end{bmatrix} \tag{95}$$

$$\tilde{U}_{[k]} = [\, I_{q_k} \;\; 0 \,] \left(\begin{bmatrix} -B_M^T Y^T \\ B_N^T \end{bmatrix} W^T \tilde{P} + \begin{bmatrix} 0 \\ R_N C_N \end{bmatrix} W^{-1} \right) \tag{96}$$

According to (70) and (71), the matrices $\bar{N}_{[k]}$ and $\bar{M}_{[k]}$ from $k=1$ to $k=m-1$ are exactly the same as those for the state-feedback(Ito and Freeman, 1999*b*). The backstepping procedure to find L_k, R_k and s_k satisfying $\bar{M}_{[k]} < 0$ (or $\bar{N}_{[k]} < 0$) can be carried out for all entries of x_M but x_m as it is done for the state-feedback case.

Lemma 2 *Let k is any integer belonging to $[1, m-1]$.*
(i) Assume that $\bar{N}_{[k-1]}(\bar{x}_{[k-1]}) < 0$ hold for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$ unless $k=1$. There always exists a scalar-valued smooth function $s_k(\bar{x}_{[k]})$ such that

$$\bar{N}_{[k]}(\bar{x}_{[k]}) < 0 \tag{97}$$

is satisfied for all $\bar{x}_{[k]} \in \mathcal{R}^k$.
(ii) Assume that $\bar{M}_{[k-1]}(\bar{x}_{[k-1]}) < 0$ hold for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$ unless $k=1$. There always exist a scalar-valued smooth function $s_k(\bar{x}_{[k]})$ and a $\mathbf{C}^0$ function $\lambda_k(\bar{x}_{[k]})$ such that

$$\bar{M}_{[k]}(\bar{x}_{[k]}) < 0, \quad \lambda_k(\bar{x}_{[k]}) > 0 \tag{98}$$

are satisfied with $L_k = \lambda_k I_{p_k}$ and $R_k = 0$ for all $\bar{x}_{[k]} \in \mathcal{R}^k$.

Let $\tilde{J}_k(\bar{x}_{[k-1]}) \in \mathcal{R}^{1\times 1}$ be defined with

$$\tilde{\Psi}_k - \tilde{\Phi}_k^T \bar{N}_{[k-1]}^{-1} \tilde{\Phi}_k \quad \text{for} \quad m+1 \le k \le n \tag{99}$$

We also define $J_k(\bar{x}_{[k-1]}) \in \mathcal{R}^{1\times 1}$, $E_k(\bar{x}_{[k-1]}) \in \mathcal{R}^{1\times 2p_k}$ and $F_k(\bar{x}_{[k-1]}) \in \mathcal{R}^{2p_k \times 2p_k}$ as

$$\Psi_k - \Phi_k^T \bar{M}_{[k-1]}^{-1} \Phi_k = \begin{bmatrix} J_k & E_k \\ E_k^T & F_k \end{bmatrix} \quad \text{for} \quad m+1 \le k \le n \tag{100}$$

The matrices $\tilde{J}_m(\bar{x}_{[m]})$, $J_m(\bar{x}_{[m]})$, $E_m(\bar{x}_{[m]})$ and $F_m(\bar{x}_{[m]})$ are defied in the same way. Using the Schur complements of (99) and (100) , we have the following.

Lemma 3 *Let k is any integer belonging to $[m+1, n]$.*
(i) Assume that $\bar{N}_{[k-1]}(\bar{x}_{[k-2]}) < 0$ is satisfied for all $\bar{x}_{[k-2]} \in \mathcal{R}^{k-2}$. Then, $\bar{N}_{[k]}(\bar{x}_{[k-1]}) < 0$ holds for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$ if and only if

$$\tilde{J}_k < 0 \tag{101}$$

is satisfied for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$.
(ii) Assume that $\bar{M}_{[k-1]}(\bar{x}_{[k-2]}) < 0$ is satisfied for all $\bar{x}_{[k-2]} \in \mathcal{R}^{k-2}$. Then, $\bar{M}_{[k]}(\bar{x}_{[k-1]}) < 0$ holds for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$ if and only if

$$F_k < 0, \; J_k - E_k F_k^{-1} E_k^T < 0, \quad \text{when } p_k \neq 0 \tag{102}$$

$$J_k < 0, \qquad \text{when } p_k = 0 \tag{103}$$

are satisfied for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$.
(iii) The statements (i) and (ii) are true for $k = m$ by replacing $\bar{x}_{[m-1]}$ with $\bar{x}_{[m]}$.

