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ON COMPLEMENTARY DUALS—BOTH FIXED POINTS (II)—

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Abstract

We consider a quadratic minimization (primal) problem with both fixed endpoints and its associated maximization (dual) problem from a view point of complementarity. We show that a new complementary identity produces the pair with an equality condition. The condition is a linear system of $(2n+1)$-equation on $(2n+1)$-variable. The system yields a couple of solutions; one is a minimum solution to n-variable primal and the other is a maximum one to $(n+1)$-variable dual. Both the solutions turn out to be complementary. The optimal solution is characterized by the backward Fibonacci sequence. The duality is enhanced through conjugate function. The optimal solution is also given by dynamic programming.

1. Introduction

A wide class of dynamic optimization problems has been analyzed by R. Bellman and others [2, 3, 5, 4, 6, 8, 9, 10, 11, 12]. Recently a duality in quadratic programming has been established through several approaches [1, 7, 18, 19, 20, 21].

In this paper we propose a *new* complementary identity

$$(\mathrm{C}_n)\quad (c-x_1)\mu_1 + x_1(\mu_1-\mu_2) + \sum_{k=2}^{n}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_n-d)\mu_{n+1}$$
$$= c\mu_1 - d\mu_{n+1}.$$

(The *old* one was in [20]). This identity with an *elementary inequality* with equality condition

$$(\mathrm{EI})\quad 2xy \le x^2+y^2 \qquad \text{on } R^2;\ x=y$$

generates a pair (of primal and dual) with an equality condition. The condition is a linear system of $(2n+1)$-equation on $(2n+1)$-variable. The system yields a couple of solutions; one is a minimum solution to n-variable primal and the other is a maximum one to $(n+1)$-variable dual. Both the solutions are characterized by the Fibonacci sequence [13, 15, 22, 25].

Section 2 shows a complementary identity for the first three pairs and an n-th pair. Section 3 presents the three pairs with an equality condition. Section 4 solves the n-th paired problem—n-variable minimization and $(n+1)$-variable maximization—. Section 5 shows the complementary duality for the n-th pair. In Section 6, we discuss the duality through conjugate function [14, 16, 23]. Section 7 solves both the third pair and the n-th pair through dynamic programming [2, 17, 18, 24].

2. Complementary identities

First we present three elementary complementary identities, which take a fundamental role in analyzing respective pairs of primal and dual.

Let two real constants c, d be given. First we consider a pair of one-variable x and two-variable (λ, μ). Then an identity

$$(\mathrm{C}_1) \quad (c-x)\lambda + x(\lambda-\mu) + (x-d)\mu = c\lambda - d\mu$$

holds true.

Second we consider a pair of two-variable (x, y) and three-variable (λ, μ, ν). Then an identity

$$(\mathrm{C}_2) \quad (c-x)\lambda + x(\lambda-\mu) + (x-y)\mu + y(\mu-\nu) + (y-d)\nu = c\lambda - d\nu$$

holds true.

Third we consider three-variable/four-variable case. Let us divide a 3-dimensional vector and a 4-dimensional vector

$$(x, y, z), \qquad (\lambda, \mu, \nu, \xi)$$

into 7-dimensional ones

$$\begin{array}{ccccccc} (c-x, & x, & x-y, & y, & y-z, & z, & z-d), \\ (\lambda, & \lambda-\mu, & \mu, & \mu-\nu, & \nu, & \nu-\xi, & \xi), \end{array}$$

respectively. It turns out that the *inner product* is $c\lambda - d\xi$. Thus an identity

$$\begin{aligned} (\mathrm{C}_3) \quad & (c-x)\lambda + x(\lambda-\mu) + (x-y)\mu + y(\mu-\nu) + (y-z)\nu + z(\nu-\xi) + (z-d)\xi \\ & = c\lambda - d\xi \end{aligned}$$

holds true.

Finally we consider a general case. Let us divide an n-dimensional vector and an $(n+1)$-dimensional vector

$$(x_1, x_2, x_3, \ldots, x_k, \ldots, x_n),$$

$$(\mu_1, \mu_2, \mu_3, \ldots, \mu_k, \ldots, \mu_n, \mu_{n+1})$$

into $(2n+1)$-dimensional ones

$$\begin{aligned} &(c - x_1, x_1, x_1 - x_2, x_2, x_2 - x_3, \ldots, \\ &\quad x_{k-1}, x_{k-1} - x_k, x_k, \ldots, x_{n-1}, x_{n-1} - x_n, x_n, x_n - d), \\ &(\mu_1, \mu_1 - \mu_2, \mu_2, \mu_2 - \mu_3, \mu_3, \ldots, \\ &\quad \mu_{k-1} - \mu_k, \mu_k, \mu_k - \mu_{k+1}, \ldots, \mu_{n-1} - \mu_n, \mu_n, \mu_n - \mu_{n+1}, \mu_{n+1}), \end{aligned}$$

respectively. Then we make an inner product of resulting two ones. It turns out to be $c\mu_1 - d\mu_{n+1}$:

$$(\mathrm{C}_n) \quad (c - x_1)\mu_1 + x_1(\mu_1 - \mu_2) + \sum_{k=2}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})]$$

$$+ (x_{n-1} - x_n)\mu_n + x_n(\mu_n - \mu_{n+1}) + (x_n - d)\mu_{n+1} = c\mu_1 - d\mu_{n+1}.$$

This identity is called *complementary*.

3. Three pairs

In this paper as a pair of *primal*[1] and *dual*, we take n-variable optimization problems.

We present first three pairs as follows. The first pair is

$$(\mathrm{P}_1) \quad \begin{array}{ll} \text{minimize} & (c - x)^2 + x^2 + (x - d)^2 \\ \text{subject to} & \text{(i)} \quad x \in R^1 \end{array}$$

$$(\mathrm{D}_1) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2] - 2d\mu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu) \in R^2. \end{array}$$

The second is

$$(\mathrm{P}_2) \quad \begin{array}{ll} \text{minimize} & (c - x)^2 + x^2 + (x - y)^2 + y^2 + (y - d)^2 \\ \text{subject to} & \text{(i)} \quad (x, y) \in R^2 \end{array}$$

$$(\mathrm{D}_2) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2] - 2d\nu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu, \nu) \in R^3. \end{array}$$

The third is

$$(\mathrm{P}_3) \quad \begin{array}{ll} \text{minimize} & (c - x)^2 + x^2 + (x - y)^2 + y^2 + (y - z)^2 + z^2 + (z - d)^2 \\ \text{subject to} & \text{(i)} \quad (x, y, z) \in R^3 \end{array}$$

$$(\mathrm{D}_3) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2 + (\nu - \xi)^2 + \xi^2] - 2d\xi \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu, \nu, \xi) \in R^4. \end{array}$$

[1] Two nouns *primal* and *dual* mean *primal problem* and *dual problem*, respectively.

