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Weak Order Implicit Stochastic Runge-Kutta Methods for Stochastic Differential Equations with a Scalar Wiener Process

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Abstract

New implicit stochastic Runge-Kutta schemes of weak order 1 or 2 are proposed for stochastic differential equations with a scalar Wiener process, which are derivative-free, which attain order 2 or 4 for ordinary differential equations, and which are A-stable in mean square for a linear test equation in some general settings. They are sought in a transparent way and their convergence order and stability properties are confirmed in numerical experiments.

1 Introduction

Numerical methods for stochastic differential equations (SDEs) have been developed for several decades [7, 15, 16, 24, 26, 28], and much progress in this subject has been recently reported [2, 4, 10, 17]. In the progress, we are especially concerned with numerical methods with good stability properties.

In designing such methods, implicit Runge-Kutta methods for SDEs are one of the very attractive candidates since it is well known that implicit Runge-Kutta methods for ordinary differential equations (ODEs) are the most effective way to overcome stiff problems. In fact, many researchers have proposed implicit stochastic Runge-Kutta (SRK) methods. Let us introduce some of them. As said in [4], since methods that copes with stiffness in the purely deterministic setting can be appropriate implicit numerical methods for SDEs with additive noise, we concentrate on the case of multiplicative noise in the sequel. Milstein *et al.* [18, 19] have proposed the balanced implicit method and implicit methods with a bounded random variable. Tian and Burrage [27] have proposed diagonally implicit SRK methods with two stages. These methods are fully implicit methods for strong solutions, and techniques used in the methods are devoted to avoiding possible unboundedness of numerical solutions which is caused by the use of normal random variables. On the other hand, for weak solutions the fully implicit Euler scheme with a bounded random variable has been proposed [10].

Incidentally, Komori [11] and Rößler [20, 22] have proposed very general SRK families to obtain weak solutions in the last few years. Each of them has extended the rooted tree analysis invented by Butcher [6] to seek order conditions of each of their SRK families, and they have succeeded in obtaining new derivative-free SRK schemes. Komori [12, 13] has proposed new weak second-order SRK schemes for not only commutative SDEs but also non-commutative SDEs, whereas Rößler [21, 23] has proposed new weak second-order SRK schemes for commutative SDEs or SDEs with a scalar Wiener process. Thus, we can see that the families are general enough to obtain new weak SRK schemes. Since all their schemes are explicit, however, our present aim is to seek new fully implicit schemes that have good stability properties for SDEs with a scalar Wiener process on the basis of Komori's SRK family.

The organization of the present paper is as follows. In the next section we will basic notations, definitions and concepts. In Section 3 we will find a solution of order conditions for each of implicit SRK methods with one or two stages in a general form including free parameters. In Section 4 we will decide the values of the free parameters and obtain schemes possessing good stability properties. In Section 5 we will give numerical experiments to confirm convergence order and stability properties for the schemes. In the last section we will give the summary and remarks.

2 Preliminaries

In this section we introduce an SRK family which gives a weak approximation to a solution for SDEs with a scalar Wiener process, an expression of weak order conditions for it, and the concepts of stability for SRK methods.

2.1 SRK family

First of all, we introduce some notations and the definition of weak order. Consider the autonomous d-dimensional SDE

$$\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{g}_0(\boldsymbol{y}(t))\mathrm{d}t + \boldsymbol{g}_1(\boldsymbol{y}(t)) \circ \mathrm{d}W(t), \quad \boldsymbol{y}(0) = \boldsymbol{x}_0, \tag{2. 1}$$

where $W(s)$ is a scalar Wiener process, $\boldsymbol{x}_0$ is independent of $W(t)-W(0)$ for $t \geq 0$, and $\circ$ means the Stratonovich formulation. When the following Lipschitz condition is satisfied, the SDE has exactly one continuous global solution on the entire interval $[0,\infty)$ ([1], p. 113): there exists a positive constant K such that, for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^d$,

$$||\boldsymbol{g}_0(\boldsymbol{x}) - \boldsymbol{g}_0(\boldsymbol{y})|| + ||\boldsymbol{g}_1(\boldsymbol{x}) - \boldsymbol{g}_1(\boldsymbol{y})|| \leq K||\boldsymbol{x} - \boldsymbol{y}||.$$

For a given time T_{end}, let τ_n be an equidistant grid point nh $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\mathrm{def}}{=} T_{end}/M < 1$ (M is a natural number) and $\boldsymbol{y}_n$ a discrete approximation to the solution $\boldsymbol{y}(\tau_n)$ of (2. 1). The initial approximate random variable $\boldsymbol{y}_0$ is supposed to have the same probability law with all moments finite as that of $\boldsymbol{x}_0$. In addition, let $C_P^L(\boldsymbol{R}^d, \boldsymbol{R})$ be the totality of L times continuously differentiable $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^d$, all of whose partial derivatives of order less than or equal to L have polynomial growth. Then, the definition of weak order is given as follows [3].

Definition 2.1 *Suppose that discrete approximations $\boldsymbol{y}_n$ are given by a scheme. Then, we say that the scheme is of weak (global) order q if for each $G \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$, $C > 0$ (independent of h) and $\delta > 0$ exist such that*

$$|E[G(\boldsymbol{y}(\tau_M)] - E[G(\boldsymbol{y}_M)]| \leq Ch^q, \quad h \in (0, \delta).$$

In order to obtain an approximate solution $\boldsymbol{y}_{n+1}$ of the solution $\boldsymbol{y}(t_{n+1})$ when $\boldsymbol{y}_n$ is given, we consider the SRK family given by

$$\boldsymbol{y}_{n+1} \;=\; \boldsymbol{y}_n + \sum_{i=1}^{s}\sum_{j=0}^{1} c_i^{(j)} \boldsymbol{Y}_i^{(j)}, \quad \boldsymbol{Y}_{i_a}^{(j_a)} = \tilde{\eta}_{i_a}^{(j_a)} \boldsymbol{g}_{j_a}(\boldsymbol{y}_n + \sum_{i_b=1}^{s}\sum_{j_b=0}^{1} \alpha_{i_a i_b}^{(j_a,j_b)} \boldsymbol{Y}_{i_b}^{(j_b)}) \tag{2. 2}$$

$(1 \leq i_a \leq s,\ j_a = 0, 1)$, where the constants $c_i^{(j)}$ and $\alpha_{i_a i_b}^{(j_a,j_b)}$ are defined by the Butcher tableau and where each $\tilde{\eta}_{i_a}^{(j_a)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\tilde{\eta}_{i_a}^{(j_a)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j_a = 0), \\ K_2 h^k & (j_a = 1) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. Note that although this is a specific formulation of a generalized SRK family given in [12], it is sufficiently useful since (2. 1) has a scalar Wiener process only and we consider weak second order at most in the sequel.

2.2 Weak order conditions by bi-colored rooted trees

In this subsection we give a brief introduction of an expression of weak order conditions by bi-colored rooted trees (BRTs).

First, we introduce the bi-colored rooted tree (BRT) and a function on its set.