From (79) and (80), the function $\tilde{J}_k$ is given by

$$\tilde{J}_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \bullet_{m,k-1,k-1}) \tag{104}$$

This implies that there always exist a scalar-valued smooth function $s_k(\bar{x}_{[k-1]})$ such that $\tilde{J}_k(\bar{x}_{[k-1]}) < 0$ is satisfied for all $\bar{x}_{[k-1]}$ ($\bar{x}_{[m]}$ in the $k = m$ case). Hence, we can obtain the following.

Theorem 3 *The nominal nonlinear system Σ_{01} can be globally uniformly asymptotically stabilized by the reduced-order dynamic feedback law (16-17) with a smooth function K.*

Note that A_N and A_{MN} are allowed to depend on x_M if (39) is satisfied. The result of (Kanellakopoulos, 1991) without the adaptive mechanism can be considered as the special case $x_M = x_1$ of the above theorem. Theorem 3, however, employs domination instead of exact cancelation.

As for robust stabilization, from (83-86), the matrices J_k is given by

$$J_k = 2P_k(a_{kk} + a_{k,k+1}s_k) + \diamondsuit_{m,k-1,k-1} \tag{105}$$

for $m \le k \le n$.

Lemma 4 *Let k is any integer belonging to $[m+1, n]$. Suppose that $\mathbf{C}^0$ function matrices $L_k(x_{[m]}, \hat{x}_{[k-2]})$ and $R_k(x_{[m]}, \hat{x}_{[k-2]})$ belong to $\mathbf{L}_k$ and $\mathbf{R}_k$, respectively. Then, there always exist a scalar-valued smooth function $s_k(\bar{x}_{[k-1]})$ such that $J_k - E_k F_k^{-1} E_k^T < 0$ is satisfied for all $\bar{x}_{[k-1]} \in \mathcal{R}^{k-1}$. This fact is also true for $k = m$ by replacing $\bar{x}_{[m-1]}$ with $\bar{x}_{[m]}$.*

Due to (83-86), for $m \leq k \leq n$, the matrices E_k and F_k defined with $R_k = 0$ are

$$E_k(x_{[m]}, \hat{x}_{[k-1]}) = \left[\, \diamondsuit_{m,k-1,k-1} \;\; (C_{kk}^T + \diamondsuit_{m,k-1,k-1}) L_k \,\right] \tag{106}$$

$$F_m(x_{[m]}) = \quad \text{for } k = m \tag{107}$$

$$\begin{bmatrix} -L_m - B_{mm}^T Y_m^T [H^{-1}] Y_m B_{mm} & \star_{m,0} C_{m,-}^T L_m \\ * & -L_m - L_m C_{m,-} \bar{F}_{m22} C_{m,-}^T L_m \end{bmatrix} \tag{108}$$

$$F_k(x_{[m]}, \hat{x}_{[k-2]}) = \quad \text{for } k \geq m+1 \tag{109}$$

$$\begin{bmatrix} -L_k - B_{N,-,k}^T W_{\langle k-m \rangle}^T \tilde{P}_{\langle k-m \rangle} ([H^{-1}]_{\langle k-m \rangle} + \bar{F}_{k11}) \tilde{P}_{\langle k-m \rangle} W_{\langle k-m \rangle} B_{N,-,k} & (\star_{m,0} - B_{Nk}^T W^T \tilde{P} \bar{F}_{k12}) C_{k,-}^T L_k \\ * & -L_k - L_k (C_{N,k,-} W_{[k-m]}^{-1} [H^{-1}]_{[k-m]} W_{[k-m]}^{-T} C_{N,k,-}^T + C_{k,-} \bar{F}_{k22} C_{k,-}^T) L_k \end{bmatrix}$$

$$\bar{F}_k(x_{[m]}, \bar{x}_{[k-2]}) = \begin{bmatrix} 0 & \bullet_{m,k-2,k-1} \\ \bar{U}_k & \diamondsuit_{m,k-3,k-1} \\ 0 & \diamondsuit_{m,k-3,k-1} \end{bmatrix}^T \bar{M}_{[k-1]}^{-1} \begin{bmatrix} 0 & \bullet_{m,k-2,k-1} \\ \bar{U}_k & \diamondsuit_{m,k-3,k-1} \\ 0 & \diamondsuit_{m,k-3,k-1} \end{bmatrix} = \begin{bmatrix} \bar{F}_{k11} & \bar{F}_{k12} \\ \bar{F}_{k12} & \bar{F}_{k22} \end{bmatrix} \tag{110}$$

$$\bar{U}_k(x_{[m]}, \hat{x}_{[k-3]}) = -\tilde{U}_{[k-1]} H^{-1} \begin{bmatrix} 0 \\ I_{n-k+1} \end{bmatrix} \tag{111}$$

$$\bar{F}_{m22}(x_{[m]}) = \begin{bmatrix} \bullet_{m,0,0} \\ 0 \\ 0 \end{bmatrix}^T \bar{M}_{[m-1]}^{-1} \begin{bmatrix} \bullet_{m,0,0} \\ 0 \\ 0 \end{bmatrix} \tag{112}$$