3.1. (P_1) **vs** (D_1)

Let us consider the first pair:

$$(\mathrm{P}_1) \quad \begin{array}{ll} \text{minimize} & (c-x)^2 + x^2 + (x-d)^2 \\ \text{subject to} & \text{(i)} \quad x \in R^1 \end{array}$$

$$(\mathrm{D}_1) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2] - 2d\mu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu) \in R^2. \end{array}$$

Then it turns out that both are dual to each other. It holds that

$$\begin{aligned} & 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2] - 2d\mu \\ & \quad \le (c-x)^2 + x^2 + (x-d)^2 \end{aligned}$$

for any feasible pair $(x; \lambda, \mu)$. An equality condition is

$$(\mathrm{EC}_1) \quad \begin{array}{ll} c - x = \lambda, & x = \lambda - \mu \\ x - d = \mu. & \end{array}$$

The equality condition (EC_1) is a linear system of 3-equation on 3-variable.

Let (x, λ, μ) be a solution. Then both sides become a common value with five expressions:

$$\begin{aligned} & (c-x)^2 + x^2 + (x-d)^2 \\ & = c(c-x) - d(x-d) \\ (5\mathrm{V}_1) \quad & = 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2] - 2d\mu \\ & = \lambda^2 + (\lambda-\mu)^2 + \mu^2 \\ & = c\lambda - d\mu. \end{aligned}$$

Then the primal (P_1) has a minimum value

$$\begin{aligned} m_1 &= (c-x)^2 + x^2 + (x-d)^2 \\ &= c(c-x) - d(x-d) \end{aligned}$$

at x, while the dual (D_1) has a maximum value

$$\begin{aligned} M_1 &= 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2] - 2d\mu \\ &= \lambda^2 + (\lambda-\mu)^2 + \mu^2 \\ &= c\lambda - d\mu \end{aligned}$$

at (λ, μ).

Lemma 3.1. (EC_1) *has indeed a unique solution:*

(1) $$x = \frac{1}{3}(c + d)$$

(2) $$(\lambda, \mu) = \frac{1}{3}(2c - d, c - 2d).$$

Proof. From (EC_1), we have a pair of linear systems of 1-equation on 1-variable and of 2-equation on 2-variable:

$$(EQ_1) \quad 3x = c + d \qquad \begin{aligned} 2\lambda - \mu &= c \\ -\lambda + 2\mu &= -d. \end{aligned}$$

The left system has a solution (1), while the right has a solution (2). □

The primal (P_1) has a minimum value

$$m_1 = c(c - \hat{x}) - d(\hat{x} - d) = \frac{1}{3}(2c^2 - 2cd + 2d^2)$$

at a path (point)

$$\hat{x} = \frac{1}{3}(c + d).$$

The dual (D_1) has a maximum value

$$M_1 = c\lambda^* - d\mu^* = \frac{1}{3}(2c^2 - 2cd + 2d^2)$$

at a path

$$(\lambda^*, \mu^*) = \frac{1}{3}(2c - d, c - 2d).$$

3.2. (P_2) vs (D_2)

Let us consider the second:

$$(P_2) \quad \begin{aligned} &\text{minimize} \quad (c - x)^2 + x^2 + (x - y)^2 + y^2 + (y - d)^2 \\ &\text{subject to} \quad \text{(i)} \quad (x, y) \in R^2 \end{aligned}$$

$$(D_2) \quad \begin{aligned} &\text{Maximize} \quad 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2] - 2d\nu \\ &\text{subject to} \quad \text{(i)} \quad (\lambda, \mu, \nu) \in R^2. \end{aligned}$$

Then both are dual to each other. It holds that

$$2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2] - 2d\nu \\ \le (c-x)^2 + x^2 + (x-y)^2 + y^2 + (y-d)^2$$

for any feasible pair $(x, y; \lambda, \mu, \nu)$. An equality condition is

$$(\mathrm{EC}_2) \quad \begin{aligned} c - x &= \lambda, & x &= \lambda - \mu \\ x - y &= \mu, & y &= \mu - \nu \\ y - d &= \nu. \end{aligned}$$

The equality condition (EC_2) is a linear system of 5-equation on 5-variable.

Let $(x, y, \lambda, \mu, \nu)$ be a solution of (EC_2). Then both sides become a common value with five expressions:

$$(5\mathrm{V}_2) \quad \begin{aligned} &(c-x)^2 + x^2 + (x-y)^2 + y^2 + (y-d)^2 \\ &= c(c-x) - d(y-d) \\ &= 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2] - 2d\nu \\ &= \lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 \\ &= c\lambda - d\nu. \end{aligned}$$

Then the primal (P_2) has a minimum value

$$m_2 = (c-x)^2 + x^2 + (x-y)^2 + y^2 + (y-d)^2 \\ = c(c-x) - d(y-d)$$

at (x, y), while the dual (D_2) has a maximum value

$$M_2 = 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2] - 2d\nu \\ = \lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 \\ = c\lambda - d\nu$$

at (λ, μ, ν).

LEMMA 3.2. *The system* (EC_2) *has indeed a unique solution:*

$$(x, y) = \frac{1}{8}(3c + d, c + 3d) \tag{3}$$

$$(\lambda, \mu, \nu) = \frac{1}{8}(5c - d, 2c - 2d, c - 5d). \tag{4}$$

PROOF. From (EC_2), we have a pair of linear systems of 2-equation on 2-variable and of 3-equation on 3-variable:

$$(\mathrm{EQ}_2)\quad \begin{array}{rr} 3x - y = c & 2\lambda - \mu = c \\ -x + 3y = d & -\lambda + 3\mu - \nu = 0 \\ & -\mu + 2\nu = -d. \end{array}$$

The left system has a solution (3), while the right has a solution (4). □

The primal (P_2) has a minimum value

$$m_2 = c(c - \hat{x}) - d(\hat{y} - d) = \frac{1}{8}(5c^2 - 2cd + 5d^2)$$

at a path

$$(\hat{x}, \hat{y}) = \frac{1}{8}(3c + d, c + 3d).$$

The dual (D_2) has a maximum value

$$M_2 = c\lambda^* - d\mu^* = \frac{1}{8}(5c^2 - 2cd + 5d^2)$$

at a path

$$(\lambda^*, \mu^*, \nu^*) = \frac{1}{8}(5c - d, 2c - 2d, c - 5d).$$

3.3. (P_3) **vs** (D_3)

Let us consider the third:

$$(\mathrm{P}_3)\quad \begin{array}{ll} \text{minimize} & (c - x)^2 + x^2 + (x - y)^2 + y^2 + (y - z)^2 + z^2 + (z - d)^2 \\ \text{subject to} & \text{(i)} \quad (x, y, z) \in R^3 \end{array}$$

$$(\mathrm{D}_3)\quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2 + (\nu - \xi)^2 + \xi^2] - 2d\xi \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu, \nu, \xi) \in R^4. \end{array}$$

Then both are dual to each other. It holds that

$$\begin{aligned} &2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2 + (\nu - \xi)^2 + \xi^2] - 2d\xi \\ &\quad \le (c - x)^2 + x^2 + (x - y)^2 + y^2 + (y - z)^2 + z^2 + (z - d)^2 \end{aligned}$$

for any feasible pair $(x, y, z; \lambda, \mu, \nu)$. An equality condition is

$$(\mathrm{EC}_3) \quad \begin{array}{ll} c - x = \lambda, & x = \lambda - \mu \\ x - y = \mu, & y = \mu - \nu \\ y - z = \nu, & z = \nu - \xi \\ z - d = \xi. & \end{array}$$

The equality condition (EC_3) is a linear system of 7-equation on 7-variable.

