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SOME SIMILARITY BETWEEN CONTRACTIONS AND KANNAN MAPPINGS II

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Abstract

In Kikkawa-Suzuki [Some similarity between contractions and Kannan mappings, Fixed Point Theory Appl. doi:10.1155/2007/49749], we discussed a similarity between contractions and Kannan mappings. In this paper, we continue to discuss a similarity between contractions and generalized Kannan mappings—M-Kannan mappings.

1. Introduction

The Banach contraction principle [1] is a very famous theorem in nonlinear analysis and has many useful applications and generalizations. See [2, 4–7, 12, 13, 35, 36] and others.

Theorem 1 ([1]). *Let (X,d) be a complete metric space. Let T be a contraction on X, i.e., there exists $r \in [0,1)$ satisfying*

$$d(Tx,Ty) \le rd(x,y)$$

for all $x, y \in X$. Then T has a unique fixed point.

On the other hand, in 1969, Kannan [9] proved the following fixed point theorem.

Theorem 2 ([9]). *Let (X,d) be a complete metric space. Let T be a Kannan mapping on X, i.e., there exists $\alpha \in [0,\frac{1}{2})$ such that*

$$d(Tx,Ty) \le \alpha(d(x,Tx) + d(y,Ty))$$

for all $x, y \in X$. Then T has a unique fixed point.

Contractions are always continuous and Kannan mappings are not necessarily continuous. This is a very big difference between both mappings. Also, we note that Kannan's fixed point theorem is not an extension of the Banach contraction principle.

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That is, there exist a contraction which is not Kannan, and a Kannan mapping which is not a contraction. Thus, we cannot compare both conditions directly.

We know that a metric space X is complete if and only if every Kannan mapping has a fixed point, while there exists a metric space X such that X is not complete and every contraction on X has a fixed point; see [3, 15]. Thus, the Banach contraction principle does not characterize the metric completeness of X. We can say that the notion of contractions is stronger in a sense. Recently Suzuki [30] proved a slight generalization of the Banach contraction principle which characterizes the metric completeness of X. See also [10, 31].

THEOREM 3 ([30]). *Define a nonincreasing function θ from $[0,1)$ onto $\left(\frac{1}{2},1\right]$ by*

$$\theta(r)=\begin{cases}1 & \text{if } 0\le r\le \frac{1}{2}(\sqrt{5}-1),\\ \dfrac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1)\le r\le \dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}}\le r<1.\end{cases}$$

Then for a metric space (X,d), the following are equivalent:

(i) *X is complete.*

(ii) *Every mapping T on X satisfying the following has a fixed point:*

- *There exists $r\in[0,1)$ such that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$ for all $x,y\in X$.*

REMARK. $\theta(r)$ is the best constant for every r.

Furthermore Kikkawa and Suzuki [11] proved a Kannan version of Theorem 3.

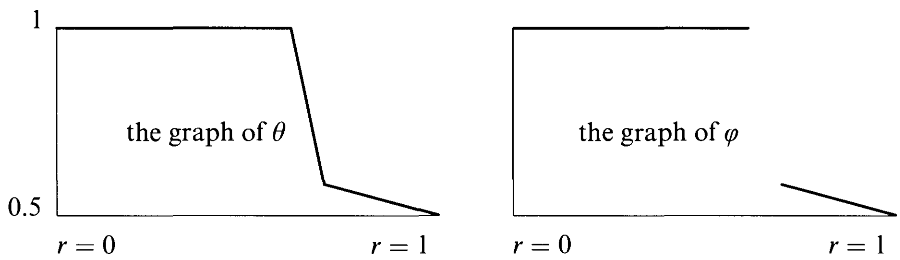
THEOREM 4 ([11]). *Define a nonincreasing function φ from $[0,1)$ into $\left(\frac{1}{2},1\right]$ by*

$$\varphi(r)=\begin{cases}1 & \text{if } 0\le r<\dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}}\le r<1.\end{cases}$$

Let (X,d) be a complete metric space and let T be a mapping on X. Let $\alpha\in\left[0,\frac{1}{2}\right)$ and put $r:=\frac{\alpha}{1-\alpha}\in[0,1)$. Assume that

$$(1)\qquad \varphi(r)d(x,Tx)\le d(x,y)\quad \text{implies}\quad d(Tx,Ty)\le \alpha d(x,Tx)+\alpha d(y,Ty)$$

for all $x,y\in X$, Then T has a unique fixed point.



