Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 70, 2023, pp. 1–23

EXPLICIT DESCRIPTION OF CONTACT TRANSFORMATIONS OF SECOND ORDER

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Abstract

The aim of this note is to explicitly describe contact transformations on the 2-jet space with one dependent variable using the canonical coordinate system.

1. Introduction

Let (M, N, p) be a fibered manifold with m-dimensional fibers. Namely, $p : M \to N$ is a surjective submersion such that $\dim N = n$ and $\dim M = m + n$. Let $J^k(M, N, p)$ denote the bundle of k-jets of local sections of (M, N, p). The k-jet bundle $J^k(M, N, p)$ has a canonical differential system C^k. Then a contact transformation of $J^k(M, N, p)$ is defined by a diffeomorphism which preserves the canonical differential system C^k. A classical theorem due to Bäcklund ([1]) demonstrates that the pseudo-group of local contact transformations of $J^k(M, N, p)$ is isomorphic with the pseudo-group of local contact transformations of $J^1(M, N, p)$ if $m = 1$ and with the pseudo-group of local diffeomorphisms (i.e. point transformations) of M if $m \geq 2$. In [25], Yamaguchi proved the above statement in its global form through the geometrization of jet bundles. Thus there is a marked distinction between $m = 1$ and $m \geq 2$. In particular, in case $m = 1$, $J^1(M, N, p)$ is a $2n + 1$-dimensional standard contact manifold. In this case, it is well-known that every local contact transformation on $J^1(M, N, p)$ can not be a prolonged point transformation.

Under the above historical background, in the present note, we describe explicitly the contact transformations of $J^2(M, N, p)$ in case $m = 1$ in terms of the local canonical coordinate system. Our expression enables us to recognize specifically the difference between the pseudo-group of prolonged first-order contact transformations and the pseudo-group of prolonged point transformations. We would like to mention that our expression can be applied to a practical approach for the discovery of interesting pseudo-groups. In case $n = 1$, there have been studied deeply various subgroups of contact pseudo-groups and the corresponding equivalence problems of ordinary differential equations, e.g. point transformations, fiber-preserving maps, area-preserving maps, Painlevé type transformations, web type transformations, etc ([5], [6], [21], [8], [12]). On the other hand, in case $n \geq 2$, such previous studies are limited ([13]). Because of this situation, the applicability of our present work is expected. We are also interested in the application of our expression to transformations of geometric solutions with various properties, e.g. geometric singularities for solutions ([14], [15], [16]).

Now we would like to touch on the significance of the present note. The topics covered in this note have been discussed by great pioneers in past ([1], [2], [3], [7], [9], [10]). However, their classical articles are not written very carefully and are not very easy to read. Therefore, it is important to clearly explain these topics from a modern perspective. In regard to this matter, our argument is elementary and very detailed. In this sense, it can be expected that this note will play a role of a lecture note in this research field.

Throughout the present note, we always assume the differentiability of class C^∞ and sometimes use the terminology in [25].

Acknowledgments. The first author was supported by JSPS KAKENHI Grant Number 20K03633.

2. Contact transformations of the 1-jet bundle

Let M be a manifold of dimension $n+1$. We consider the Grassmann bundle $J(M,n)$ over M consisting of all n-dimensional contact elements to M. Namely, $J(M,n)$ is defined by $J(M,n) = \bigcup_{x\in M} J_x$, $J_x = \mathrm{Gr}(T_x(M),n)$, where $\mathrm{Gr}(T_x(M),n)$ denotes the Grassmann manifold of n-dimensional subspaces of $T_x(M)$. Let $\Pi^1_0 : J(M,n) \to M$ be the bundle projection. The canonical differential system C on $J(M,n)$ is the differential system of corank 1 defined by

$$C(u) = (\Pi^1_0)_*^{-1}(u) = \{v \in T_u(J(M,n)) \mid (\Pi^1_0)_*(v) \in u\} \subset T_u(J(M,n)) \xrightarrow{(\Pi^1_0)_*} T_x(M),$$

where $\Pi^1_0(u) = x$ for $u \in J(M,n)$. The pair $(J(M,n),C)$ is called the geometric 1-jet space. A diffeomorphism Φ of $J(M,n)$ onto itself is called a contact transformation of first order if it preserves the canonical differential system C, i.e., $\Phi_*C = C$. We emphasize that $(J(M,n),C)$ is the standard contact manifold of dimension $2n+1$. Hence $(J(M,n),C)$ has a local canonical system $(J^1(n,1),C^1)$. Namely, $J^1(n,1)$ is the coordinate space $J^1(n,1) := \{(x_i,z,p_i) \mid 1 \le i \le n\}$ and $C^1 := \{\varpi_0 = 0\}$ is the canonical contact structure described by the defining 1-form $\varpi_0 := dz - \sum_{i=1}^n p_i\,dx_i$. This local canonical system $(J^1(n,1),C^1)$ is called the 1-jet space for n independent and one dependent variables. In the same way as above, if a (local) diffeomorphism $\phi : J^1(n,1) \to J^1(n,1)$ satisfies $\phi_*C^1 = C^1$, then ϕ is called a (local) contact transformation of first order. Now we put $J^0(n,1) := \{(x_i,z) \mid 1 \le i \le n\}$. Then the natural projection $\pi^1_0 : J^1(n,1) \to J^0(n,1)$ defined by $\pi^1_0(x_i,z,p_i) = (x_i,z)$ gives a local structure of $\Pi^1_0 : J(M,n) \to M$. In this note, we mainly use the above (local) fibration π^1_0 for the purpose of calculating the explicit form of the following prolonged diffeomorphisms