Let $(x, y, z, \lambda, \mu, \nu, \xi)$ be a solution of (EC_3). Then both sides become a common value with five expressions:

$$\begin{aligned} & (c-x)^2 + x^2 + (x-y)^2 + y^2 + (y-z)^2 + z^2 + (z-d)^2 \\ & = c(c-x) - d(z-d) \\ (5\mathrm{V}_3) \quad & = 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 + (\nu-\xi)^2 + \xi^2] - 2d\xi \\ & = \lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 + (\nu-\xi)^2 + \xi^2 \\ & = c\lambda - d\xi. \end{aligned}$$

Then the primal (P_3) has a minimum value

$$\begin{aligned} m_3 &= (c-x)^2 + x^2 + (x-y)^2 + y^2 + (y-z)^2 + z^2 + (z-d)^2 \\ &= c(c-x) - d(z-d) \end{aligned}$$

at (x, y, z), while the dual (D_3) has a maximum value

$$\begin{aligned} M_3 &= 2c\lambda - [\lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 + (\nu-\xi)^2 + \xi^2] - 2d\xi \\ &= \lambda^2 + (\lambda-\mu)^2 + \mu^2 + (\mu-\nu)^2 + \nu^2 + (\nu-\xi)^2 + \xi^2 \\ &= c\lambda - d\xi \end{aligned}$$

at (λ, μ, ν, ξ).

LEMMA 3.3. *The system* (EC_3) *has indeed a unique solution:*

$$(x, y, z) = \frac{1}{21}(8c + d, 3c + 3d, c + 8d) \tag{5}$$

$$(\lambda, \mu, \nu, \xi) = \frac{1}{21}(13c - d, 5c - 2d, 2c - 5d, c - 13d). \tag{6}$$

PROOF. From (EC_3), we have a pair of linear systems of 3-equation on 3-variable and of 4-equation on 4-variable:

$$
(\mathrm{EQ}_3) \quad
\begin{aligned}
3x - y &= c & 2\lambda - \mu &= c \\
-x + 3y - z &= 0 & -\lambda + 3\mu - \nu &= 0 \\
-y + 3z &= d & -\mu + 3\nu - \xi &= 0 \\
 & & -\nu + 2\xi &= -d.
\end{aligned}
$$

The left system has a solution (5), while the right has a solution (6). □

The primal (P_3) has a minimum value

$$m_3 = c(c - \hat{x}) - d(\hat{z} - d) = \frac{1}{21}(13c^2 - 2cd + 13d^2)$$

at a path

$$(\hat{x}, \hat{y}, \hat{z}) = \frac{1}{21}(8c + d, 3c + 3d, c + 8d).$$

The dual (D_3) has a maximum value

$$M_3 = c\lambda^* - d\xi^* = \frac{1}{21}(13c^2 - 2cd + 13d^2)$$

at a path

$$(\lambda^*, \mu^*, \nu^*, \xi^*) = \frac{1}{21}(13c - d, 5c - 2d, 2c - 5d, c - 13d).$$

Here we note that the first eight *Fibonacci numbers* appear:

1, 1, 2, 3, 5, 8, 13, 21.

The *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation

$$(\mathrm{Fibo}): \quad x_{n+2} - x_{n+1} - x_n = 0 \qquad x_1 = 1, \ x_0 = 0.$$

Table 1. Fibonacci sequence $\{F_n\}$

n	$\cdots$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
F_n	$\cdots$	-1	1	0	1	1	2	3	5	8	13	21	34	55	89

n	12	13	14	15	16	17	18	19	$\cdots$
F_n	144	233	377	610	987	1597	2584	4181	$\cdots$

4. (P_n) **vs** (D_n)

Now let us consider the n-variable pair, where $n \geq 2$. First we present the n-th complementary identity, which takes a fundamental role in analyzing a pair of primal and dual.

Let $x = \{x_k\}_0^{n+1}$, $\mu = \{\mu_k\}_1^{n+1}$ be a pair of sequences of real number with $x_0 = c$, $x_{n+1} = d$. Then an identity

$$(\mathrm{C}_n) \quad c\mu_1 - d\mu_{n+1} = \sum_{k=1}^{n}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_n - d)\mu_{n+1}$$

holds true. This identity yields a pair of minimization problem

$$(\mathrm{P}_n) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n}[(x_{k-1} - x_k)^2 + x_k^2] + (x_n - x_{n+1})^2 \\ \text{subject to} & \text{(i)} \quad x \in R^{n+2}, \qquad \text{(ii)} \quad x_0 = c, \ x_{n+1} = d \end{array}$$

and a maximization problem

$$(\mathrm{D}_n) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \displaystyle\sum_{k=1}^{n}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_{n+1}^2 - 2d\mu_{n+1} \\ \text{subject to} & \text{(i)} \quad \mu \in R^{n+1}. \end{array}$$

Then both are dual to each other. It holds that

$$\begin{aligned} & 2c\mu_1 - \sum_{k=1}^{n}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_{n+1}^2 - 2d\mu_{n+1} \\ & \qquad \leq (c - x_1)^2 + x_1^2 + \sum_{k=2}^{n}[(x_{k-1} - x_k)^2 + x_k^2] + (x_n - d)^2 \end{aligned}$$

for any feasible pair (x, μ). An equality condition is

$$(\mathrm{EC}_n) \quad \begin{array}{lll} c - x_1 = \mu_1, & x_1 = \mu_1 - \mu_2 & \\ x_{k-1} - x_k = \mu_k, & x_k = \mu_k - \mu_{k+1} & 2 \leq k \leq n \\ x_n - d = \mu_n. & & \end{array}$$

The equality condition (EC_n) is a linear system of $(2n+1)$-equation on $(2n+1)$-variable.

Let (x, μ) be a solution of (EC_n). Then both sides become a common value with five expressions:

$$
\begin{aligned}
& (c - x_1)^2 + x_1^2 + \sum_{k=2}^{n} [(x_{k-1} - x_k)^2 + x_k^2] + (x_n - d)^2 \\
& = c(c - x_1) - d(x_n - d) \\
(5\mathrm{V}_n) \quad & = 2c\mu_1 - \sum_{k=1}^{n} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_{n+1}^2 - 2d\mu_{n+1} \\
& = \sum_{k=1}^{n} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_{n+1}^2 \\
& = c\mu_1 - d\mu_{n+1}.
\end{aligned}
$$

The primal (P_n) has a minimum value

$$
\begin{aligned}
m_n &= (c - x_1)^2 + x_1^2 + \sum_{k=2}^{n} [(x_{k-1} - x_k)^2 + x_k^2] + (x_n - d)^2 \\
&= c(c - x_1) - d(x_n - d)
\end{aligned}
$$

at x, while the dual (D_n) has a maximum value

$$
\begin{aligned}
M_n &= 2c\mu_1 - \sum_{k=1}^{n} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_{n+1}^2 - 2d\mu_{n+1} \\
&= \sum_{k=1}^{n} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_{n+1}^2 \\
&= c\mu_1 - d\mu_{n+1}
\end{aligned}
$$

at μ.