Definition 2.2 (BRT) *A BRT with a root ⓙ (colored with a label j $(= 0, 1)$) is a tree recursively defined in the following manner:*

1) *$\tau^{(j)}$ is the primitive tree having only a vertex ⓙ.*

2) *If $t_1, \ldots, t_k$ are BRTs, then $[t_1, \ldots, t_k]^{(j)}$ is also a BRT with the root ⓙ.*

The totality of BRTs is denoted by T. Examples of BRTs are indicated in Fig. 1.

$\tau^{(1)}$ $\quad$ $[\tau^{(0)}, \tau^{(1)}]^{(1)}$

Figure 1: Examples of BRTs

Definition 2.3 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows:*

$$\Phi(\tau^{(j)}; s) = \int_{\tau_n}^{s} \circ \mathrm{d}W_j(s_1), \qquad \Phi(t; s) = \int_{\tau_n}^{s} \prod_{i=1}^{k} \Phi(t_i; s_1) \circ \mathrm{d}W_j(s_1) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)},$$

where $W_j(s_1)$ stands for s_1 if $j = 0$, or $W(s_1)$ if $j = 1$.

For ease of notation we will denote $\Phi(t; \tau_{n+1})$ by $\Phi(t)$.

Next, we introduce another function to relate T to the formula parameters of (2. 2).

Definition 2.4 (Elementary numerical weight $\tilde{\Phi}(t)$ on T) *Let s be the stage number of (2. 2). An elementary numerical weight of $t \in T$ is given in the following manner:*

i) *Trace the vertices of t in the direction from the root to upper vertices. Then, for the root vertex, prepare an index i_1 and set $\Theta = c_{i_1}^{(j_1)} \tilde{\eta}_{i_1}^{(j_1)}$ if the color is j_1. For each vertex except the root vertex, prepare a new index i_{k+1}, multiply Θ by $\alpha_{i_k i_{k+1}}^{(j_k, j_{k+1})} \tilde{\eta}_{i_{k+1}}^{(j_{k+1})}$ if the color is j_{k+1}, and reset Θ by it, where i_k and j_k mean the index and the color of the parent vertex, respectively.*

ii) *Define $\tilde{\Phi}(t)$ by the summation of Θ over $i.$ from 1 to s.*

For example,

$$\tilde{\Phi}\left([\tau^{(0)}, \tau^{(1)}]^{(1)}\right) = \sum_{i_1, i_2, i_3 = 1}^{s} c_{i_1}^{(1)} \tilde{\eta}_{i_1}^{(1)} \alpha_{i_1 i_2}^{(1,0)} \tilde{\eta}_{i_2}^{(0)} \alpha_{i_1 i_3}^{(1,1)} \tilde{\eta}_{i_3}^{(1)}.$$

The definitions above are slightly simpler than those in [13]. This is because we deal with the specific formulation of the SRK family as we have said in the previous subsection.

Now, we can give weak order conditions. Let $\rho(t)$ be the number of vertices of $t \in T$ and $r(t)$ the number of vertices of t with the color 0, and suppose that any component of $\boldsymbol{g}_j$ belongs to $C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$ $(j = 0, 1)$ and the regularity of the time discrete approximation is satisfied [10, 12]. In addition, if the following are satisfied, the time discrete approximation $\boldsymbol{y}_M$ converges to the $\boldsymbol{y}(\tau_M)$ with weak (global) order q as $h \to 0$:

$$E\left[\prod_{j=1}^{L} \tilde{\Phi}(t_j)\right] = E\left[\prod_{j=1}^{L} \Phi(t_j)\right] \tag{2. 3}$$

for any $t_1, \ldots, t_L \in T$ $(1 \leq L \leq 2q)$ satisfying $\sum_{j=1}^{L}(\rho(t_j) + r(t_j)) \leq 2q$ and

$$E\left[\tilde{\Phi}(t)\right] = 0 \tag{2. 4}$$

for any $t \in T$ satisfying $\rho(t) + r(t) = 2q + 1$. For details about calculations for the expectations of both sides of (2. 3), see [13].

2.3 Stability for SRK methods

In this subsection we introduce some concepts of stability. As preliminaries, we begin with the concepts of stability in mean square for SDEs.

Suppose that $\boldsymbol{g}_0(\boldsymbol{0}) = \boldsymbol{g}_1(\boldsymbol{0}) = \boldsymbol{0}$ and $\boldsymbol{x}_0$ is a constant with probability 1 in (2. 1). Then, $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ is the unique solution of (2. 1) if $\boldsymbol{x}_0 = \boldsymbol{0}$. This trivial solution is called the equilibrium position. The definition of stability in mean square is given as follows ([1], p. 188).

Definition 2.5 *The equilibrium position is said to be stable in mean square if, for every* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$\sup_{0 \leq t < \infty} E[||\boldsymbol{y}(t)||^2 \leq \epsilon \qquad \text{for} \ \ ||\boldsymbol{x}_0|| \leq \delta.$$

Further if

$$\lim_{t \to \infty} E[||\boldsymbol{y}(t)||^2] = 0 \qquad \text{for all} \ \ \boldsymbol{x}_0 \text{ in a neighborhood of } \boldsymbol{y} = \boldsymbol{0},$$

the equilibrium position is said to be asymptotically stable in mean square.

Let us consider the case that $\boldsymbol{g}_0(\boldsymbol{y}) = A\boldsymbol{y}$ and $\boldsymbol{g}_1(\boldsymbol{y}) = B\boldsymbol{y}$, where $d \times d$ real matrices A and B are diagonalizable and commute, that is, they are simultaneously diagonalizable ([9], p. 50). Then, asymptotic mean square stability of the equilibrium position is equivalent to

$$\max_{1 \leq i \leq d} \left\{ \Re(\lambda_i(A)) + \Big(\Re(\lambda_i(B))\Big)^2 \right\} < 0, \tag{2. 5}$$

where $\lambda_i(A)$ and $\lambda_i(B)$ denote the ith eigenvalues of A and B, respectively [14]. If we set $\lambda = \lambda_j(A)$ and $\sigma = \lambda_j(B)$ for j such that $\Re(\lambda_j(A)) + (\Re(\lambda_j(B)))^2$ equals the expression in the left-hand side of (2. 5), asymptotic mean square stability in the multi-dimensional linear SDE is equivalent to that in the scalar SDE

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \sigma y(t) \circ \mathrm{d}w(t), \qquad y(0) = x_0, \qquad \lambda, \sigma \in \boldsymbol{C}, \tag{2. 6}$$

which is one of the scalar SDEs obtained from the multi-dimensional linear SDE by diagonalization and for which (2. 5) corresponds to

$$\Re(\lambda) + \Big(\Re(\sigma)\Big)^2 < 0. \tag{2. 7}$$

Thus, when we consider stability, we will devote ourselves to dealing with (2. 6), provided that $x_0 \neq 0$ with probability 1.

When a one-step scheme is applied to (2. 6), it is expressed by

$$y_{n+1} = R_1(h, \tilde{\boldsymbol{\eta}}, \lambda, \sigma) y_n$$

in general [5], where $\tilde{\boldsymbol{\eta}}$ stands for a vector of random variables and the pth moment of each of its elements is supposed to be a monomial of h for $p(= 1, 2, \ldots)$. We will call R_1 the amplification factor [14]. The numerical counterpart of asymptotic mean square stability is given as follows [5, 25].