Remark. $\varphi(r)$ is the best constant for every r.

We note that θ and φ are similar, but $\theta(r) \neq \varphi(r)$ for some r. Since $\theta(r) \le \varphi(r)$ for every r, we can say that Kannan is stronger in another sense. The authors were very surprised by Theorem 4 because they guessed that $\theta(r)$ is best in Theorem 4 when they were proving it. Then they proved another theorem where $\theta(r)$ is best.

Theorem 5 ([11]). *Define a function θ as in Theorem 3. Let (X,d) be a complete metric space and let T be a mapping on X. Suppose that there exists $r \in [0,1)$ such that*

$$(2) \qquad \theta(r)d(x,Tx) \le d(x,y) \qquad \textit{implies} \qquad d(Tx,Ty) \le r\max\{d(x,Tx), d(y,Ty)\}$$

for all $x, y \in X$. Then T has a unique fixed point.

Remark. $\theta(r)$ is the best constant for every r.

We call a mapping T on X *M-Kannan* if there exists $r \in [0,1)$ such that

$$d(Tx,Ty) \le r\max\{d(x,Tx), d(y,Ty)\}$$

for all $x, y \in X$. By Theorems 3–5, we can guess that the notion of M-Kannan mappings is more similar to that of contractions than that of Kannan mappings is.

Using the notion of τ-distances, Suzuki [23] considered some weaker contractions and Kannan mappings and proved the following (Theorem 6):

- If T is a contraction with respect to a τ-distance, then T is Kannan with respect to another τ-distance.
- If T is Kannan with respect to a τ-distance, then T is a contraction with respect to another τ-distance.

That is, the τ-distance versions of both conditions are equivalent.

So, from the above-mentioned thing, it is a very natural question whether the τ-distance versions of contractions and M-Kannan mappings are equivalent. In this paper, we shall give the positive answer to the question. Therefore we can still guess that M-Kannan is more similar to contraction than Kannan is.

2. Preliminaries

Throughout this paper we denote by $\mathbf{N}$ the set of positive integers.

In [18], Suzuki introduced the notion of τ-distance in order to generalize the results of Kada *et al* [8], Tataru [35], Zhong [36, 37] and others. Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ1) $p(x,z) \le p(x,y) + p(y,z)$ for all $x, y, z \in X$;

(τ2) $\eta(x,0) = 0$ and $\eta(x,t) \ge t$ for all $x \in X$ and $t \in [0,\infty)$, and η is concave and continuous in its second variable;

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w,x) \le \liminf_n p(w, x_n)$ for all $w \in X$;

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. See [8, 14, 16–29, 32–34] for useful examples and theorems. The following is a key lemma in this paper.

LEMMA 1 ([23]). *Let (X, d) be a metric space and let p be a τ-distance on X. Let T be a mapping on X and let u be a point of X such that*

$$\lim_{m,n\to\infty} p(T^m u, T^n u) = 0.$$

Then for every $x \in X$, $\lim_k p(T^k u, x)$ and $\lim_k p(x, T^k u)$ exist. Moreover, define functions β and γ from X into $[0,\infty)$ by

$$\beta(x) = \lim_{k\to\infty} p(T^k u, x) \qquad \text{and} \qquad \gamma(x) = \lim_{k\to\infty} p(x, T^k u).$$

Then the following hold:

(i) *A function q_1 from $X \times X$ into $[0,\infty)$ defined by*

$$q_1(x,y) = \beta(x) + \beta(y)$$

is a symmetric τ-distance on X.

(ii) *A function q_2 from $X \times X$ into $[0,\infty)$ defined by*

$$q_2(x,y) = \gamma(x) + \beta(y)$$

is a τ-distance on X.