LEMMA 4.1. *The system* (EC_n) *has indeed a unique solution:*

$$
(7) \qquad \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_{n-2} \\ \hat{x}_{n-1} \\ \hat{x}_n \end{pmatrix} = \frac{1}{F_{2n+2}} \begin{pmatrix} F_{2n}c + F_2 d \\ F_{2n-2}c + F_4 d \\ F_{2n-4}c + F_6 d \\ \vdots \\ F_6 c + F_{2n-4} d \\ F_4 c + F_{2n-2} d \\ F_2 c + F_{2n} d \end{pmatrix}
$$

$$\mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \\ \vdots \\ \mu_{n-2}^* \\ \mu_{n-1}^* \\ \mu_n^* \\ \mu_{n+1}^* \end{pmatrix} = \frac{1}{F_{2n+2}} \begin{pmatrix} F_{2n+1}c - F_1 d \\ F_{2n-1}c - F_3 d \\ F_{2n-3}c - F_5 d \\ \vdots \\ F_7 c - F_{2n-5} d \\ F_5 c - F_{2n-3} d \\ F_3 c - F_{2n-1} d \\ F_1 c - F_{2n+1} d \end{pmatrix}. \tag{8}$$

PROOF. From (EC_n), we have a pair of linear systems of n-equation on n-variable and $(n+1)$-equation on $(n+1)$-variable:

$$(EQ_n) \qquad \begin{aligned} 3x_1 - x_2 &= c \\ -x_1 + 3x_2 - x_3 &= 0 \\ &\vdots \\ -x_{n-2} + 3x_{2n-1} - x_n &= 0 \\ -x_{n-1} + 3x_n &= d \end{aligned} \qquad \begin{aligned} 2\mu_1 - \mu_2 &= c \\ -\mu_1 + 3\mu_2 - \mu_3 &= 0 \\ &\vdots \\ -\mu_{n-2} + 3\mu_{n-1} - \mu_n &= 0 \\ -\mu_{n-1} + 3\mu_n - \mu_{n+1} &= 0 \\ -\mu_n + 2\mu_{n+1} &= -d. \end{aligned}$$

The left system has a solution $\hat{x}$ in (7), while the right has a solution μ^* in (8).

In fact, the left system is written as

$$Ax = b$$

where x, b are n-vectors:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix}, \qquad b = \begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ d \end{pmatrix}$$

and A is an $n \times n$-matrix:

$$A=\begin{pmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 3 \end{pmatrix}.$$

Further A has the inverse

$$A^{-1}=\frac{1}{F_{2n+2}}\begin{pmatrix} F_{2n} & F_{2n-2} & F_{2n-4} & F_{2n-6} & \cdots & F_8 & F_6 & F_4 & F_2 \\ F_{2n-2} & 3F_{2n-2} & 3F_{2n-4} & 3F_{2n-6} & \cdots & 3F_8 & 3F_6 & 3F_4 & F_4 \\ F_{2n-4} & 3F_{2n-4} & 8F_{2n-4} & 8F_{2n-6} & \cdots & 8F_8 & 8F_6 & 3F_6 & F_6 \\ F_{2n-6} & 3F_{2n-6} & 8F_{2n-6} & 21F_{2n-6} & \cdots & 21F_8 & 8F_8 & 3F_8 & F_8 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_8 & 3F_8 & 8F_8 & 21F_8 & \cdots & 21F_{2n-6} & 8F_{2n-6} & 3F_{2n-6} & F_{2n-6} \\ F_6 & 3F_6 & 8F_6 & 8F_8 & \cdots & 8F_{2n-6} & 8F_{2n-4} & 3F_{2n-4} & F_{2n-4} \\ F_4 & 3F_4 & 3F_6 & 3F_8 & \cdots & 3F_{2n-6} & 3F_{2n-4} & 3F_{2n-2} & F_{2n-2} \\ F_2 & F_4 & F_6 & F_8 & \cdots & F_{2n-6} & F_{2n-4} & F_{2n-2} & F_{2n} \end{pmatrix}.$$

Thus a unique solution $x=A^{-1}b$ is specified in (7).

On the other hand, the right is

$$B\mu = f$$

where $\mu,\ f$ are $(n+1)$-vectors:

$$\mu=\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{n-1} \\ \mu_n \\ \mu_{n+1} \end{pmatrix}, \qquad f=\begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -d \end{pmatrix}$$

and B is an $(n+1)\times(n+1)$-matrix:

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Further B has the inverse

$$B^{-1} = \frac{1}{F_{2n+2}} \begin{pmatrix} F_{2n+1} & F_{2n-1} & F_{2n-3} & F_{2n-5} & \cdots & F_7 & F_5 & F_3 & F_1 \\ F_{2n-1} & 2F_{2n-1} & 2F_{2n-3} & 2F_{2n-5} & \cdots & 2F_7 & 2F_5 & 2F_3 & F_3 \\ F_{2n-3} & 2F_{2n-3} & 5F_{2n-3} & 5F_{2n-5} & \cdots & 5F_7 & 5F_5 & 2F_5 & F_5 \\ F_{2n-5} & 2F_{2n-5} & 5F_{2n-5} & 13F_{2n-5} & \cdots & 13F_7 & 5F_7 & 2F_7 & F_7 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_7 & 2F_7 & 5F_7 & 13F_7 & \cdots & 13F_{2n-5} & 5F_{2n-5} & 2F_{2n-5} & F_{2n-5} \\ F_5 & 2F_5 & 5F_5 & 5F_7 & \cdots & 5F_{2n-5} & 5F_{2n-3} & 2F_{2n-3} & F_{2n-3} \\ F_3 & 2F_3 & 2F_5 & 2F_7 & \cdots & 2F_{2n-5} & 2F_{2n-3} & 2F_{2n-1} & F_{2n-1} \\ F_1 & F_3 & F_5 & F_7 & \cdots & F_{2n-5} & F_{2n-3} & F_{2n-1} & F_{2n+1} \end{pmatrix}.$$

Thus a unique solution $\mu = B^{-1}f$ is specified in (8). □

LEMMA 4.2. *The primal* (P_n) *has a minimum value*

$$m_n = c(c - \hat{x}_1) - d(\hat{x}_n - d) = \frac{1}{F_{2n+2}}(F_{2n+1}c^2 - 2cd + F_{2n+1}d^2)$$

at the path $\hat{x}$. *The dual* (D_n) *has a maximum value*

$$M_n = c\mu_1^* - d\mu_{n+1}^* = \frac{1}{F_{2n+2}}(F_{2n+1}c^2 - 2cd + F_{2n+1}d^2)$$

at the path μ^*.