Definition 2.6 *The numerical scheme is said to be MS-stable for a particular h, λ and σ if*

$$R_2(h, \lambda, \sigma) \stackrel{\text{def}}{=} E[|R_1(h, \tilde{\boldsymbol{\eta}}, \lambda, \sigma)|^2] < 1.$$

MS-stability means that $E[|y_n|^2] \to 0$ as $n \to \infty$ for the given h, $\tilde{\boldsymbol{\eta}}$, λ and σ. Further, the counterpart of deterministic A-stability is given as follows [8].

Definition 2.7 *The numerical scheme is said to be A-stable in mean square if it is MS-stable for any h when (2. 7) holds.*

3 Implicit SRK methods

On the basis of (2. 2), we derive implicit SRK methods in a general form including free parameters. The order conditions will be clearly arranged with help of BRTs, and by utilizing some results in ordinary Runge-Kutta, they will be solved in a transparent way.

3.1 Method with one stage

We derive a weak first-order implicit SRK method with one stage. For this, let us start with simplifying assumptions, which are helpful for simplification of order conditions [11, 13].

In relation to $\tau^{(0)}$, $\tau^{(1)}$ and $[\tau^{(1)}]^{(1)}$, assume that the following equations hold:

$$\sum c_{i_1}^{(0)} \tilde{\eta}_{i_1}^{(0)} = h, \qquad \sum c_{i_1}^{(1)} \tilde{\eta}_{i_1}^{(1)} = \triangle\tilde{W}, \qquad \sum c_{i_1}^{(1)} \tilde{\eta}_{i_1}^{(1)} \tilde{\alpha}_{i_1 i_2}^{(1,1)} \tilde{\eta}_{i_2}^{(1)} = \frac{(\triangle\tilde{W})^2}{2}, \tag{3. 1}$$

where $\triangle\tilde{W}$ is a bounded random variables satisfying

$$E\left[(\triangle\tilde{W})^k\right] = \begin{cases} 0 & (k = 1, 3), \\ h & (k = 2), \\ O(h^2) & (k \geq 4). \end{cases} \tag{3. 2}$$

Further, when we set $\tilde{\eta}_i^{(j)} = h$ $(j = 0)$ or $\triangle\tilde{W}$ $(j = 1)$, we have $E[\tilde{\Phi}(t)] = 0$ for $\tau^{(1)}$, $t = [\tau^{(1)}]^{(0)}$, $[\tau^{(0)}]^{(1)}$, $[[\tau^{(1)}]^{(1)}]^{(1)}$ and $[\tau^{(1)}, \tau^{(1)}]^{(1)}$. Thus, the following statement is true for the BRTs who appear in (2. 3) and (2. 4) when $q = 1$:

Statement 3.1 *The expectation of an elementary numerical weight or the product of those is equal to* 0 *if the odd number of vertices are of the color* 1.

Table 1: Conditions to satisfy for weak order 1

No.	Condition	No.	Condition	No.	Condition
1	$\sum c_i^{(0)} = 1$	2	$\sum c_i^{(1)} = 1$	3	$\sum c_i^{(1)} A_i^{(1,1)} = \frac{1}{2}$

Since the counterpart of this statement is always true in the elementary weights [13], the number of order conditions to solve decreases if the statement holds.

Now, because the statement holds, (2. 4) automatically holds when $q = 1$. In addition, from (3. 1) and (3. 2), (2. 3) holds when $q = 1$. Thus, we can seek weak first-order SRK methods by solving (3. 1) under (3. 2).

By setting $A_{i_1}^{(j_1,j_2)} \stackrel{\text{def}}{=} \sum_{i_2=1}^{s} \alpha_{i_1 i_2}^{(j_1,j_2)}$ for ease of notation, we obtain the three conditions for weak order 1 shown in Table 1. The system of Conditions 2 and 3 has the same algebraic structure as that of order conditions for ordinary Runge-Kutta methods to attain order 2 for ODEs. Hence, the stage number s has to be at least 1. When we set $s = 1$, the solution of the system is uniquely determined ([6], p. 201): $c_1^{(1)} = 1$ and $A_1^{(1,1)} = \frac{1}{2}$. From Condition 1 we have $c_1^{(0)} = 1$. On the other hand, $A_1^{(0,0)}$, $A_1^{(0,1)}$ and $A_1^{(1,0)}$ can take any value. By setting $A_i^{(0,0)} = 1/2$, we obtain an implicit method with stage one:

$$\begin{array}{c|c} \left[\alpha_{i_a i_b}^{(0,0)}\right] & \left[\alpha_{i_a i_b}^{(1,0)}\right] \\ \left[\alpha_{i_a i_b}^{(0,1)}\right] & \left[\alpha_{i_a i_b}^{(1,1)}\right] \\ \hline \left(\boldsymbol{c}^{(0)}\right)^\top & \left(\boldsymbol{c}^{(1)}\right)^\top \end{array} = \begin{array}{c|c} \frac{1}{2} & \alpha_{11}^{(1,0)} \\ \alpha_{11}^{(0,1)} & \frac{1}{2} \\ \hline 1 & 1 \end{array}, \qquad (3.\ 3)$$

which is of weak order 1 and which is of order 2 for ODEs. Here, note that $\alpha_{11}^{(0,1)}$ and $\alpha_{11}^{(1,0)}$ are free parameters.

3.2 Method with two stages

We have solved the order conditions for weak order 1. In this subsection, we consider implicit SRK methods with two stages and find a solution of the order conditions for weak order 2 in a similar way.

In relation to $\tau^{(0)}$, $\tau^{(1)}$, $[\tau^{(1)}]^{(1)}$, $[\tau^{(1)}]^{(0)}$ and $[\tau^{(0)}]^{(1)}$, let us assume that the following equations hold:

$$\begin{cases} \sum c_i^{(0)} \tilde{\eta}_i^{(0)} = h, \quad \sum c_i^{(1)} \tilde{\eta}_i^{(1)} = \triangle W, \quad \sum c_{i_1}^{(1)} \tilde{\eta}_{i_1}^{(1)} \tilde{\alpha}_{i_1 i_2}^{(1,1)} \tilde{\eta}_{i_2}^{(1)} = \dfrac{(\triangle W)^2}{2}, \\ \sum c_{i_1}^{(0)} \tilde{\eta}_{i_1}^{(0)} \tilde{\alpha}_{i_1 i_2}^{(0,1)} \tilde{\eta}_{i_2}^{(1)} = \sum c_{i_1}^{(1)} \tilde{\eta}_{i_1}^{(1)} \tilde{\alpha}_{i_1 i_2}^{(1,0)} \tilde{\eta}_{i_2}^{(0)} = \dfrac{h \triangle W}{2}, \end{cases} \qquad (3.\ 4)$$

where $\triangle W$ is a bounded random variable satisfying

$$E\left[(\triangle W)^k\right] = \begin{cases} 0 & (k = 1, 3, 5), \\ (k-1)h^{k/2} & (k = 2, 4), \\ O(h^3) & (k \geq 6). \end{cases}$$

Further, we set $\tilde{\eta}_i^{(j)} = h$ $(j = 0)$ or $\triangle W$ $(j = 1)$. Then, the statement 3.1 holds for the BRTs who appear in (2. 3) and (2. 4) when $q = 2$. Thus, since (2. 4) automatically holds when $q = 2$, we can restrict our attention to (2. 3). Furthermore, since 14 order conditions in relation to $\tau^{(0)}$, $\tau^{(1)}$, $[\tau^{(1)}]^{(1)}$, $[\tau^{(1)}]^{(0)}$ and $[\tau^{(0)}]^{(1)}$ are satisfied by (3. 4) as shown in [11, 13], all we have to do is to solve the system of seventeen equations in Tables 1 and 2, which include the simplifying conditions.