([20]). The first-order contact prolongation of a (local) diffeomorphism $\varphi : J^0(n,1) \to J^0(n,1)$ is the unique (local) contact transformation $\varphi^{(1)} : J^1(n,1) \to J^1(n,1)$ satisfying $\pi^1_0 \circ \varphi^{(1)} = \varphi \circ \pi^1_0$. In the rest of this section, we calculate the explicit form of $\varphi^{(1)}$ in terms of the canonical coordinate of $J^1(n,1)$. Let $\varphi^{(1)} : (x_i, z, p_i) \mapsto (X_i(x_i, z), Z(x_i, z), P_i = P_i(x_i, z, p_i))$ denote the first-order contact prolongation of a diffeomorphism $\varphi : (x_i, z) \mapsto (X_i(x_i, z), Z(x_i, z))$. This prolongation $\varphi^{(1)}$ of φ satisfies ${\varphi^{(1)}}_*(C^1) = C^1$, that is,

$$dZ - \sum_{i=1}^n P_i\, dX_i = f\left(dz - \sum_{i=1}^n p_i\, dx_i\right) \qquad \text{for a nonzero function } f(x_i, z, p_i)$$

$$\iff \sum_{i=1}^n Z_{x_i}\, dx_i + Z_z\, dz - \sum_{i=1}^n P_i\left(\sum_{j=1}^n (X_i)_{x_j} dx_j + (X_i)_z dz\right) = f\left(dz - \sum_{i=1}^n p_i\, dx_i\right)$$

$$\iff \left(Z_z - \sum_{i=1}^n P_i (X_i)_z\right) dz + \sum_{j=1}^n \left(Z_{x_j} - \sum_{i=1}^n P_i (X_i)_{x_j}\right) dx_j = f\left(dz - \sum_{j=1}^n p_j\, dx_j\right).$$

Hence the coordinate functions X_i and Z satisfy

$$\begin{cases} Z_{x_1} - \sum_{i=1}^n P_i (X_i)_{x_1} = -fp_1 \\ \vdots \\ Z_{x_n} - \sum_{i=1}^n P_i (X_i)_{x_n} = -fp_n \\ Z_z - \sum_{i=1}^n P_i (X_i)_z = f \end{cases} \iff \begin{cases} \sum_{i=1}^n P_i (X_i)_{x_1} - fp_1 = Z_{x_1} \\ \vdots \\ \sum_{i=1}^n P_i (X_i)_{x_n} - fp_n = Z_{x_n} \\ \sum_{i=1}^n P_i (X_i)_z + f = Z_z. \end{cases}$$

These conditions can be summarized as

$$\begin{pmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{pmatrix} \begin{pmatrix} P_1 \\ \vdots \\ P_n \\ -f \end{pmatrix} = \begin{pmatrix} Z_{x_1} \\ \vdots \\ Z_{x_n} \\ Z_z \end{pmatrix}. \tag{1}$$

We put

$$K(\varphi) = \begin{pmatrix} (X_1)_{x_1} & \cdots & (X_1)_{x_n} & (X_1)_z \\ \vdots & & \vdots & \vdots \\ (X_n)_{x_1} & \cdots & (X_n)_{x_n} & (X_n)_z \\ p_1 & \cdots & p_n & -1 \end{pmatrix}, \qquad \mathcal{K}(\varphi) = \left(\begin{array}{cccc:c} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 & Z_{x_1} \\ \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n & Z_{x_n} \\ (X_1)_z & \cdots & (X_n)_z & -1 & Z_z \end{array}\right).$$

If there exists a solution $(P_1, \dots, P_n, -f)$ of the equation (1), $(P_1, \dots, P_n, -f)$ is unique by the following argument. We assume the existence of a solution of (1). Then an equality $\operatorname{rank} K(\varphi) = \operatorname{rank} \mathscr{K}(\varphi)$ holds. Here, if the rank of $K(\varphi)$ is less than $n+1$, then all minors of order $(n+1) \times (n+1)$ of $\mathscr{K}(\varphi)$ are equal to 0. In particular, we have $\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} \\ (X_1)_z & \cdots & (X_n)_z & Z_z \end{vmatrix} = 0$. Since φ is a diffeomorphism, this is a contradiction. Thus we have an equality $\operatorname{rank} K(\varphi) = \operatorname{rank} \mathscr{K}(\varphi) = n+1$, i.e., $|K(\varphi)| \neq 0$. By Cramer's rule, we obtain the expression of the unique solution of (1):