5. Duality

In this section, we show that (P_n) and (D_n) are dual to each other. It turns out that the duality is based upon the complementary identity and an elementary inequality with equality

(9) $$2xy \le x^2 + y^2 \qquad \text{on } R^2;\ x = y.$$

Let $x = \{x_k\}_0^{n+1}$, $\mu = \{\mu_k\}_1^{n+1}$ be a pair of sequences of real number with $x_0 = c$, $x_{n+1} = d$. Then an identity

$$(\mathrm{C}_n) \quad c\mu_1 - d\mu_{n+1} = \sum_{k=1}^{n}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_n - d)\mu_{n+1}$$

holds true. We make it double. Then an identity

$$2c\mu_1 - 2d\mu_{n+1} = \sum_{k=1}^{n}[2(x_{k-1} - x_k)\mu_k + 2x_k(\mu_k - \mu_{k+1})] + 2(x_n - d)\mu_{n+1}$$

with the elementary inequality (9) yields

$$2c\mu_1 - 2d\mu_{n+1} \leq \sum_{k=1}^{n}[(x_{k-1} - x_k)^2 + x_k^2] + (x_n - x_{n+1})^2 + \sum_{k=1}^{n}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_{n+1}^2.$$

Thus we have an inequality

$$2c\mu_1 - \sum_{k=1}^{n}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_{n+1}^2 - 2d\mu_{n+1} \leq \sum_{k=1}^{n}[(x_{k-1} - x_k)^2 + x_k^2] + (x_n - x_{n+1})^2$$

for any feasible pair (x, μ). The sign of equality holds iff

$$(\mathrm{EC}_n) \quad \begin{array}{lll} c - x_1 = \mu_1, & x_1 = \mu_1 - \mu_2 & \\ x_{k-1} - x_k = \mu_k, & x_k = \mu_k - \mu_{k+1} & 2 \leq k \leq n \\ x_n - d = \mu_{n+1}. & & \end{array}$$

Thus we have a pair of minimization problem

$$(\mathrm{P}_n) \quad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n}[(x_{k-1} - x_k)^2 + x_k^2] + (x_n - x_{n+1})^2 \\ \text{subject to} & \text{(i)} \quad x \in R^{n+2}, \qquad \text{(ii)} \quad x_0 = c,\ x_{n+1} = d \end{array}$$

and a maximization problem

$$(\mathrm{D}_n) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_{n+1}^2 - 2d\mu_{n+1} \\ \text{subject to} & \text{(i)} \quad \mu \in R^{n+1}. \end{array}$$

Hence both are dual to each other.

6. Conjugate dual

Let f be a differentiable convex function on R^1. Then a *conjugate function* f^* is defined by

$$f^*(\lambda) = \underset{x \in R^1}{\mathrm{Max}}[\lambda x - f(x)] \qquad \lambda \in R^1. \tag{10}$$

Then it holds that

$$\lambda x \le f(x) + f^*(\lambda) \qquad (x, \lambda) \in R^2. \tag{11}$$

The sign of equality holds iff

$$f'(x) = \lambda. \tag{12}$$

In the following, we assume that three convex functions f, g, h are given. We show that three identities generate their respective pairs of primal and dual with equality condition.

The first identity

$$(\mathrm{C}_1) \quad c\lambda - d\mu = (c - x)\lambda + x(\lambda - \mu) + (x - d)\mu$$

yields a pair

$$(\mathrm{CP}_1) \quad \begin{array}{ll} \text{minimize} & f(c - x) + g(x) + h(x - d) \\ \text{subject to} & \text{(i)} \quad x \in R^1 \end{array}$$

$$(\mathrm{CD}_1) \quad \begin{array}{ll} \text{Maximize} & c\lambda - [f^*(\lambda) + g^*(\lambda - \mu) + h^*(\mu)] - d\mu \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu) \in R^2. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_1) \quad \begin{array}{ll} f'(c - x) = \lambda, & g'(x) = \lambda - \mu \\ h'(x - d) = \mu. & \end{array}$$

The second identity

$$(\mathrm{C}_2) \quad c\lambda - d\nu = (c - x)\lambda + x(\lambda - \mu) + (x - y)\mu + y(\mu - \nu) + (y - d)\nu$$

yields a pair

$$(\mathrm{CP}_2)\quad \begin{array}{ll} \text{minimize} & f(c-x)+g(x)+f(x-y)+g(y)+h(y-d) \\ \text{subject to} & \text{(i)}\quad (x,y)\in R^2 \end{array}$$

$$(\mathrm{CD}_2)\quad \begin{array}{ll} \text{Maximize} & c\lambda-[f^*(\lambda)+g^*(\lambda-\mu)+f^*(\mu)+g^*(\mu-\nu)+h^*(\nu)]-d\nu \\ \text{subject to} & \text{(i)}\quad (\lambda,\mu,\nu)\in R^3. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_2)\quad \begin{array}{ll} f'(c-x)=\lambda, & g'(x)=\lambda-\mu \\ f'(x-y)=\mu, & g'(y)=\mu-\nu \\ h'(y-d)=\nu. & \end{array}$$

The third identity

$$(\mathrm{C}_3)\quad c\lambda-d\xi=(c-x)\lambda+x(\lambda-\mu)+(x-y)\mu+y(\mu-\nu)+(y-z)\nu + z(\nu-\xi)+(z-d)\xi$$

yields a pair

$$(\mathrm{CP}_3)\quad \begin{array}{ll} \text{minimize} & f(c-x)+g(x)+f(x-y)+g(y)+f(y-z)+g(z)+h(z-d) \\ \text{subject to} & \text{(i)}\quad (x,y,z)\in R^3 \end{array}$$

$$(\mathrm{CD}_3)\quad \begin{array}{ll} \text{Maximize} & c\lambda-[f^*(\lambda)+g^*(\lambda-\mu)+f^*(\mu)+g^*(\mu-\nu)+f^*(\nu) \\ & \quad +g^*(\nu-\xi)+h^*(\xi)]-d\xi \\ \text{subject to} & \text{(i)}\quad (\lambda,\mu,\nu,\xi)\in R^4. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_3)\quad \begin{array}{ll} f'(c-x)=\lambda, & g'(x)=\lambda-\mu \\ f'(x-y)=\mu, & g'(y)=\mu-\nu \\ f'(y-z)=\nu, & g'(z)=\nu-\xi \\ h'(z-d)=\xi. & \end{array}$$

Finally the n-th identity

$$(\mathrm{C}_n)\quad c\mu_1-d\mu_{n+1}=\sum_{k=1}^{n}[(x_{k-1}-x_k)\mu_k+x_k(\mu_k-\mu_{k+1})]+(x_n-d)\mu_{n+1}$$

yields a pair

$$(\mathrm{CP}_n)\quad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n}[f(x_{k-1}-x_k)+g(x_k)]+h(x_n-x_{n+1}) \\ \text{subject to} & \text{(i)}\quad x\in R^{n+2}, \qquad \text{(ii)}\quad x_0=c,\ x_{n+1}=d \end{array}$$

$$(\mathrm{CD}_n)\quad \begin{array}{ll} \text{Maximize} & c\mu_1-\sum_{k=1}^{n}[f^*(\mu_k)+g^*(\mu_k-\mu_{k+1})]-h^*(\mu_{n+1})-d\mu_{n+1} \\ \text{subject to} & \text{(i)}\quad \mu\in R^{n+1}. \end{array}$$

An equality condition is

$$(\mathrm{CEC}_n)\quad \begin{array}{lll} f'(c-x_1)=\mu_1, & g'(x_1)=\mu_1-\mu_2 & \\ f'(x_{k-1}-x_k)=\mu_k, & g'(x_k)=\mu_k-\mu_{k+1} & 2\le k\le n \\ h'(x_n-d)=\mu_{n+1}. & & \end{array}$$

7. Dynamic programming

We show that dynamic programming solves both primal and dual.