Table 2: Conditions to satisfy for weak order 2

No.	Condition	No.	Condition
4	$\sum c_i^{(0)} A_i^{(0,1)} = \frac{1}{2}$	11	$\sum c_i^{(1)} A_i^{(1,0)} A_i^{(1,1)} = \frac{1}{4}$
5	$\sum c_i^{(1)} A_i^{(1,0)} = \frac{1}{2}$	12	$\sum c_{i_1}^{(1)} \alpha_{i_1 i_2}^{(1,1)} \alpha_{i_2 i_3}^{(1,1)} A_{i_3}^{(1,1)} = \frac{1}{24}$
6	$\sum c_i^{(0)} A_i^{(0,0)} = \frac{1}{2}$	13	$\sum c_{i_1}^{(1)} \alpha_{i_1 i_2}^{(1,1)} \left(A_{i_2}^{(1,1)}\right)^2 = \frac{1}{12}$
7	$\sum c_{i_1}^{(1)} \alpha_{i_1 i_2}^{(1,1)} A_{i_2}^{(1,0)} = \frac{1}{4}$	14	$\sum c_{i_1}^{(1)} A_{i_1}^{(1,1)} \alpha_{i_1 i_2}^{(1,1)} A_{i_2}^{(1,1)} = \frac{1}{8}$
8	$\sum c_{i_1}^{(0)} \alpha_{i_1 i_2}^{(0,1)} A_{i_2}^{(1,1)} = \frac{1}{4}$	15	$\sum c_i^{(1)} \left(A_i^{(1,1)}\right)^3 = \frac{1}{4}$
9	$\sum c_{i_1}^{(1)} \alpha_{i_1 i_2}^{(1,0)} A_{i_2}^{(0,1)} = 0$	16	$\sum c_{i_1}^{(1)} \alpha_{i_1 i_2}^{(1,1)} A_{i_2}^{(1,1)} = \frac{1}{6}$
10	$\sum c_i^{(0)} \left(A_i^{(0,1)}\right)^2 = \frac{1}{2}$	17	$\sum c_i^{(1)} \left(A_i^{(1,1)}\right)^2 = \frac{1}{3}$

The system of Conditions 2, 3, 12, 13, 14, 15, 16 and 17 has the same algebraic structure as that of the order conditions for ordinary Runge-Kutta methods to attain order 4 for ODEs ([6], pp. 90-91). Hence, the stage number s has to be at least 2. When we set $s = 2$, the solution of the system of the eight conditions is uniquely determined ([6], p. 202):

$$\begin{array}{c|c} \boldsymbol{A}^{(1,1)} & \left[\alpha_{i_a i_b}^{(1,1)}\right] \\ \hline & \left(\boldsymbol{c}^{(1)}\right)^\top \end{array} = \begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}. \tag{3. 5}$$

From Conditions 5, 11 and this, we have $A_1^{(1,0)} = A_2^{(1,0)} = 1/2$, which also satisfies Condition 7. By noting that $A_1^{(0,1)} = A_2^{(0,1)}$ leads to a conflict with Conditions 1, 4 and 10, we obtain from them

$$c_1^{(0)} = \frac{1}{2\delta_0}, \quad c_2^{(0)} = \frac{2\delta_0 - 1}{2\delta_0}, \quad A_2^{(0,1)} = \frac{A_1^{(0,1)} - 1}{2A_1^{(0,1)} - 1} \quad (A_1^{(0,1)} \neq \frac{1}{2}),$$

where $\delta_0 \stackrel{\mathrm{def}}{=} 2\left(A_1^{(0,1)}\right)^2 - 2A_1^{(0,1)} + 1$. Then, we have

$$A_2^{(0,0)} = \frac{\delta_0 - A_1^{(0,0)}}{2\delta_0 - 1}, \quad \alpha_{21}^{(0,1)} = \frac{\delta_0 - 2\alpha_{11}^{(0,1)}}{2\left(2\delta_0 - 1\right)}, \quad \alpha_{21}^{(1,0)} = -\alpha_{11}^{(1,0)} + \frac{1 - A_1^{(0,1)}}{\delta_0}$$

from Conditions 6, 8 and 9, respectively. Since $c_1^{(0)} = c_2^{(0)} = 1/2$ and $A_2^{(0,0)} = 1 - A_1^{(0,0)}$ if $A_1^{(0,1)} = 0$ or 1, we can take the set of values in the right-hand side of (3. 5) as a set of values of $A_i^{(0,0)}$'s, $\alpha_{i_a i_b}^{(0,0)}$'s and $c_i^{(0)}$'s, which enables a scheme to attain order 4 for ODEs. For $A_1^{(0,1)} = 0$ or 1, we finally obtain

$$\begin{array}{c|c} \left[\alpha_{i_a i_b}^{(0,0)}\right] & \left[\alpha_{i_a i_b}^{(1,0)}\right] \\ \left[\alpha_{i_a i_b}^{(0,1)}\right] & \left[\alpha_{i_a i_b}^{(1,1)}\right] \\ \hline \left(\boldsymbol{c}^{(0)}\right)^\top & \left(\boldsymbol{c}^{(1)}\right)^\top \end{array} = \begin{array}{cc|cc} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} & \alpha_{11}^{(1,0)} & \frac{1}{2} - \alpha_{11}^{(1,0)} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} & 1 - \delta_1 & \delta_1 - \frac{1}{2} \\ \alpha_{11}^{(0,1)} & \delta_2 & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} - \alpha_{11}^{(0,1)} & \frac{1}{2} - \delta_2 & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \tag{3. 6}$$

as a solution of all the order conditions. Here, $\delta_1 \stackrel{\text{def}}{=} A_1^{(0,1)} + \alpha_{11}^{(1,0)}$ and $\delta_2 \stackrel{\text{def}}{=} A_1^{(0,1)} - \alpha_{11}^{(0,1)}$. Note that the set of values of $c_i^{(1)}$'s and $\alpha_{i_a i_b}^{(1,1)}$'s is unique for weak order 2.

4 Stability analysis

In the previous section we have found solutions of the order conditions for SRK methods with one or two stages, which includes the free parameters. In this section, let us consider how to decide the values of the parameters and seek schemes possessing good stability properties.