Here, the following expressions are used.

$$B_N = [\, B_{Nm+1} | B_{Nm+2} | \cdots | B_{Nn} \,] = \left[\begin{array}{c|c|c|c} B_{N,-,m+1} & 0 & \cdots & 0 \\ & B_{N,-,m+2} & \ddots & \vdots \\ & & & 0 \\ & & \cdots & B_{N,-,n} \end{array} \right], \quad B_{N,-,n} = B_{nn} \tag{113}$$

$$C = \left[\begin{array}{c} C_{1,-} \,|\, 0 \cdots 0 \\ \hline C_{2,-} \,|\, 0 \;\vdots \\ \hline \vdots \quad \ddots \; 0 \\ \hline C_{n,-} \end{array} \right], C_N = \left[\begin{array}{c} C_{N,1,-} \,|\, 0 \cdots 0 \\ \hline C_{N,2,-} \,|\, 0 \;\vdots \\ \hline \vdots \quad \ddots \; 0 \\ \hline C_{N,n,-} \end{array} \right] \tag{114}$$

$$[H^{-1}]_{\langle k \rangle} = \left[\begin{array}{c|c} [H^{-1}]_{kk} & \star_{0,0} \\ \hline \star_{0,0} & [H^{-1}]_{\langle k-1 \rangle} \end{array} \right], \quad [H^{-1}]_{\langle 1 \rangle} = H^{-1}, \quad [H^{-1}]_{\langle n-m \rangle} = [H^{-1}]_{n-m,n-m} \tag{115}$$

$$\tilde{P}_{\langle k \rangle} = \left[\begin{array}{c|c} \tilde{P}_k & 0 \\ \hline 0 & \tilde{P}_{\langle k+1 \rangle} \end{array} \right], \quad \tilde{P}_{\langle 1 \rangle} = \tilde{P}, \quad \tilde{P}_{\langle n-m \rangle} = \tilde{P}_{n-m} \tag{116}$$

$$W_{\langle k \rangle} = \left[\begin{array}{c|c} W_{kk} & 0 \\ \hline \star_{0,0} & W_{\langle k+1 \rangle} \end{array} \right], \quad W_{\langle 1 \rangle} = W, \quad W_{\langle n-m \rangle} = W_{n-m,n-m} \tag{117}$$

Lemma 5 *(i) For $k = m$: Suppose that $R_m = 0$, $p_m \neq 0$. Assume that $\bar{M}_{[m-1]}(\bar{x}_{[m-1]}) < 0$ holds for all $\bar{x}_{[m-1]} \in \mathcal{R}^{m-1}$. There exists a scalar-valued $\mathbf{C}^0$ function $\lambda_k(x_{[m]})$ such that*

$$\lambda_k(x_{[m]}) > 0, \quad F_k(x_{[m]}) < 0 \tag{118}$$

are satisfied for all $x_{[m]} \in \mathcal{R}^m$ with $L_k(x_{[m]}) = \lambda_k(x_{[m]}) I_{p_k}$ if

$$\lambda_{max}\left(-B_{mm}^T Y_m^T [H^{-1}] Y_m B_{mm}\right) \lambda_{max}\left(-C_{k,-} \bar{F}_{k22} C_{k,-}^T\right) \leq \frac{1}{4} \tag{119}$$

holds for all $x_{[m]} \in \mathcal{R}^m$.
(ii) For $m+1 \leq k \leq n$: Suppose that $R_k = 0$, $p_k \neq 0$. Assume that $\bar{M}_{[k-1]}(\bar{x}_{[k-2]}) < 0$ holds for all $\bar{x}_{[k-2]} \in \mathcal{R}^{k-2}$. There exists a scalar-valued $\mathbf{C}^0$ function $\lambda_k(x_{[m]}, \hat{x}_{[k-2]})$ such that

$$\lambda_k(x_{[m]}, \hat{x}_{[k-2]}) > 0, \quad F_k(x_{[m]}, \hat{x}_{[k-2]}) < 0 \tag{120}$$

are satisfied for all $(x_{[m]}, \hat{x}_{[k-2]}) \in \mathcal{R}^m \times \mathcal{R}^{k-2}$ *with* $L_k(x_{[m]}, \hat{x}_{[k-2]}) = \lambda_k(x_{[m]}, \hat{x}_{[k-2]}) I_{p_k}$ *if*