7.1. The third pair

$$(\mathrm{CP}_3)\quad \begin{array}{ll} \text{minimize} & f(c-x)+g(x)+f(x-y)+g(y)+f(y-z)+g(z)+h(z-d) \\ \text{subject to} & \text{(i)}\quad (x,y,z)\in R^3 \end{array}$$

$$(\mathrm{CD}_3)\quad \begin{array}{ll} \text{Maximize} & c\lambda-[f^*(\lambda)+g^*(\lambda-\mu)+f^*(\mu)+g^*(\mu-\nu)+f^*(\nu) \\ & \qquad +g^*(\nu-\xi)+h^*(\xi)]-d\xi \\ \text{subject to} & \text{(i)}\quad (\lambda,\mu,\nu,\xi)\in R^4. \end{array}$$

7.1.1. Primal (CP_3)

Let U be the minimum value of (CP_3). Let $u_1(x)$ be the minimum value of two-variable subproblem:

$$(\mathrm{SP}_2)\quad \begin{array}{ll} \text{minimize} & f(x-y)+g(y)+f(y-z)+g(z)+h(z-d) \\ \text{subject to} & \text{(i)}\quad (y,z)\in R^2. \end{array}$$

Let $u_2(y)$ be the minimum value of one-variable subproblem:

$$(\mathrm{SP}_1) \quad \begin{array}{ll} \text{minimize} & f(y-z)+g(z)+h(z-d) \\ \text{subject to} & \text{(i)} \quad z \in R^1. \end{array}$$

Finally let $u_3(z) := h(z-d)$. Then we have a recursive formula

$$\begin{aligned} u_3(z) &= h(z-d) \\ u_2(y) &= \min_{z \in R^1}[f(y-z)+u_3(z)] \\ u_1(x) &= \min_{y \in R^1}[f(x-y)+g(y)+u_2(y)] \\ U &= \min_{x \in R^1}[f(c-x)+g(x)+u_1(x)]. \end{aligned}$$

Now let us solve the forementioned problem:

$$(\mathrm{P}_3) \quad \begin{array}{ll} \text{minimize} & (c-x)^2+x^2+(x-y)^2+y^2+(y-z)^2+z^2+(z-d)^2 \\ \text{subject to} & \text{(i)} \quad (x,y,z) \in R^3. \end{array}$$

Then the recursive formula

$$\begin{aligned} u_3(z) &= z^2+(z-d)^2 \\ u_2(y) &= \min_{z \in R^1}[(y-z)^2+u_3(z)] \\ u_1(x) &= \min_{y \in R^1}[(x-y)^2+y^2+u_2(y)] \\ U &= \min_{x \in R^1}[(c-x)^2+x^2+u_1(x)] \end{aligned}$$

has a solution

$$\begin{aligned} u_3(z) &= \frac{1}{2}(2z^2-2zd+d^2) \\ u_2(y) &= \frac{1}{3}(2y^2-2yd+2d^2), \qquad \hat{z}(y) = \frac{1}{3}(y+d) \\ u_1(x) &= \frac{1}{8}(5x^2-2xd+5d^2), \qquad \hat{y}(x) = \frac{1}{8}(3x+d) \\ U &= \frac{1}{21}(13c^2-2cd+13d^2), \qquad \hat{y}(x) = \frac{1}{21}(8c+d). \end{aligned}$$

Thus we get a minimum point $(\hat{x}, \hat{y}, \hat{z})$, where

$$\hat{x} = \hat{x}(c) = \frac{1}{21}(8c + d)$$

$$\hat{y} = \hat{y}(\hat{x}) = \frac{1}{8}(3\hat{x} + d) = \frac{1}{21}(3c + 3d)$$

$$\hat{z} = \hat{z}(\hat{y}) = \frac{1}{3}(\hat{y} + d) = \frac{1}{21}(c + 8d).$$

Hence (P_3) attains a minimum $U = \frac{1}{21}(13c^2 - 2cd + 13d^2)$ at a point

$$(\hat{x}, \hat{y}, \hat{z}) = \frac{1}{21}(8c + d, 3c + 3d, c + 8d).$$

See (5).

7.1.2. Dual (CD_3)

Let V be the maximum value of (CD_3). Let $v_1(\lambda)$ be the minimum value of three-variable subproblem:

$$(SD_3) \quad \begin{array}{ll} \text{minimize} & [f^*(\lambda) + g^*(\lambda - \mu) + f^*(\mu) + g^*(\mu - \nu) + f^*(\nu) \\ & \quad + g^*(\nu - \xi) + h^*(\xi)] + d\xi \\ \text{subject to} & \text{(i)} \quad (\mu, \nu, \xi) \in R^3. \end{array}$$

Let $v_2(\mu)$ be the minimum value of two-variable subproblem:

$$(SD_2) \quad \begin{array}{ll} \text{minimize} & [f^*(\mu) + g^*(\mu - \nu) + f^*(\nu) + g^*(\nu - \xi) + h^*(\xi)] + d\xi \\ \text{subject to} & \text{(i)} \quad (\nu, \xi) \in R^2. \end{array}$$

Let $v_3(\nu)$ be the minimum value of one-variable subproblem:

$$(SD_1) \quad \begin{array}{ll} \text{minimize} & [f^*(\nu) + g^*(\nu - \xi) + h^*(\xi)] + d\xi \\ \text{subject to} & \text{(i)} \quad \xi \in R^1. \end{array}$$

Finally let $v_4(\xi) := h^*(\xi) + d\xi$. Then we have a recursive formula

$$v_4(\xi) = h^*(\xi) + d\xi$$

$$v_3(\nu) = \min_{\xi \in R^1}[f^*(\nu) + g^*(\nu - \xi) + v_4(\xi)]$$

$$v_2(\mu) = \min_{\nu \in R^1}[f^*(\mu) + g^*(\mu - \nu) + v_3(\nu)]$$

$$v_1(\lambda) = \min_{\mu \in R^1} [f^*(\lambda) + g^*(\lambda - \mu) + v_2(\mu)]$$

$$V = \operatorname*{Max}_{\lambda \in R^1} [c\lambda - v_1(\lambda)].$$

Now let us solve the forementioned problem:

$$(\mathrm{D}_3) \quad \begin{array}{ll} \text{Maximize} & 2c\lambda - [\lambda^2 + (\lambda - \mu)^2 + \mu^2 + (\mu - \nu)^2 + \nu^2 + (\nu - \xi)^2 + \xi^2] - 2d\xi \\ \text{subject to} & \text{(i)} \quad (\lambda, \mu, \nu, \xi) \in R^4. \end{array}$$