4.1 MS-stable scheme with one stage

First of all, we represent the amplification factor for (2. 2) in a general form. When we apply (2. 2) to (2. 6), by utilizing Cramer's rule we obtain

$$R_1(h, \tilde{\boldsymbol{\eta}}, \lambda, \sigma) = \frac{\det\left(I - \tilde{D}\tilde{B}\tilde{A} + \tilde{D}\tilde{B}\mathbf{1}\boldsymbol{c}^\top\right)}{\det\left(I - \tilde{D}\tilde{B}\tilde{A}\right)}, \tag{4. 1}$$

where

$$\tilde{A} \stackrel{\text{def}}{=} \begin{bmatrix} \alpha_{11}^{(0,0)} & \alpha_{11}^{(0,1)} & \cdots & \alpha_{1s}^{(0,0)} & \alpha_{1s}^{(0,1)} \\ \alpha_{11}^{(1,0)} & \alpha_{11}^{(1,1)} & \cdots & \alpha_{1s}^{(1,0)} & \alpha_{1s}^{(1,1)} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{s1}^{(0,0)} & \alpha_{s1}^{(0,1)} & \cdots & \alpha_{ss}^{(0,0)} & \alpha_{ss}^{(0,1)} \\ \alpha_{s1}^{(1,0)} & \alpha_{s1}^{(1,1)} & \cdots & \alpha_{ss}^{(1,0)} & \alpha_{ss}^{(1,1)} \end{bmatrix}, \quad \begin{aligned} \tilde{D} &\stackrel{\text{def}}{=} \operatorname{diag}(\tilde{\eta}_1^{(0,0)}, \tilde{\eta}_1^{(1,1)}, \ldots, \tilde{\eta}_s^{(0,0)}, \tilde{\eta}_s^{(1,1)}), \\ \tilde{B} &\stackrel{\text{def}}{=} \operatorname{diag}(\lambda, \sigma, \ldots, \lambda, \sigma), \\ \boldsymbol{c} &\stackrel{\text{def}}{=} \left[c_1^{(0)}, c_1^{(1)}, \ldots, c_s^{(0)}, c_s^{(1)}\right]^\top \end{aligned}$$

and $\mathbf{1}$ stands for a $2s$-dimensional column vector of 1's.

For ease of notation, denote by $D_1(h, \tilde{\boldsymbol{\eta}}, \lambda, \sigma)$ and $N_1(h, \tilde{\boldsymbol{\eta}}, \lambda, \sigma)$ the denominator and the numerator in the right-hand side of (4. 1), respectively. From (3. 3),

$$D_1(h, \triangle\tilde{W}, \lambda, \sigma) = \left(\frac{\lambda h}{2} - 1\right)\left(\frac{\sigma\triangle\tilde{W}}{2} - 1\right) - \alpha_{11}^{(0,1)}\alpha_{11}^{(1,0)}\lambda h\sigma\triangle\tilde{W}, \tag{4. 2}$$

$$N_1(h, \triangle W, \lambda, \sigma) = \left(\frac{\lambda h}{2} + 1\right)\left(\frac{\sigma\triangle\tilde{W}}{2} + 1\right) - (\alpha_{11}^{(0,1)}\alpha_{11}^{(1,0)} + \delta_3)\lambda h\sigma\triangle\tilde{W}, \tag{4. 3}$$

where $\delta_3 \stackrel{\text{def}}{=} 1 - \alpha_{11}^{(0,1)} - \alpha_{11}^{(1,0)}$. Now, we can see that $R_1(h, \triangle\tilde{W}, \lambda, \sigma)$ is regarded as a function of $z \stackrel{\text{def}}{=} h\lambda$ and $\sigma\triangle\tilde{W}$, say, $\tilde{R}_1(z, \sigma\triangle\tilde{W})$.

Suppose that a pair $(\alpha_{11}^{(0,1)}, \alpha_{11}^{(1,0)})$ is given. In addition, denote $\Re(z)$ and $\Im(z)$ by z_r and z_i. Then, if $|\tilde{R}_1(\mathrm{i}z_i, \mathrm{i}\Im(\sigma)\triangle\tilde{W})| = 1$ and $\tilde{R}_1(z, \mathrm{i}\Im(\sigma)\triangle\tilde{W})$ is analytic for any z such that $z_r < 0$, $|\tilde{R}_1(z, \mathrm{i}\Im(\sigma)\triangle\tilde{W})| < 1$ holds in the region $z_r < 0$. This equation is equivalent to

$$\begin{aligned} &\left\{\Re\Big(N_1(h, \triangle\tilde{W}, \mathrm{i}\Im(\lambda), \mathrm{i}\Im(\sigma))\Big)\right\}^2 + \left\{\Im\Big(N_1(h, \triangle\tilde{W}, \mathrm{i}\Im(\lambda), \mathrm{i}\Im(\sigma))\Big)\right\}^2 \\ = \;&\left\{\Re\Big(D_1(h, \triangle\tilde{W}, \mathrm{i}\Im(\lambda), \mathrm{i}\Im(\sigma))\Big)\right\}^2 + \left\{\Im\Big(D_1(h, \triangle\tilde{W}, \mathrm{i}\Im(\lambda), \mathrm{i}\Im(\sigma))\Big)\right\}^2. \end{aligned}$$

Since

$$\Re\Big(N_1(h,\triangle\tilde{W},\mathrm{i}\Im(\lambda),\mathrm{i}\Im(\sigma))\Big)-\Re\Big(D_1(h,\triangle\tilde{W},\mathrm{i}\Im(\lambda),\mathrm{i}\Im(\sigma))\Big)=z_i\Im(\sigma)\triangle\tilde{W}\delta_3,$$
$$\Im\Big(N_1(h,\triangle\tilde{W},\mathrm{i}\Im(\lambda),\mathrm{i}\Im(\sigma))\Big)+\Im\Big(D_1(h,\triangle\tilde{W},\mathrm{i}\Im(\lambda),\mathrm{i}\Im(\sigma))\Big)=0$$

from (4. 2) and (4. 3), the equation is satisfied when $\delta_3 = 0$. By noting that $|R_1(h,\triangle\tilde{W},\lambda,\mathrm{i}\Im(\sigma))| < 1$ with probability 1 means $R_2(h,\lambda,\mathrm{i}\Im(\sigma)) < 1$, we can say that it yields an A-stable scheme in the case of $\Re(\sigma) = 0$ to find a pair $(\alpha_{11}^{(0,1)}, \alpha_{11}^{(1,0)})$ which satisfies $\delta_3 = 0$ and for which $\tilde{R}_1(z,\mathrm{i}\Im(\sigma)\triangle\tilde{W})$ is analytic in the region $z_r < 0$.