$$\lambda_{max}\left(-B_{N,-,k}^T W_{\langle k-m\rangle}^T \tilde{P}_{\langle k-m\rangle}([H^{-1}]_{\langle k-m\rangle} + \bar{F}_{k11})\tilde{P}_{\langle k-m\rangle} W_{\langle k-m\rangle} B_{N,-,k}\right) \times$$
$$\lambda_{max}\left(-C_{N,k,-} W_{[k-m]}^{-1}[H^{-1}]_{[k-m]} W_{[k-m]}^{-T} C_{N,k,-}^T - C_{k,-}\bar{F}_{k22}C_{k,-}^T\right) \leq \frac{1}{4} \qquad (121)$$

holds for all $(x_{[m]}, \hat{x}_{[k-2]}) \in \mathcal{R}^m \times \mathcal{R}^{k-2}$.

Since the entries of B and C matrices represent the nonlinear bounds of uncertainties, the conditions (119) and (121) are considered as the nonlinear size of tolerable uncertainties. We obtain the following

Theorem 4 *Suppose that* B *and* C *satisfy (119) for* $k = m$ *and (121) for all* $k = m+1, \ldots, n$. *Assume that the uncertainty* Σ_Δ *only has static uncertain components* Δ_{is} *and* Δ_{ir}. *Then, the system* Σ_{P1} *can be globally uniformly asymptotically stabilized for any admissible uncertainty by the reduced-order dynamic feedback law (16-17) with a smooth function* K.

We can always achieve robust stabilization for the following class of uncertain systems.

Theorem 5 *Suppose that* $B_{mm} = 0$ *and* $B_N = 0$.

(i) *If the uncertainty* Σ_Δ *only has static uncertain components* Δ_{is} *and* Δ_{ir}, *the system* Σ_{P1} *can be globally uniformly asymptotically stabilized for any admissible uncertainty by the reduced-order dynamic feedback law (16-17) with a smooth function* K.

(ii) *If the uncertainty* Σ_Δ *has dynamic uncertain components* Δ_{id}, *the system* Σ_{P1} *can be semi-globally uniformly asymptotically stabilized for any admissible uncertainty by the reduced-order dynamic feedback law (16-17) with a smooth function* K.

Global robust stabilizability against dynamic uncertainties is not always achievable if the nonlinear size of uncertainty is prescribed *a priori*. However, if we relax the robustness requirement, a stability robustness in terms of Input-to-State Stability(ISS) can be obtained.

Theorem 6 *Assume that* B_{mm} *is uniformly bounded and* $B_N = 0$. *Then, the system* Σ_{01} *can be ISS stabilized by the reduced-order dynamic feedback law (16-17) with a smooth function* K.

Note that A_N and A_{MN} are allowed to depend on x_M if (39) is satisfied. The difference between full-order observer and reduced-order observer is clearly seen in Theorem 5 and 6. For *robust* stabilization, it has been demonstrated that observer design cannot be completely separated from feedback-gain design(Ito and Krstić, 1999). The observer must be designed strong enough by taking into account the size of uncertainty. The reduced-order observer does not have any dynamics for x_M so that this part of the system cannot be made robust by the observer. This is why the constraint on B_{mm} is required in Theorem 5 and 6 in order to guarantee that robust stabilization is *always* achievable for *arbitrarily* large admissible uncertainties. The constraint is not required in the full-order observer case(Ito and Krstić, 1999). In contrast, there are no difference between the reduced-order observer and the full-order observer in achieving nominal stabilization as shown in Theorem 3.

6 Static partial-state feedback

This section focuses on static partial-state feedback control for the class of uncertain systems defined with Σ_{P2}.

Consider the static feedback law as

$$u = K(x_N)x_N \tag{122}$$

We consider the diffeomorphism between $x \in \mathcal{R}^n$ and $\chi \in \mathcal{R}^n$ as follows:

$$\chi = S(x_N)x \ . \tag{123}$$

The time-derivative of χ is obtained as

$$\dot{\chi} = \left[\frac{\partial S}{\partial x_{m+1}}x, \frac{\partial S}{\partial x_{m+2}}x, \cdots, \frac{\partial S}{\partial x_n}x\right]\dot{x}_N + S(x_N)\dot{x} = T(x_M, x_N)\dot{x} \ . \tag{124}$$