We denote by $\tau(X)$ the set of all τ-distances on a metric space (X, d). A τ-distance p on X is called *symmetric* if $p(x,y) = p(y,x)$ for all $x, y \in X$. We also denote by $\tau_0(X)$ the set of all symmetric τ-distances on X. It is obvious that $d \in \tau_0(X) \subset \tau(X)$. We denote by $TC(X)$ the set of all mappings T on X such that there exist $p \in \tau(X)$ and $r \in [0,1)$ satisfying

$$p(Tx, Ty) \le r p(x,y)$$

for all $x, y \in X$. We define sets $TK(X)$, $TC_0(X)$, $TK_0(X)$, $TC_2(X)$ and $TC_3(X)$ of mappings on X as follows: $T \in TK(X)$ if and only if there exist $p \in \tau(X)$ and $\alpha \in [0, \frac{1}{2})$ satisfying either of the following holds:

$$p(Tx, Ty) \le \alpha(p(Tx, x) + p(Ty, y))$$

for all $x, y \in X$, or

$$p(Tx, Ty) \le \alpha(p(Tx, x) + p(y, Ty))$$

for all $x, y \in X$. $T \in TC_0(X)$ if and only if there exist $p \in \tau_0(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le rp(x, y)$$

for all $x, y \in X$. $T \in TK_0(X)$ if and only if there exist $p \in \tau_0(X)$ and $\alpha \in [0, \frac{1}{2})$ satisfying

$$p(Tx, Ty) \le \alpha(p(Tx, x) + p(Ty, y))$$

for all $x, y \in X$. $T \in TC_2(X)$ if and only if there exist $p \in \tau(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le rp(y, x)$$

for all $x, y \in X$. $T \in TC_3(X)$ if and only if there exist $\ell \in \mathbf{N}$, $p \in \tau(X)$ and $r \in [0, 1)$ satisfying

$$p(T^\ell x, T^\ell y) \le rp(x, y)$$

for all $x, y \in X$. We recall that a mapping T on X belongs to $TC(X)$ if and only if T is a contraction with respect to some τ-distance p on X [18], and a mapping T on X belongs to $TK(X)$ if and only if T is Kannan with respect to some τ-distance p on X [20].

We know that the above six sets of mappings coincide completely.

Theorem 6 ([23]). *Let (X, d) be a metric space. Then*

$$TC_0(X) = TC(X) = TC_2(X) = TC_3(X) = TK_0(X) = TK(X)$$

holds.

We also know the following fixed point theorem.

Theorem 7 ([18, 20]). *Let (X, d) be a complete metric space and let T be a mapping on X belonging to $TC(X)$. Then T has a unique fixed point.*

3. Main result

In this section, we prove our main result. We define sets $TM(X)$ and $TM_0(X)$ of mappings on X as follows: $T \in TM(X)$ if and only if there exist $p \in \tau(X)$ and $r \in [0, 1)$ satisfying either of the following holds:

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(Ty, y)\}$$

for all $x, y \in X$, or

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(y, Ty)\}$$

for all $x, y \in X$. $T \in TM_0(X)$ if and only if there exist $p \in \tau_0(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(Ty, y)\}$$

for all $x, y \in X$.

The following is our main result.

THEOREM 8. *Let (X, d) be a metric space. Then*

$$TC_0(X) = TC(X) = TM_0(X) = TM(X)$$

holds.

In order to prove it, we need some lemmas. In the following lemmas and the proof of Theorem 8, we define sets $TM_1(X)$ and $TM_2(X)$ of mappings on X as follows: $T \in TM_1(X)$ if and only if there exist $p \in \tau(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(Ty, y)\}$$

for all $x, y \in X$. $T \in TM_2(X)$ if and only if there exist $p \in \tau(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(y, Ty)\}$$

for all $x, y \in X$. We note $TM(X) = TM_1(X) \cup TM_2(X)$.

LEMMA 2. *For every metric space X,*

$$TM_1(X) \subset TC_3(X)$$

holds.

PROOF. Fix $T \in TM_1(X)$. Then there exist $p \in \tau(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(Ty, y)\}$$

for all $x, y \in X$. So, $p(T^2x, Tx) \le r \max\{p(T^2x, Tx), p(Tx, x)\}$ holds for $x \in X$. If $p(T^2x, Tx) > p(Tx, x)$, we have $1 \le r$. This is a contradiction. Thus,

$$p(T^2x, Tx) \le rp(Tx, x)$$

for all $x \in X$. Using this, we have $p(T^{n+1}x, T^n x) \le r^n p(Tx, x)$ for $n \in \mathbf{N}$ and $x \in X$. Fix $u \in X$. Then for $m, n \in \mathbf{N}$, we have

$$\begin{aligned} p(T^m u, T^n u) &\le r \max\{p(T^m u, T^{m-1}u), p(T^n u, T^{n-1}u)\} \\ &\le r \max\{r^{m-1}, r^{n-1}\} p(Tu, u) \end{aligned}$$