Let $\alpha_{11}^{(1,0)}$ be $1-\alpha_{11}^{(0,1)}$ to satisfy $\delta_3 = 0$ and $\triangle W$ a two-point distributed random variable with $P(\triangle W = \pm\sqrt{h}) = 1/2$. Then, $R_2(h,\Re(\lambda),\Re(\sigma)) - 1$ is expressed by

$$R_2(h,\Re(\lambda),\Re(\sigma))-1=\frac{N_2\left(z_r,v_r;\alpha_{11}^{(0,1)}\right)}{D_2\left(z_r,v_r;\alpha_{11}^{(0,1)}\right)},$$

where D_2 and N_2 are polynomials (including the parameter $\alpha_{11}^{(0,1)}$) with respect to z_r and $v_r \stackrel{\text{def}}{=} h(\Re(\sigma))^2$. Since it is possible to re-scale D_2 and N_2 by an arbitrary factor, let us assume that D_2 and N_2 are re-scaled such that the coefficient of the principal term in D_2 is equal to $(1-4\alpha_{11}^{(0,1)}(1-\alpha_{11}^{(0,1)}))^4$. By some calculations, we can obtain

$$\begin{aligned}&N_2\left(z_r,-z_r;\alpha_{11}^{(0,1)}\right)\\&\quad=8z_r^2\left\{(2\alpha_{11}^{(0,1)}+1)(2\alpha_{11}^{(0,1)}-3)\delta_4^4z_r^3+4\delta_4^2z_r^2+4(\delta_4^2+4)z_r+16(\delta_4^2-3)\right\},\end{aligned}$$

where $\delta_4 \stackrel{\text{def}}{=} 2\alpha_{11}^{(0,1)}-1$. In the right-hand side, thus, only if $\alpha_{11}^{(0,1)} = 1/2$, it follows that the coefficients of z_r^3 and z_r^5 are non-negative, whereas those of z_r^2 and z_r^4 are non-positive. Thus, let us set the value of $\alpha_{11}^{(0,1)}$ to 1/2.

When $\alpha_{11}^{(0,1)} = \alpha_{11}^{(1,0)} = 1/2$, we can immediately see from (4. 2) that $D_1(h,\triangle\tilde{W},\lambda,\mathrm{i}\Im(\sigma)) \neq 0$ in the region $z_r < 0$ and $D_1(h,\triangle\tilde{W},\Re(\lambda),\Re(\sigma)) > 0$ in the region $z_r < -v_r$. In addition,

$$N_2\left(z_r,v_r;\frac{1}{2}\right)=128\left\{(z_r-2)^2z_r+v_r(4-z_r)\right\}<128z_r^2(z_r-3)<0$$

in the region $z_r < -v_r$. Consequently, (3. 3) for $\alpha_{11}^{(0,1)} = \alpha_{11}^{(1,0)} = 1/2$ is A-stable not only in the case of $\Re(\sigma) = 0$ but also in the case of $\Im(\lambda) = \Im(\sigma) = 0$. In the sequel, this scheme will be called ISRK1.

4.2 MS-stable scheme with two stages

In a similar way, let us decide the values of the three free parameters included in the solution of the order conditions in Subsection 3.2. Since one of them, $A_1^{(0,1)}$, takes 0 or 1, we deal with each case separately.

Let us begin with the case of $A_1^{(0,1)} = 0$. From (3. 6) and $A_1^{(0,1)} = 0$,

$$\begin{aligned}D_1(h,\triangle W,\lambda,\sigma)=&\left(\frac{z^2}{12}-\frac{z}{2}+1\right)\left(\frac{(\sigma\triangle W)^2}{12}+1\right)+\left(\frac{\sqrt{3}-2}{6}z^2+\frac{z}{2}-1\right)\frac{\sigma\triangle W}{2}\\&+z\left(\frac{(1-\sqrt{3})z+2\sqrt{3}}{12}\delta_5-\delta_6\right)(\sigma\triangle W)^2+z(2-z)\sigma\triangle W\delta_6,\quad(4.\ 4)\end{aligned}$$

$$N_1(h,\triangle W,\lambda,\sigma) = \left(\frac{z^2}{12}+\frac{z}{2}+1\right)\left(\frac{(\sigma\triangle W)^2}{12}+1\right)+\left(\frac{2-\sqrt{3}}{6}z^2+\frac{z}{2}+1\right)\frac{\sigma\triangle W}{2}$$
$$+\,z\left(\frac{(\sqrt{3}-1)z+2\sqrt{3}}{12}\delta_5+\delta_6\right)(\sigma\triangle W)^2+z(2+z)\sigma\triangle W\delta_6, \quad (4.\ 5)$$

where $\delta_5 \stackrel{\text{def}}{=} \alpha_{11}^{(0,1)}+\alpha_{11}^{(1,0)}-1/2$ and $\delta_6 \stackrel{\text{def}}{=} \alpha_{11}^{(0,1)}\left(1-2\alpha_{11}^{(1,0)}\right)$. Similarly to the previous subsection, we can see that it yields an A-stable scheme in the case of $\Re(\sigma)=0$ to find a pair $(\alpha_{11}^{(0,1)},\alpha_{11}^{(1,0)})$ which satisfies $\delta_5=0$ and for which $\tilde{R}_1(z,\mathrm{i}\Im(\sigma)\triangle W)$ is analytic in the region $z_r<0$.

Let $\alpha_{11}^{(1,0)}$ be $1/2-\alpha_{11}^{(0,1)}$ to satisfy $\delta_5=0$ and $\triangle W$ a three-point distributed random variable with $P(\triangle W=\pm\sqrt{3h})=1/6$ and $P(\triangle W=0)=2/3$. Then, let us consider the polynomial $D_2\left(z_r,v_r;\alpha_{11}^{(0,1)}\right)$, provided that D_2 and N_2 are re-scaled such that the coefficient of the principal term in D_2 is equal to 1.

Suppose that $D_2\left(z_r,v_r;\alpha_{11}^{(0,1)}\right)$ does not vanish in the region $z_r<0$, and then let us devote ourselves to $N_2\left(z_r,v_r;\alpha_{11}^{(0,1)}\right)$. By utilizing (4. 5), we can obtain

$$N_2\left(z_r,v_r;\alpha_{11}^{(0,1)}\right)=128\left(2-\sqrt{3}+24\left(\alpha_{11}^{(0,1)}\right)^2\right)^2 v_r^3z_r^{12}+24\left(1+16\left(\alpha_{11}^{(0,1)}\right)^2\right)v_r^4z_r^{11}+R,$$

where R stands for the remainder terms with orders less than those of the first two terms in the right-hand side. The substitution of $v_r=-z_r$ into the expression in the right-hand side of the equation yields

$$-8\left(9216\left(\alpha_{11}^{(0,1)}\right)^4+48(31-16\sqrt{3})\left(\alpha_{11}^{(0,1)}\right)^2+109-64\sqrt{3}\right)z_r^{15}+O(z_r^{14}).$$

Thus, if

$$\left(\alpha_{11}^{(0,1)}\right)^2<\frac{-31+16\sqrt{3}+\sqrt{-15+32\sqrt{3}}}{384}\ (\approx 8.00\times10^{-3}), \quad (4.\ 6)$$

the coefficient of z_r^{15} is positive. Then, further, the coefficients of $z_r^{n_o}$ also become positive and the coefficients of $z_r^{n_e}$ become negative for $n_o=1,3,\ldots,13$ and $n_e=0,2,\ldots,14$. Thus, when (4. 6) is satisfied, $N_2\left(z_r,-z_r;\alpha_{11}^{(0,1)}\right)<0$ holds in the region $z_r<0$. Finally, by plotting the region of MS-stability for several values of $\alpha_{11}^{(0,1)}$ which satisfy (4. 6), we have found that the region of MS-stability becomes large when $\alpha_{11}^{(0,1)}=0$. In the sequel, (3. 6) for $A_1^{(0,1)}=\alpha_{11}^{(0,1)}=0$ and $\alpha_{11}^{(1,0)}=1/2$ will be called ISRK2. In Fig. 2, the dark-colored part indicates the region of MS-stability for ISRK2, whereas the part enclosed by the two straight lines $v_r=-z_r$ and $v_r=0$ indicates the region in which (2. 7) holds. Here, note that the line $v_r=-z_r$ is included in the region of MS-stability since $N_2\left(z_r,-z_r;1/2\right)<0$.