Note that the matrix T may contain unmeasured signal x_M even if S only contains x_N. We now choose a diffeomorphism $S(x_N)$ in a particular form of

$$S^{-1}(x_N) = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & s_{m+1} & 1 & 0 & \ddots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & 0 & \ddots & s_{n-1} & 1 \end{array}\right] \tag{125}$$

where all s_i, $i = m+1, m+2, \ldots, n$ are smooth functions of x_N. These functions are chosen such that their dependence on x_N is consistent with

$$s_i(x_{m+[i]}) \ \text{ if } \ m+1 \le i \le n \tag{126}$$

Here, $x_{m+[i]}$ denotes

$$x_{m+[i]} = \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_{m+i} \end{bmatrix} \tag{127}$$

and $x_{m+[n-m]} = x_N$. Due to (126), we have

$$S^{-1}(x_N) = \begin{bmatrix} I & 0 \\ 0 & S_N^{-1}(x_N) \end{bmatrix}, \quad S(x_N) = \begin{bmatrix} I & 0 \\ 0 & S_N(x_N) \end{bmatrix}, \quad T(x_N) = \begin{bmatrix} I & 0 \\ 0 & T_N(x_N) \end{bmatrix} \tag{128}$$

$$S_N(x_N) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -s_{m+1} & 1 & 0 & \cdots & 0 \\ s_{m+1}s_{m+2} & -s_{m+2} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (-1)^{n-m-1}s_{m+1}\cdots s_{n-1} & \cdots & s_{n-2}s_{n-1} & -s_{n-1} & 1 \end{bmatrix} \tag{129}$$

$$T_N(x_N) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ *_{m+1,m+1} & 1 & 0 & \cdots & 0 \\ *_{m+2,m+2} & *_{m+2,m+2} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ *_{n-1,n-1} & \cdots & \cdots & *_{n-1,n-1} & 1 \end{bmatrix} \tag{130}$$

where $*_{m+i,m+j}$ denotes any function depending only on $x_{m+[i]}$ and the functions s_{m+1} through s_{m+j} and their partial derivatives. Here, the function T only depends on the measured state x_N. We now consider a feedback gain (122) in the form of

$$K = \left[(-1)^{n-m-1} s_{m+1} \cdots s_n \ \cdots \ -s_{n-1} s_n \ s_n \right] \tag{131}$$

By using the matrices

$$\hat{A} := [A \ G], \quad \hat{S} := \left[\begin{array}{c|c} \multicolumn{2}{c}{S^{-1}} \\ \hline 0 \cdots 0 & s_n \end{array} \right]$$

the following is straightforward from Ito and Freeman (1999*b*).

Theorem 7 *(i) Suppose that there exist a constant matrix $P = P^T > 0$ such that*

$$N(x_N) := \hat{S}^T \hat{A}^T T^T P + P T \hat{A} \hat{S} < 0 \tag{132}$$

is satisfied for all x_N in $\mathcal{R}^{n-m}$. Then, the nominal nonlinear system Σ_{02} is globally uniformly asymptotically stabilized by the static feedback law (122). Furthermore, a Lyapunov function is given by $V = \chi^T P \chi$.
(ii) Suppose that there exist a constant matrix $P = P^T > 0$ and scaling functions $L \in \mathbf{L}$ and $R \in \mathbf{R}$ such that

$$M(x_N) = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + P T \hat{A} \hat{S} & P T B + S^{-T} C^T R^T & S^{-T} C^T L \\ B^T T^T P + R C S^{-1} & -L & 0 \\ L C S^{-1} & 0 & -L \end{bmatrix} < 0 \tag{133}$$

is satisfied for all x_N in $\mathcal{R}^{n-m}$. Then, the nonlinear system Σ_{P2} is globally uniformly asymptotically stabilized for any admissible uncertainty Σ_Δ by the static feedback law (122). Furthermore, a Lyapunov function is given by $V = \chi^T P \chi + \sum_{i=1}^{n} \lambda_{id} V_{\Delta i}(x_{\Delta_i})$.

Let $N_{[k]}$ and $M_{[k]}$ be defined by adding subscript $[k]$ to every matrix in (132) and (133), respectively. The individual matrices are defined in the same way as in Section 4 except that

$$\hat{S}_{[k]} = \begin{cases} \left[\begin{array}{c|c} \multicolumn{2}{c}{S_{[k]}^{-1}} \\ \hline 0 \cdots 0 & 0 \end{array} \right] & \text{for} \quad k = m \\ \left[\begin{array}{c|c} \multicolumn{2}{c}{S_{[k]}^{-1}} \\ \hline 0 \cdots 0 & s_k \end{array} \right] & \text{for} \quad m+1 \le k \le n \end{cases} \tag{134}$$

$$P = \left[\begin{array}{c|cccc} P_{[m]} & & 0 & & \\ \hline & P_{m+1} & 0 & \cdots & 0 \\ 0 & 0 & P_{m+2} & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ & 0 & \cdots & 0 & P_n \end{array} \right], \quad P_{[k]} = \begin{cases} P_{[m]} & \text{for} \quad k = m \\ \left[\begin{array}{c|c} P_{[k-1]} & 0 \\ \hline 0 & P_k \end{array} \right] & \text{for} \quad m+1 \le k \le n \end{cases} \tag{135}$$

where $P_{[m]}$ need not be diagonal. The dependence of SD scaling matrices are chosen as

$$L_k(x_{m+1}), \quad R_k(x_{m+1}) \ \text{ for } 1 \le k \le m \tag{136}$$

$$L_k(x_{m+[k]}), \quad R_k(x_{m+[k]}) \ \text{ for } m+1 \le k \le n \tag{137}$$