and hence $\lim_{m,n} p(T^m u, T^n u) = 0$. So, by Lemma 1, $\beta(x) = \lim_k p(T^k u, x)$ is well-defined for every $x \in X$, and a function q from $X \times X$ into $[0, \infty)$ defined by

$$q(x, y) = \beta(x) + \beta(y)$$

for $x, y \in X$ is a τ-distance. Since

$$\begin{aligned} p(Tx, x) &\le \lim_{k\to\infty} (p(Tx, T^k u) + p(T^k u, x)) \\ &\le \lim_{k\to\infty} (r \max\{p(Tx, x), p(T^k u, T^{k-1} u)\} + p(T^k u, x)) \\ &= rp(Tx, x) + \beta(x), \end{aligned}$$

we have

$$p(Tx, x) \le \frac{1}{1-r}\beta(x)$$

for $x \in X$. Fix $\ell \in \mathbf{N}$ with $\frac{r^\ell}{1-r} \le r$. Then we have

$$\begin{aligned} \beta(T^\ell x) &= \lim_{k\to\infty} p(T^k u, T^\ell x) \\ &\le \lim_{k\to\infty} r \max\{p(T^k u, T^{k-1} u), p(T^\ell x, T^{\ell-1} x)\} \\ &\le \lim_{k\to\infty} r \max\{p(T^k u, T^{k-1} u), r^{\ell-1} p(Tx, x)\} \\ &= r^\ell p(Tx, x) \\ &\le \frac{r^\ell}{1-r}\beta(x) \\ &\le r\beta(x) \end{aligned}$$

for all $x \in X$. So, we have

$$q(T^\ell x, T^\ell y) = \beta(T^\ell x) + \beta(T^\ell y) \le r(\beta(x) + \beta(y)) = rq(x, y)$$

for $x, y \in X$. This implies $T \in TC_3(X)$. □

LEMMA 3. *For every metric space* X,

$$TM_2(X) \subset TC_3(X)$$

holds.

PROOF. Fix $T \in TM_2(X)$. Then there exist $p \in \tau(X)$ and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r \max\{p(Tx, x), p(y, Ty)\}$$

for all $x, y \in X$. Since

$$p(Tx, T^2x) \le r \max\{p(Tx, x), p(Tx, T^2x)\}$$

and

$$p(T^2x, Tx) \le r \max\{p(T^2x, Tx), p(x, Tx)\},$$

we have $p(Tx, T^2x) \le rp(Tx, x)$ and $p(T^2x, Tx) \le rp(x, Tx)$ for $x \in X$. Thus,

$$p(T^{2m}x, T^{2m+1}x) \le r^{2m}p(x, Tx),$$

$$p(T^{2m}x, T^{2m-1}x) \le r^{2m-1}p(x, Tx),$$

$$p(T^{2m+1}x, T^{2m}x) \le r^{2m}p(Tx, x)$$

and

$$p(T^{2m+1}x, T^{2m+2}x) \le r^{2m+1}p(Tx, x)$$

hold for $m \in \mathbf{N}$ and $x \in X$. Fix $u \in X$. Then for $m, n \in \mathbf{N}$, we have

$$\begin{aligned} p(T^m u, T^n u) &\le r \max\{p(T^m u, T^{m-1}u), p(T^{n-1}u, T^n u)\} \\ &\le \max\{r^m, r^n\}(p(Tu, u) + p(u, Tu)) \end{aligned}$$

and hence $\lim_{m,n} p(T^m u, T^n u) = 0$. So, by Lemma 1, $\beta(x) = \lim_k p(T^k u, x)$ and $\gamma(x) = \lim_k p(x, T^k u)$ are well-defined for every $x \in X$, and a function q from $X \times X$ into $[0, \infty)$ defined by

$$q(x, y) = \gamma(x) + \beta(y)$$

for $x, y \in X$ is a τ-distance. Since

$$\begin{aligned} p(x, Tx) &\le \lim_{k \to \infty} (p(x, T^k u) + p(T^k u, Tx)) \\ &\le \lim_{k \to \infty} (p(x, T^k u) + r \max\{p(T^k u, T^{k-1}u), p(x, Tx)\}) \\ &= \gamma(x) + rp(x, Tx), \end{aligned}$$