The rest of our works in the case of $A_1^{(0,1)}=0$ is to show that $D_1(h,\triangle W,\lambda,\mathrm{i}\Im(\sigma))$ and $D_1(h,\triangle W,\Re(\lambda),\Re(\sigma))$ do not vanish in the region $z_r<0$ when $\alpha_{11}^{(0,1)}=0$ and $\alpha_{11}^{(1,0)}=1/2$. If $D_1(h,\triangle W,\lambda,\mathrm{i}\Im(\sigma))=0$, from (4. 4) the following two equations must hold simultaneously:

$$\begin{cases}(z_r^2-z_i^2-6z_r+12)\left((\Im(\sigma)\triangle W)^2-12\right)=12z_i\Im(\sigma)\triangle W(2(2-\sqrt{3})z_r-3),\\(z_r-3)z_i\left((\Im(\sigma)\triangle W)^2-12\right)=-6\Im(\sigma)\triangle W((2-\sqrt{3})(z_r^2-z_i^2)-3z_r+6).\end{cases}$$

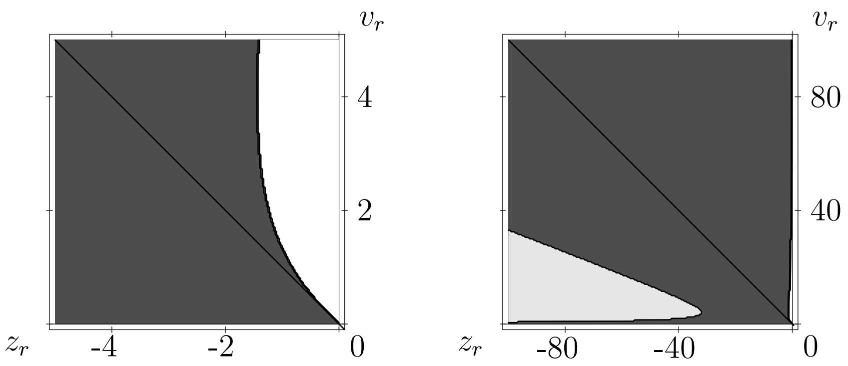

Figure 2: Region of MS-stability for ISRK2 when $\Im(\lambda) = \Im(\sigma) = 0$

Since it is clear that these do not hold simultaneously when $\Im(\sigma)\triangle W = 0$, we proceed with calculations, provided that $\Im(\sigma)\triangle W \neq 0$. From the equations we have

$$\frac{2-\sqrt{3}}{3}(z_r^2 + z_i^2)^2 + (2 - z_r)((5 - 2\sqrt{3})z_r^2 - 6z_r + 12) + ((2\sqrt{3} - 5)z_r + 4(\sqrt{3} - 1))z_i^2 = 0.$$

Since the expression in the left-hand side is positive in the region $z_r < 0$, we can see that $D_1(h, \triangle W, \lambda, \mathrm{i}\Im(\sigma)) \neq 0$ holds there.

On the other hand, if $D_1(h, \triangle W, \Re(\lambda), \Re(\sigma)) = 0$, we have

$$(6(2\sqrt{3} - 3)\Re(\sigma)\triangle W + \delta_7)z_r^2 - 6\delta_7 z_r + 12\delta_7 = 0,$$

where $\delta_7 \stackrel{\text{def}}{=} (\Re(\sigma)\triangle W - 3)^2 + 3$. The discriminant of the quadratic equation with respect to z_r, that is, $-3\delta_7(24(2\sqrt{3} - 3)\Re(\sigma)\triangle W + \delta_7)$ is negative since $\delta_7 > 0$ and

$$24(2\sqrt{3}-3)\Re(\sigma)\triangle W+\delta_7 = (\Re(\sigma)\triangle W+3(8\sqrt{3}-13))^2+39(48\sqrt{3}-83) \geq 39(48\sqrt{3}-83) > 0.$$

In addition, by taking $D_1(h, \triangle W, \Re(\lambda), \Re(\sigma)) > 0$ when $z_r = 0$ into account, we can see that $D_1(h, \triangle W, \Re(\lambda), \Re(\sigma)) > 0$.

Finally, we consider the case of $A_1^{(0,1)} = 1$. We can proceed with calculations in a similar way. No value of $\alpha_{11}^{(0,1)}$, however, exists such that the coefficient of z_r^{15} in $N_2\left(z_r, -z_r; \alpha_{11}^{(0,1)}\right)$ is negative. Thus, differently from the previous case, $N_2\left(z_r, -z_r; \alpha_{11}^{(0,1)}\right) < 0$ does not always hold in the region $z_r < 0$.

5 Numerical experiments

We show the results of numerical experiments to confirm the weak order of the schemes found in the previous section and their properties of MS-stability.

The following SDE is considered:

$$\mathrm{d}\boldsymbol{y}(t) = \left(A - \frac{1}{2}B^2\right)\boldsymbol{y}(t)\mathrm{d}t + B\boldsymbol{y}(t) \circ \mathrm{d}W(t), \quad \boldsymbol{y}(0) = \boldsymbol{x}_0,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}.$$