From

$$A_{[m]} = A_M, \quad B_{[m]} = B_M, \quad C_{[m]} = C_M \tag{138}$$

and the definition of $\hat{S}_{[m]}$, $N_{[m]}$ and $M_{[m]}$ are given by

$$N_{[m]}(x_{m+1}) = A_M^T P_M + P_M A_M, \quad M_{[m]}(x_{m+1}) = \begin{bmatrix} M_{S[m]} & \begin{matrix} C_M^T L_{[m]} \\ 0 \end{matrix} \\ \begin{matrix} L_{[m]} C_M & 0 \end{matrix} & -L_{[m]} \end{bmatrix} \tag{139}$$

$$M_{S[m]}(x_{m+1}) = \begin{bmatrix} A_M^T P_M + P_M A_M & P_M B_M + C_M^T R_{[m]}^T \\ B_M^T P_M + R_{[m]} C_M & -L_{[m]} \end{bmatrix} \tag{140}$$

This section uses the following assumptions.

Assumption 1 *There exists a constant symmetric matrix $P_{[m]} > 0$ such that $N_{[m]}(x_{m+1}) < 0$ is satisfied for all $x_{m+1} \in \mathcal{R}$.*

Assumption 2 *There exist a constant symmetric matrix $P_{[m]} > 0$, $\mathbf{C}^0$ functions $L_{[m]}(x_{m+1}) \in \mathbf{L}_{[m]}$ and $R_{[m]}(x_{m+1}) \in \mathbf{R}_{[m]}$ such that $M_{[m]}(x_{m+1}) < 0$ is satisfied for all $x_{m+1} \in \mathcal{R}$.*

According to Theorem 7, Assumption 1 says that the x_M-subsystem is globally uniformly asymptotically stable. Assumption 2 requires the x_M-subsystem is globally uniformly asymptotically stable for all admissible uncertainties appearing in the x_M-subsystem. It is obvious that the properties in Theorem 2 are satisfied for $M_{[k]}$ and $N_{[k]}$ of this section. Thereby, the following procedures are proposed.
Nominal backstepping : Solve $N_{[k]} < 0$ for s_k from $k = m+1$ through $k = n$.
Robust backstepping : Solve $M_{[k]} < 0$ for $\{s_k, L_k, R_k\}$ from $k = m+1$ through $k = n$.
Note that the backstepping is only required for the measured part of the state. We can prove the following.

Lemma 6 *Suppose $1 \le k \le n-m$.*
(i) $N_{[m+k]}(x_{m+[k]}) < 0$ is equivalent to

$$\begin{bmatrix} N_{[m]}(x_{m+1}) & \tilde{\Phi}_{m+1}(x_{m+1}) \\ \tilde{\Phi}_{m+1}^T(x_{m+1}) & \tilde{\Psi}_{m+1}(x_{m+1}) \end{bmatrix} < 0 \quad for \quad k = 1 \tag{141}$$

$$\begin{bmatrix} N_{[m+k-1]}(x_{m+[k-1]}) & \tilde{\Phi}_{m+k}(x_{m+[k]}) \\ \tilde{\Phi}_{m+k}^T(x_{m+[k]}) & \tilde{\Psi}_{m+k}(x_{m+[k]}) \end{bmatrix} < 0 \quad for \quad 2 \le k \le n-m \; , \tag{142}$$

where $\tilde{\Phi}_{m+k}$ depends only on $(s_{m+1}, \cdots, s_{k-1})$ and their partial derivatives. The symmetric matrix $\tilde{\Psi}_{m+k}$ depends on s_{m+k}.
(ii) $M_{[m+k]}(x_{m+[k]}) < 0$ is equivalent to

$$\begin{bmatrix} M_{[m]}(x_{m+1}) & \Phi_{m+1}(x_{m+1}) \\ \Phi_{m+1}^T(x_{m+1}) & \Psi_{m+1}(x_{m+1}) \end{bmatrix} < 0 \quad for \quad k = 1 \tag{143}$$

$$\begin{bmatrix} M_{[m+k-1]}(x_{m+[k-1]}) & \Phi_{m+k}(x_{m+[k]}) \\ \Phi_{m+k}^T(x_{m+[k]}) & \Psi_{m+k}(x_{m+[k]}) \end{bmatrix} < 0 \quad for \quad 2 \le k \le n-m \; , \tag{144}$$

where Φ_{m+k} depends only on $(L_{[m+k]}, R_{[m+k]})$ and $(s_{m+1}, \cdots, s_{m+k-1})$ and their partial derivatives. The symmetric matrix Ψ_{m+k} depends on (L_{m+k}, R_{m+k}) and s_{m+k}.