we have

$$p(x, Tx) \le \frac{1}{1 - r}\gamma(x)$$

for $x \in X$. Since

$$\begin{aligned} p(Tx, x) &\le \lim_{k \to \infty} (p(Tx, T^k u) + p(T^k u, x)) \\ &\le \lim_{k \to \infty} (r \max\{p(Tx, x), p(T^{k-1}u, T^k u)\} + p(T^k u, x)) \\ &= rp(Tx, x) + \beta(x), \end{aligned}$$

we also have

$$p(Tx, x) \leq \frac{1}{1-r}\beta(x)$$

for $x \in X$. Fix $\ell \in \mathbf{N}$ with $\frac{r^{2\ell+1}}{1-r} \leq \sqrt{r}$. Then we have

$$\begin{aligned}
\beta(T^{2\ell+1}x) &= \lim_{k\to\infty} p(T^k u, T^{2\ell+1}x) \\
&\leq \lim_{k\to\infty} r \max\{p(T^k u, T^{k-1}u), p(T^{2\ell}x, T^{2\ell+1}x)\} \\
&\leq \lim_{k\to\infty} r \max\{p(T^k u, T^{k-1}u), r^{2\ell} p(x, Tx)\} \\
&= r^{2\ell+1} p(x, Tx) \\
&\leq \frac{r^{2\ell+1}}{1-r}\gamma(x) \\
&\leq \sqrt{r}\gamma(x)
\end{aligned}$$

and

$$\begin{aligned}
\gamma(T^{2\ell+1}x) &= \lim_{k\to\infty} p(T^{2\ell+1}x, T^k u) \\
&\leq \lim_{k\to\infty} r \max\{p(T^{2\ell+1}x, T^{2\ell}x), p(T^{k-1}u, T^k u)\} \\
&\leq \lim_{k\to\infty} r \max\{r^{2\ell} p(Tx, x), p(T^{k-1}u, T^k u)\} \\
&= r^{2\ell+1} p(Tx, x) \\
&\leq \frac{r^{2\ell+1}}{1-r}\beta(x) \\
&\leq \sqrt{r}\beta(x).
\end{aligned}$$

Therefore we obtain

$$q(T^{2\ell+1}x, T^{2\ell+1}y) = \gamma(T^{2\ell+1}x) + \beta(T^{2\ell+1}y) \leq \sqrt{r}(\beta(x) + \gamma(y)) = \sqrt{r}q(y, x)$$

for all $x, y \in X$ and hence

$$q(T^{4\ell+2}x, T^{4\ell+2}y) \leq \sqrt{r}q(T^{2\ell+1}y, T^{2\ell+1}x) \leq rq(x, y)$$

for $x, y \in X$. This implies $T \in TC_3(X)$. □

PROOF OF THEOREM 8. By Theorem 6 and Lemmas 2 and 3, we have

$$TM(X) \subset TC_3(X) = TC_0(X) = TK_0(X).$$

Also, it is obvious

$$TK_0(X) \subset TM_0(X) \subset TM(X).$$

Therefore we obtain the desired result. □

4. Additional results

In [31], we proved a mapping in Theorem 3 belongs to $TC(X)$. Motivated by this thing, we shall prove that mappings in Theorems 4 and 5 also belong to $TC(X)$.

THEOREM 9. *Let (X,d) be a metric space. Let T be a mapping on X satisfying the assumption of Theorem 4. Then T belongs to $TC(X)$.*

PROOF. Let T be a mapping on X satisfying (1) for some $\alpha \in [0, \frac{1}{2})$. Put $r := \frac{\alpha}{1-\alpha} \in [0,1)$ and fix $u \in X$. We proved in [11] the following:

- For every $x \in X$, $\{T^n x\}$ is a Cauchy sequence in X.
- If the limit of $\{T^n x\}$ exists, then the limit is a fixed point of T.
- $d(T^{n+1}u, Tx) \le \alpha d(T^n u, T^{n+1}u) + \alpha d(x, Tx)$ holds for sufficiently large $n \in \mathbf{N}$ provided $x \in X$ satisfies $x \ne \lim_j T^j u$.

Therefore we can define a function β from X into $[0,\infty)$ by $\beta(x) = \lim_n d(T^n u, x)$. From Lemma 1, a function p from $X \times X$ into $[0,\infty)$ by $p(x,y) = \beta(x) + \beta(y)$ is a τ-distance.