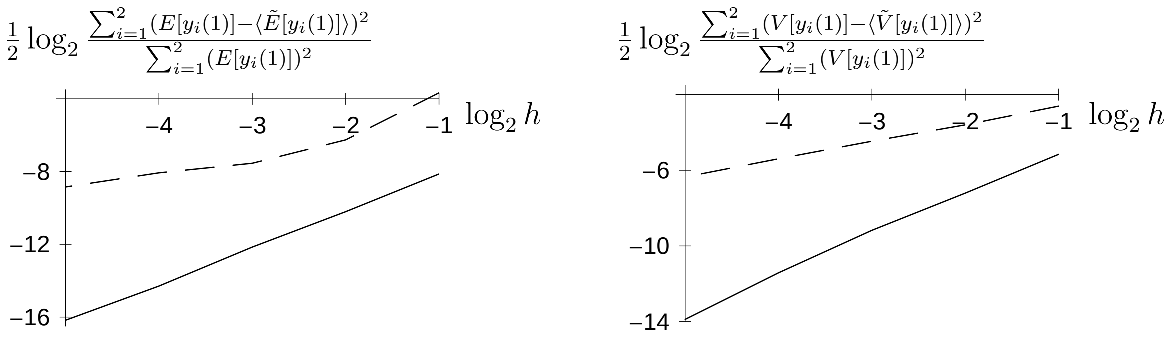

Figure 3: Relative errors in the first test

In the first test, we set $a_1 = -3$, $a_2 = -2$, $b = 1/2$ and $\boldsymbol{x}_0 = [1\ 0]^\top$ with probability 1. Then, we sought $\boldsymbol{y}_M$ by means of the schemes, and calculated the arithmetic means and the arithmetic variances of the ith element of $\boldsymbol{y}_M$ as estimates of $E[y_i(1)]$ and variances $V[y_i(1)]$ $(i = 1, 2)$, respectively. On the other hand, their exact values were sought from $\mathrm{d}E[\boldsymbol{y}(t)]/\mathrm{d}t = AE[\boldsymbol{y}(t)]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 2 & 0 \\ -3 & -\frac{7}{4} & 1 \\ 0 & -6 & -\frac{15}{4} \end{bmatrix}\begin{bmatrix} E[y_1^2(t)] \\ E[y_1(t)y_2(t)] \\ E[y_2^2(t)] \end{bmatrix}.$$

In an experiment, 16 sample sets were considered and 16×10^6 independent trajectories were simulated for one sample set. Thus, we obtained 16 estimates for each of $E[y_i(1)]$ and $V[y_i(1)]$ $(i = 1, 2)$. The results are indicated in Figure 3 and Table 3. In the figure and the table, an estimator of an unknown quantity θ and an arithmetic mean of 16 estimates $\tilde{\theta}$'s are denoted by $\hat{\theta}$ and $\langle\tilde{\theta}\rangle$ [13]. The dash or solid line means ISRK1 or ISRK2, respectively. The figure illustrates that each scheme achieves each weak order expected theoretically.

Table 3: Biases and standard deviations in the first test

Scheme	h	Bias				Standard deviation			
		$\hat{E}[y_1(1)]$	$\hat{E}[y_2(1)]$	$\hat{V}[y_1(1)]$	$\hat{V}[y_2(1)]$	$\hat{E}[y_1(1)]$	$\hat{E}[y_2(1)]$	$\hat{V}[y_1(1)]$	$\hat{V}[y_2(1)]$
ISRK1	2^{-1}	$7.85e-3$	$-6.48e-2$	$4.47e-3$	$-2.77e-2$	$4.39e-5$	$9.19e-5$	$7.26e-6$	$3.05e-5$
	2^{-2}	$-4.96e-3$	$-9.67e-3$	$-5.40e-4$	$-1.41e-2$	$4.52e-5$	$1.07e-4$	$1.46e-5$	$7.99e-5$
	2^{-3}	$-4.44e-3$	$4.62e-4$	$-1.13e-3$	$-7.67e-3$	$3.43e-5$	$8.39e-5$	$1.38e-5$	$8.21e-5$
	2^{-4}	$-2.70e-3$	$1.51e-3$	$-8.03e-4$	$-4.00e-3$	$2.82e-5$	$6.94e-5$	$1.79e-5$	$1.08e-4$
	2^{-5}	$-1.47e-3$	$1.06e-3$	$-4.64e-4$	$-2.04e-3$	$3.95e-5$	$9.71e-5$	$1.69e-5$	$1.02e-4$
ISRK2	2^{-1}	$-2.60e-3$	$-1.42e-3$	$-2.01e-3$	$4.36e-3$	$5.14e-5$	$1.32e-4$	$1.14e-5$	$7.61e-5$
	2^{-2}	$-5.49e-4$	$-4.42e-4$	$-5.37e-4$	$1.03e-3$	$4.03e-5$	$1.00e-4$	$1.76e-5$	$1.10e-4$
	2^{-3}	$-1.24e-4$	$-1.34e-4$	$-1.34e-4$	$2.64e-4$	$3.88e-5$	$9.56e-5$	$1.70e-5$	$1.04e-4$
	2^{-4}	$-3.91e-5$	$-1.37e-5$	$-3.58e-5$	$5.12e-5$	$4.15e-5$	$1.02e-4$	$1.49e-5$	$8.97e-5$
	2^{-5}	$-7.92e-6$	$-8.00e-6$	$-1.13e-5$	$-9.52e-7$	$3.93e-5$	$9.64e-5$	$1.67e-5$	$1.00e-4$

In the second test, we set $a_1 = -10000$, $a_2 = -205$, $b = 3$ and $\boldsymbol{x}_0 = [0\ 1]^\top$. Then, since (2. 5) holds, the equilibrium position is asymptotically stable in mean square. In each experiment, 1×10^5 independent trajectories were simulated. The results are showed in Table 4. The table indicates that ISRK2 is not MS-stable for $h = 1/2$. In fact, it is true from Fig. 2 since the eigenvalues of $A - B^2/2$ are -129.5 and -84.5.

Table 4: Estimates in the second test

Scheme	h	$\tilde{E}[y_1(3)]$	$\tilde{E}[y_2(3)]$	$\tilde{V}[y_1(3)]$	$\tilde{V}[y_2(3)]$	$\tilde{E}[y_1(5)]$	$\tilde{E}[y_2(5)]$	$\tilde{V}[y_1(5)]$	$\tilde{V}[y_2(5)]$
ISRK1	2^{-1}	-0.003	0.911	0.000	0.000	-0.003	0.809	0.000	0.000
	2^{-2}	-0.003	0.447	0.000	0.000	-0.001	0.194	0.000	0.000
	2^{-3}	-0.000	0.007	0.000	0.000	-0.000	0.000	0.000	0.000
ISRK2	2^{-1}	-0.009	1.746	0.001	21.407	-0.013	2.049	0.004	98.331
	2^{-2}	-0.002	0.269	0.000	2.718	-0.000	0.057	0.000	0.434
	2^{-3}	-0.000	0.000	0.000	0.000	-0.000	0.000	0.000	0.000

6 Summary and remarks

First, we have considered implicit SRK methods with one or two stages, and have found solutions of the order conditions for them to be of weak order 1 or 2 in a form including the free parameters. Second, we have decided the values of the parameters such that the schemes have good stability properties. Third, we have performed the numerical experiments and have shown that the schemes achieve the convergence order and stability properties expected theoretically.

The schemes have the following features.

- The schemes ISRK1 and ISRK2 are A-stable in the case that $\Im(\sigma)$ =0 in (2. 6).
- Further, ISRK1 is A-stable also in the case of $\Re(\lambda) = \Re(\sigma) = 0$.
- Although ISRK1 and ISRK2 are of weak order 1 or 2, respectively, they are of order 2 or 4 for ODEs. For this, they can be expected to show better performance in seeking an approximation to the expectation of a solution for SDEs with small noise [13].
- When $\Im(\lambda) = \Im(\sigma) = 0$, ISRK1 and ISRK2 satisfy that the slope of the boundary curve at the origin $(z_r, v_r) = (0, 0)$ is equal to -1. In the neighborhood of the origin, thus, the region of MS-stability for them is consistent with that in which (2. 7) holds.

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