Then the recursive formula

$$v_4(\xi) = \xi^2 + 2d\xi$$

$$v_3(\nu) = \min_{\xi \in R^1} [\nu^2 + (\nu - \xi)^2 + v_4(\xi)]$$

$$v_2(\mu) = \min_{\nu \in R^1} [\mu^2 + (\mu - \nu)^2 + v_3(\nu)]$$

$$v_1(\lambda) = \min_{\mu \in R^1} [\lambda^2 + (\lambda - \mu)^2 + v_2(\mu)]$$

$$V = \operatorname*{Max}_{\lambda \in R^1} [2c\lambda - v_1(\lambda)]$$

has a solution

$$v_4(\xi) = \xi^2 + 2\xi d$$

$$v_3(\nu) = \frac{1}{2}(3\nu^2 + 2\nu d - d^2), \qquad \xi^*(\nu) = \frac{1}{2}(\nu - d)$$

$$v_2(\mu) = \frac{1}{5}(8\mu^2 + 2\mu d - 3d^2), \qquad \nu^*(\mu) = \frac{1}{5}(2\mu - d)$$

$$v_1(\lambda) = \frac{1}{13}(21\lambda^2 + 2\lambda d - 8d^2), \qquad \mu^*(\lambda) = \frac{1}{13}(5\lambda - d)$$

$$V = \frac{1}{21}(13c^2 - 2cd + 13d^2), \qquad \lambda^*(c) = \frac{1}{21}(13c - d).$$

Thus we get a maximum point $(\lambda^*, \mu^*, \nu^*, \xi^*)$, where

$$\lambda^* = \lambda^*(c) = \frac{1}{21}(13c - d)$$

$$\mu^* = \mu^*(\lambda^*) = \frac{1}{13}(5\lambda^* - d) = \frac{1}{21}(5c - 2d)$$

$$\nu^* = \nu^*(\mu^*) = \frac{1}{5}(2\mu^* - d) = \frac{1}{21}(2c - 5d)$$

$$\xi^* = \xi^*(\nu^*) = \frac{1}{2}(\nu^* - d) = \frac{1}{21}(c - 13d).$$

Hence (D_3) attains a maximum $V = \frac{1}{21}(13c^2 - 2cd + 13d^2)$ at a point

$$(\lambda^*, \mu^*, \nu^*, \xi^*) = \frac{1}{21}(13c - d, 5c - 2d, 2c - 5d, c - 13d).$$

See also (6).

7.2. The n-th pair

$$(\mathrm{CP}_n)\quad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n}[f(x_{k-1} - x_k) + g(x_k)] + h(x_n - x_{n+1}) \\ \text{subject to} & \text{(i)}\quad x \in R^{n+2}, \qquad \text{(ii)}\quad x_0 = c,\ x_{n+1} = d \end{array}$$

$$(\mathrm{CD}_n)\quad \begin{array}{ll} \text{Maximize} & c\mu_1 - \sum_{k=1}^{n}[f^*(\mu_k) + g^*(\mu_k - \mu_{k+1})] - h^*(\mu_{n+1}) - d\mu_{n+1} \\ \text{subject to} & \text{(i)}\quad \mu \in R^{n+1}. \end{array}$$

7.2.1. Primal (CP_n)

Let U be the minimum value of (CP_n). Let $u_k(x_k)$ be the minimum value of $(n-k)$-variable subproblem:

$$(\mathrm{CP}_{n-k})\quad \begin{array}{ll} \text{minimize} & \sum_{l=k+1}^{n}[f(x_{l-1} - x_l) + g(x_l)] + h(x_n - x_{n+1}) \\ \text{subject to} & \text{(i)}\quad (x_{k+1}, x_{k+2}, \ldots, x_n) \in R^{n-k}, \qquad \text{(ii)}\quad x_{n+1} = d \end{array}$$

where $1 \le k \le n-1$. Finally let $u_n(x_n) := h(x_n - d)$. Then we have a recursive formula

$$u_n(x_n) = h(x_n - d)$$

$$u_k(x_k) = \min_{x_{k+1} \in R^1}[f(x_k - x_{k+1}) + g(x_{k+1}) + u_{k+1}(x_{k+1})] \qquad 1 \le k \le n-1$$

$$U = \min_{x_1 \in R^1}[f(c - x_1) + g(x_1) + u_1(x_1)].$$

Let $\hat{x}_{k+1}(x_k)$ be a minimizer. Then a sequence $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ is called a *policy*.

Now let us solve the forementioned problem:

$$(\mathrm{P}_n)\qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n}[(x_{k-1}-x_k)^2+x_k^2]+(x_n-x_{n+1})^2 \\ \text{subject to} & \text{(i)}\quad x\in R^{n+2}, \qquad \text{(ii)}\quad x_0=c,\ x_{n+1}=d. \end{array}$$

Then the recursive formula

$$u_n(x_n)=(x_n-d)^2$$

$$u_k(x_k)=\min_{x_{k+1}\in R^1}[(x_k-x_{k+1})^2+x_{k+1}^2+u_{k+1}(x_{k+1})] \qquad 1\le k\le n-1$$

$$U=\min_{x_1\in R^1}[(c-x_1)^2+x_1^2+u_1(x_1)],$$

has a solution

$$u_n(x_n)=x_n^2-2x_nd+d^2$$

$$u_{n-1}(x_{n-1})=\frac{1}{3}(2x_{n-1}^2-2x_{n-1}d+2d^2), \qquad \hat{x}_n(x_{n-1})=\frac{1}{3}(x_{n-1}+d)$$

$$u_{n-2}(x_{n-2})=\frac{1}{8}(5x_{n-2}^2-2x_{n-2}d+5d^2), \qquad \hat{x}_{n-1}(x_{n-2})=\frac{1}{8}(3x_{n-2}+d)$$

$$\left.\begin{array}{l} u_k(x_k)=\dfrac{1}{F_{2n-2k+2}}(F_{2n-2k+1}x_k^2-2x_kd+F_{2n-2k+1}d^2), \\ \qquad\qquad \hat{x}_{k+1}(x_k)=\dfrac{1}{F_{2n-2k+2}}(F_{2n-2k}x_k+d) \end{array}\right. \qquad 1\le k\le n-3,$$

$$U=\frac{1}{F_{2n+2}}(F_{2n+1}c^2-2cd+F_{2n+1}d^2), \qquad \hat{x}_1(c)=\frac{1}{F_{2n+2}}(F_{2n}c+F_2d).$$

Thus we get a minimum point $\hat{x}=(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_k,\ldots,\hat{x}_{n-1},\hat{x}_n)$ as follows:

$$\hat{x}_1=\hat{x}_1(c)=\frac{1}{F_{2n+2}}(F_{2n}c+F_2d)$$

$$\hat{x}_2=\hat{x}_2(\hat{x}_1)=\frac{1}{F_{2n}}(F_{2n-2}\hat{x}_1+d)=\frac{1}{F_{2n+2}}(F_{2n-2}c+F_4d)$$

$$\hat{x}_3=\hat{x}_3(\hat{x}_2)=\frac{1}{F_{2n-2}}(F_{2n-4}\hat{x}_1+d)=\frac{1}{F_{2n+2}}(F_{2n-4}c+F_6d)$$

$$\vdots$$

$$\hat{x}_{k+1} = \hat{x}_{k+1}(\hat{x}_k) = \frac{1}{F_{2n-2k+2}}(F_{2n-2k}\hat{x}_k + d) = \frac{1}{F_{2n+2}}(F_{2n-2k}c + F_{2k+2}d)$$

$$\vdots$$

$$\hat{x}_{n-1} = \hat{x}_{n-1}(\hat{x}_{n-2}) = \frac{1}{F_6}(F_4\hat{x}_{n-2} + d) = \frac{1}{F_{2n+2}}(F_4c + F_{2n-2}d)$$

$$\hat{x}_n = \hat{x}_n(\hat{x}_{n-1}) = \frac{1}{F_4}(F_2\hat{x}_{n-1} + d) = \frac{1}{F_{2n+2}}(F_2c + F_{2n}d).$$