We shall show $\beta(Tx) \le r\beta(x)$ for $x \in X$. In the case where $\{T^n u\}$ does not converge to x, we have

$$\begin{aligned}\beta(Tx) &= \lim_{n\to\infty} d(T^{n+1}u, Tx) \\ &\le \lim_{n\to\infty} (\alpha d(T^n u, T^{n+1}u) + \alpha d(x, Tx)) \\ &= \alpha d(x, Tx) \\ &\le \alpha \lim_{n\to\infty} (d(x, T^n u) + d(T^n u, Tx)) \\ &= \alpha\beta(x) + \alpha\beta(Tx)\end{aligned}$$

and hence $\beta(Tx) \le r\beta(x)$. In the other case, where $\{T^n u\}$ converges to x, we have $Tx = x$. Since $\beta(x) = 0$, we have $\beta(Tx) = \beta(x) = 0 = r\beta(x)$. Therefore $\beta(Tx) \le r\beta(x)$ in both cases.

Hence we obtain $p(Tx, Ty) = \beta(Tx) + \beta(Ty) \le r\beta(x) + r\beta(y) = rp(x,y)$ for all $x, y \in X$. This implies $T \in TC(X)$. □

THEOREM 10. *Let (X,d) be a metric space. Let T be a mapping on X satisfying the assumption of Theorem 5. Then T belongs to $TC(X)$.*

PROOF. Let T be a mapping on X satisfying (2) for some $r \in [0,1)$. Fix $u \in X$. We proved in [11] the following:

- For every $x \in X$, $\{T^n x\}$ is a Cauchy sequence in X.
- If the limit of $\{T^n x\}$ exists, then the limit is a fixed point of T.
- For every $x \in X$, $d(Tx, T^2 x) \le r d(x, Tx)$ holds.
- $d(T^n u, Tx) \le r \max\{d(T^{n-1}u, T^n u), d(x, Tx)\}$ holds for sufficiently large $n \in \mathbf{N}$ provided $x \in X$ satisfies $x \ne \lim_j T^j u$.

We can define a function β from X into $[0,\infty)$ by $\beta(x) = \lim_n d(T^n u, x)$. Using Lemma 1, a function p from $X \times X$ into $[0,\infty)$ by $p(x,y) = \beta(x) + \beta(y)$ is a τ-distance.

Fix $\ell \in \mathbf{N}$ with $\frac{r^\ell}{1-r} \le r$. We shall show $\beta(T^\ell x) \le r\beta(x)$ for all $x \in X$. In the case where $\{T^n u\}$ converges to either x or $T^{\ell-1}x$, since the limit is a fixed point of T, the limit and $T^\ell x$ coincide. Thus $\beta(T^\ell x) = 0$ holds. So we have $\beta(T^\ell x) \le r\beta(x)$. In the other case, where $\{T^n u\}$ converges to neither x nor $T^{\ell-1}x$, we have

$$\begin{aligned}\beta(T^\ell x) &= \lim_{n\to\infty} d(T^n u, T^\ell x)\\ &\le \lim_{n\to\infty} r \max\{d(T^{n-1}u, T^n u), d(T^{\ell-1}x, T^\ell x)\}\\ &\le \lim_{n\to\infty} r \max\{d(T^{n-1}u, T^n u), r^{\ell-1} d(x, Tx)\}\\ &= r^\ell d(x, Tx).\end{aligned}$$

Since

$$\begin{aligned}d(x, Tx) &\le \lim_{n\to\infty} (d(x, T^n u) + d(T^n u, Tx))\\ &\le \lim_{n\to\infty} (d(x, T^n u) + r \max\{d(T^{n-1}u, T^n u), d(x, Tx)\})\\ &= \beta(x) + r d(x, Tx),\end{aligned}$$

we have $d(x, Tx) \le \frac{1}{1-r}\beta(x)$. Therefore

$$\beta(T^\ell x) \le r^\ell d(x, Tx) \le \frac{r^\ell}{1-r}\beta(x) \le r\beta(x)$$

holds for $x \in X$. We have shown $\beta(T^\ell x) \le r\beta(x)$ in both cases.

Hence we obtain

$$p(T^\ell x, T^\ell y) = \beta(T^\ell x) + \beta(T^\ell y) \le r\beta(x) + r\beta(y) = rp(x,y)$$

for all $x, y \in X$. This implies $T \in TC_3(X)$. From Theorem 6, we obtain $T \in TC(X)$. □

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