The next theorem is proved by following the argument of state-feedback control(Ito and Freeman, 1999*b*).

Lemma 7 *Let k be an integer in $[1, n-m]$.*
(i) Assume that

$$\begin{cases} N_{[m]}(x_{m+1}) < 0, \quad \forall x_{m+1} \in \mathcal{R} & if \quad k = 1 \\ N_{[m+k-1]}(x_{m+[k-1]}) < 0, \quad \forall x_{m+[k-1]} \in \mathcal{R}^{k-1} & if \quad 2 \le k \le n-m \end{cases} \tag{145}$$

holds. Then there always exist a smooth function $s_{m+k}(x_{m+[k]})$ *such that*

$$\begin{cases} \tilde{\Psi}_k(x_{m+1}) - \tilde{\Phi}_k^T(x_{m+1}) N_{[m+1]}^{-1}(x_{m+1}) \tilde{\Phi}_k(x_{m+1}) < 0 & \text{if } k = 1 \\ \tilde{\Psi}_k(x_{m+[k]}) - \tilde{\Phi}_k^T(x_{m+[k]}) N_{[m+k-1]}^{-1}(x_{m+[k-1]}) \tilde{\Phi}_k(x_{m+[k]}) < 0 & \text{if } 2 \le k \le n-m \end{cases} \tag{146}$$

is satisfied for all $x_{m+[k]} \in \mathcal{R}^k$.
(ii) Assume that

$$\begin{cases} M_{[m]}(x_{m+1}) < 0, \quad \forall x_{m+1} \in \mathcal{R} & \text{if } k = 1 \\ M_{[m+k-1]}(x_{m+[k-1]}) < 0, \quad \forall x_{m+[k-1]} \in \mathcal{R}^{k-1} & \text{if } 2 \le k \le n-m \end{cases} \tag{147}$$

holds. Then there always exist a $\mathbf{C}^0$ *function* $\lambda_{m+k}(x_{m+[k]})$ *and a smooth function* $s_{m+k}(x_{m+[k]})$ *such that*

$$\begin{cases} \Psi_k(x_{m+1}) - \Phi_k^T(x_{m+1}) M_{[m+1]}^{-1}(x_{m+1}) \Phi_k(x_{m+1}) < 0 & \text{if } k = 1 \\ \Psi_k(x_{m+[k]}) - \Phi_k^T(x_{m+[k]}) M_{[m+k-1]}^{-1}(x_{m+[k-1]}) \Phi_k(x_{m+[k]}) < 0 & \text{if } 2 \le k \le n-m \end{cases} \tag{148}$$

$$\lambda_{m+k}(x_{m+[k]}) > 0 \tag{149}$$

are satisfied for all $x_{m+[k]} \in \mathcal{R}^k$ *with* $L_k = \lambda_k I_{p_k}$ *and* $R_k = 0$.

The explicit formulas of analytical solutions $\{s_k, \lambda_k\}$ to the problems of the above theorem are the same as those of the state-feedback case(Ito and Freeman, 1999*b*). By using Lemma 7 recursively from $k = 1$ through $k = n - m$, we directly obtain the main results of this section.

Theorem 8 *If Assumption 1 is satisfied, then, the nominal nonlinear system* Σ_{02} *can be globally uniformly asymptotically stabilized by the static feedback law (122) with a smooth function* K.

Theorem 9 *Suppose that Assumption 2 is satisfied.*

(i) *Assume that the uncertainty* Σ_Δ *only has static uncertain components* Δ_{is} *and* Δ_{ir}. *The system* Σ_{P2} *can be globally uniformly asymptotically stabilized for any admissible uncertainty by the static feedback law (122) with a smooth function* K.

(ii) *Assume that the uncertainty* Σ_Δ *has dynamic uncertain components* Δ_{id}. *The system* Σ_{P2} *can be semi-globally uniformly asymptotically stabilized for any admissible uncertainty by the static feedback law (122) with a smooth function* K.