Hence (P_n) attains a minimum $U = \frac{1}{F_{2n+2}}(F_{2n+1}c^2 - 2cd + F_{2n+1}d^2)$ at a point

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_{n-2} \\ \hat{x}_{n-1} \\ \hat{x}_n \end{pmatrix} = \frac{1}{F_{2n+2}} \begin{pmatrix} F_{2n}c + F_2d \\ F_{2n-2}c + F_4d \\ F_{2n-4}c + F_6d \\ \vdots \\ F_6c + F_{2n-4}d \\ F_4c + F_{2n-2}d \\ F_2c + F_{2n}d \end{pmatrix}.$$

See also (7).

7.2.2. Dual (CD_n)

Let V be the maximum value of (CD_n). Let $v_k(\mu_k)$ be the minimum value of $(n-k+1)$-variable subproblem:

$$(\mathrm{SD}_{n-k}) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{l=k}^{n-1}[f^*(\mu_l) + g^*(\mu_l - \mu_{l+1})] + h^*(\mu_{n+1}) + d\mu_{n+1} \\ \text{subject to} & \text{(i)} \quad (\mu_{k+1}, \mu_{k+2}, \dots, \mu_{n+1}) \in R^{n-k+1} \end{array}$$

where $1 \le k \le n$. Finally let $v_{n+1}(\mu_{n+1}) := h^*(\mu_{n+1}) + d\mu_{n+1}$. Then we have a recursive formula

$$v_{n+1}(\mu_{n+1}) = h^*(\mu_{n+1}) + d\mu_{n+1}$$

$$v_k(\mu_k) = \min_{\mu_{k+1} \in R^1}[f^*(\mu_k) + g^*(\mu_k - \mu_{k+1}) + v_{k+1}(\mu_{k+1})] \qquad 1 \le k \le n$$

$$V = \operatorname*{Max}_{\mu_1 \in R^1}[c\mu_1 - v_1(\mu_1)].$$

Let $\mu^*_{k+1}(\mu_k)$ be a minimizer and $\mu^*_1(\mu_1)$ be a maximizer. Then a sequence $\{\mu^*_1, \mu^*_2, \dots, \mu^*_n\}$ is called a *policy*.

Now let us solve the forementioned problem:

$$(\mathrm{D}_n)\qquad \begin{aligned}&\text{Maximize}\quad 2c\mu_1-\sum_{k=1}^{n}[\mu_k^2+(\mu_k-\mu_{k+1})^2]-\mu_{n+1}^2-2d\mu_{n+1}\\ &\text{subject to}\quad \text{(i)}\quad \mu\in R^{n+1}.\end{aligned}$$

Then the recursive formula

$$v_{n+1}(\mu_{n+1})=\mu_{n+1}^2+2d\mu_{n+1}$$

$$v_k(\mu_k)=\min_{\mu_{k+1}\in R^1}[\mu_k^2+(\mu_k-\mu_{k+1})^2+v_{k+1}(\mu_{k+1})]\qquad 1\le k\le n$$

$$V=\operatorname*{Max}_{\mu_1\in R^1}[c\mu_1-v_1(\mu_1)]$$

has a solution

$$v_{n+1}(\mu_{n+1})=\mu_{n+1}^2+2d\mu_{n+1}$$

$$\left.\begin{aligned}v_k(\mu_k)&=\frac{1}{F_{2n-2k+2}}(F_{2n-2k+3}\mu_k^2+2\mu_kd-F_{2n-2k}d^2),\\ \mu_{k+1}^*(\mu_k)&=\frac{1}{F_{2n-2k+3}}(F_{2n-2k+1}\mu_k-d)\end{aligned}\right\}\quad 1\le k\le n,$$

$$V=\frac{1}{F_{2n+2}}(F_{2n+1}c^2-2cd+F_{2n+1}d^2),\qquad \mu_1^*(c)=\frac{1}{F_{2n+1}}(F_{2n}c-F_1d).$$

Thus we get a maximum point $\mu^*=(\mu_1^*,\mu_2^*,\dots,\mu_k^*,\dots,\mu_n^*,\mu_{n+1}^*)$, where

$$\mu_1^*=\mu_1^*(c)=\frac{1}{F_{2n+2}}(F_{2n+1}c-F_1d)$$

$$\mu_2^*=\mu_2^*(\mu_1^*)=\frac{1}{F_{2n+1}}(F_{2n-1}\mu_1^*-d)=\frac{1}{F_{2n+2}}(F_{2n-1}c-F_3d)$$

$$\mu_3^*=\mu_3^*(\mu_2^*)=\frac{1}{F_{2n-1}}(F_{2n-3}\mu_2^*-d)=\frac{1}{F_{2n+2}}(F_{2n-3}c-F_5d)$$

$$\vdots$$

$$\mu_{k+1}^*=\mu_{k+1}^*(\mu_k^*)=\frac{1}{F_{2n-2k+3}}(F_{2n-2k+1}\mu_k^*-d)=\frac{1}{F_{2n+2}}(F_{2n-2k+1}c-F_{2k+1}d)$$

$$\vdots$$

$$\mu_{n-1}^*=\mu_{n-1}^*(\mu_{n-2}^*)=\frac{1}{F_7}(F_5\mu_{n-2}^*-d)=\frac{1}{F_{2n+2}}(F_5c-F_{2n-3}d)$$

$$\mu_n^* = \mu_n^*(\mu_{n-1}^*) = \frac{1}{F_5}(F_3\mu_{n-1}^* - d) = \frac{1}{F_{2n+2}}(F_3 c - F_{2n-1}d)$$

$$\mu_{n+1}^* = \mu_{n+1}^*(\mu_n^*) = \frac{1}{F_3}(F_1\mu_n^* - d) = \frac{1}{F_{2n+2}}(F_1 c - F_{2n+1}d).$$

Hence (D_n) attains a maximum $V = \dfrac{1}{F_{2n+2}}(F_{2n+1}c^2 - 2cd + F_{2n+1}d^2)$ at a point

$$\mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \\ \vdots \\ \mu_{n-2}^* \\ \mu_{n-1}^* \\ \mu_n^* \\ \mu_{n+1}^* \end{pmatrix} = \frac{1}{F_{2n+2}} \begin{pmatrix} F_{2n+1}c - F_1 d \\ F_{2n-1}c - F_3 d \\ F_{2n-3}c - F_5 d \\ \vdots \\ F_7 c - F_{2n-5}d \\ F_5 c - F_{2n-3}d \\ F_3 c - F_{2n-1}d \\ F_1 c - F_{2n+1}d \end{pmatrix}.$$

See also (8).

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