Theorem 10 *Assume that there exist a constant matrix* $P_M > 0$, *a constant scaling* $L_{[m]} \in \mathbf{L}_{[m]}$ *and a constant number* $\nu > 0$ *such that* $M_{S[m]} + \nu I < 0$ *holds for all* $x_{m+1} \in \mathcal{R}$ *with* $R_{[m]} = 0$. *Then, the system* Σ_{02} *can be made ISS by the static feedback law (122) with a smooth function* K.
Proof : *If* $M_{S[m]} + \nu I < 0$ *holds for all* $x_{m+1} \in \mathcal{R}$, *there exists a* $\mathbf{C}^0$ *function* $C_M(x_{m+1})$ *such that* $M_{[m]} + \nu I < 0$ *holds for all* $x_{m+1} \in \mathcal{R}$ *with the constant* $L_{[m]}$ *and* $R_{[m]} = 0$. *The rest of the proof is completed in the same way as (Ito and Freeman, 1999*a*).*

The results of this section are similar to the result of Chapter 7 in Freeman and Kokotović (1996) which considers a tracking control problem without uncertainty. This section considers stabilization of uncertain systems. Assumption 1 is identical with the assumption in Freeman and Kokotović (1996) for nominal stabilization. The paper extends their observation to the robust stabilization. In addition, this section shows that controllers can be static for global stabilization against a general class of structured uncertainties, while tracking problems which cause change of the equilibrium requires dynamic controllers.

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References

Apkarian, P. and R.J. Adams (1998). Advanced gain-scheduling techniques for uncertain systems. *IEEE Trans. Contr. Syst. Tech.* **6**, 21–32.

Esfandiari, F. and H.K. Khalil (1992). Output feedback stabilization of fully linearizable systems. *Int. J. Contr* **56**, 1007–1037.

Freeman, R.A. and P.V. Kokotović (1996). *Robust nonlinear control design: State-space and Lyapunov techniques*. Birkhäuser. Boston.

Ito, H. (1998*a*). Semi-global robust nonlinear control: State-dependent scaling and computational aspects. In: *Proc. 1998 Amer. Contr. Conf.*. pp. 1663–1668.

Ito, H. (1998*b*). State-dependent scaling for robust nonlinear control: Techniques and effectiveness. In: *Proc. 37th IEEE Conf. Decis. Contr.*. pp. 4111–4114.

Ito, H. (1999). Region maximization technique in robust nonlinear control via state-dependent scaling design. In: *Proc. 1999 Amer. Contr. Conf.* to appear.

Ito, H. and M. Krstić (1999). Recursive state-dependent scaling design of robust output feedback control for global stabilization. submitted to 38th IEEE Conf. Decis. Contr.

Ito, H. and R.A. Freeman (1999*a*). Generalized state-dependent scaling: Backstepping for local optimality, global inverse optimality, and global robust stability. In: *Proc. 1999 European Contr. Conf.* to appear.

Ito, H. and R.A. Freeman (1999*b*). A new look at robust backstepping through state-dependent scaling design. In: *Proc. 1999 Amer. Contr. Conf.* to appear.

Kanellakopoulos, I. (1991). Adaptive control of Nonlinear Systems. PhD thesis. University of Illinois, Urbana.

Khalil, H.K. and F. Esfandiari (1993). Semi-global stabilization of a class of nonlinear systems using output feedback. *IEEE Trans. Autom. Contr.* **38**, 1412–1415.

Krstić, M. and Z. Li (1998). Inverse optimal design of input-to-state stabilizing nonlinear controllers. *IEEE Trans. Automat. Control* **43**, 336–350.

Krstić, M., I. Kanellakopoulos and P.V. Kokotović (1995). *Nonlinear and adaptive control design*. John Wiley & Sons. Now York.

Lin, Z. and A. Saberi (1995). Robust semi-global stabilization of minimum-phase input-output linearizable systems via partial state and output feedback. *IEEE Trans. Autom. Contr.* **40**, 1029–1041.

Lu, W.-M. and J.C. Doyle (1995). $\mathcal{H}^\infty$ control of nonlinear systems: A convex characterization. *IEEE Trans. Automat. Contr.* **40**, 1668–1675.

Sepulchre, R., M. Janković and P.V. Kokotović (1997). *Constructive nonlinear control.* Springer-Verlag. New York.

Teel, A.R. and L. Praly (1994). Global stabilizability and observability imply semi-global stabilizability by output feedback. *Syst. Contr. Lett.* **22**, 313–325.

Teel, A.R. and L. Praly (1995). Tools for semi-global stabilization by partial state feedback and output feedback. *SIAM J. Contr. and Optim.* **33**, 1443–1488.

Wu, F., X. Yang, A. Packard and G. Becker (1996). Induced $\mathcal{L}_2$-norm control for LPV systems with bounded parameter variation rates. *Int. J. Robust Nonlinear Contr.* **6**, 983–998.