$$P_i = \frac{1}{|K(\varphi)|} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}, \qquad f = -\frac{|J\varphi|}{|K(\varphi)|},$$

where $J\varphi$ is the Jacobian matrix of φ, i.e., $J\varphi := \begin{pmatrix} (X_1)_{x_1} & \cdots & (X_1)_{x_n} & (X_1)_z \\ \vdots & & \vdots & \vdots \\ (X_n)_{x_1} & \cdots & (X_n)_{x_n} & (X_n)_z \\ Z_{x_1} & \cdots & Z_{x_n} & Z_z \end{pmatrix}$.

To summarize the above discussion, we obtain the following statement.

THEOREM 2.1. *Let $\varphi : (x_i, z) \mapsto (X_i, Z)$ be a diffeomorphism and $\varphi^{(1)} : (x_i, z, p_i) \mapsto (X_i, Z, P_i)$ be the first-order contact prolongation of φ. We take any point v of $J^1(n, 1)$. The first-order contact prolongation $\varphi^{(1)}$ of φ can be defined at v if and only if $|K(\varphi)| \neq 0$ at v. Under this condition $|K(\varphi)| \neq 0$, we have the description of P_i;*

$$P_i = \frac{1}{|K(\varphi)|} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}. \tag{2}$$

Now we look a little deeper into the above condition $|K(\varphi)| \neq 0$.

PROPOSITION 2.2. *Let w be any point of $J^0(n, 1)$. The following three conditions are equivalent.*

(i) *At any point v in the fiber $(\pi_0^1)^{-1}(w)$, we have $|K(\varphi)| \neq 0$.*

(ii) *We have $(K_{(n+1)1}(\varphi), \dots, K_{(n+1)n}(\varphi)) = \mathbf{0}$ at w, where $K_{(n+1)k}$ $(1 \leq k \leq n)$ is the cofactor of the $(n+1, k)$th element of $K(\varphi)$.*

(iii) *We have $((X_1)_z, \dots, (X_n)_z) = \mathbf{0}$ at w.*

By taking the contraposition of the above, the following three conditions are also equivalent.

(i)′ *There exists a point* v *in the fiber* $(\pi_0^1)^{-1}(w)$ *such that* $|K(\varphi)| = 0$.

(ii)′ *We have* $(K_{(n+1)1}(\varphi), \ldots, K_{(n+1)n}(\varphi)) \neq \mathbf{0}$ *at* w.

(iii)′ *We have* $((X_1)_z, \ldots, (X_n)_z) \neq \mathbf{0}$ *at* w.

Proof. First of all, we prove the equivalence between (i) and (ii). We assume that (ii) is false, that is, $(K_{(n+1)1}(\varphi), \ldots, K_{(n+1)n}(\varphi)) \neq \mathbf{0}$ at w. There exists a number k $(1 \le k \le n)$ such that $K_{(n+1)k}(\varphi) \neq 0$. We use the cofactor expansion along the $(n+1)$th row to evaluate $|K(\varphi)|$, i.e., $|K(\varphi)| = K_{(n+1)1}(\varphi)p_1 + \cdots + K_{(n+1)n}(\varphi)p_n - K_{(n+1)(n+1)}(\varphi)$. By taking the point v in the fiber $(\pi_0^1)^{-1}(w)$ defined by $p_k = \dfrac{K_{(n+1)(n+1)}(\varphi)}{K_{(n+1)k}(\varphi)}$, $p_l = 0$ $(l = 1, \ldots, k-1, k+1, \ldots, n)$, we have $|K(\varphi)| = 0$ at v. Namely the statement (i) is false. Thus we obtain the implication (i) $\Rightarrow$ (ii). Conversely, we assume that (ii) is true, that is, $(K_{(n+1)1}(\varphi), \ldots, K_{(n+1)n}(\varphi)) = \mathbf{0}$ at w. Then we have $K_{(n+1)(n+1)}(\varphi) \neq 0$ at any point v in the fiber $(\pi_0^1)^{-1}(w)$ by the calculation;

$$
\begin{aligned}
0 \neq |J\varphi| &= \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_1)_{x_n} & (X_1)_z \\ \vdots & & \vdots & \vdots \\ (X_n)_{x_1} & \cdots & (X_n)_{x_n} & (X_n)_z \\ Z_{x_1} & \cdots & Z_{x_n} & Z_z \end{vmatrix} \\
&= K_{(n+1)1}(\varphi)Z_{x_1} + \cdots + K_{(n+1)n}(\varphi)Z_{x_n} + K_{(n+1)(n+1)}(\varphi)Z_z \\
&= K_{(n+1)(n+1)}(\varphi)Z_z.
\end{aligned}
$$

We have $|K(\varphi)| \neq 0$ at v by the following cofactor expansion of $|K(\varphi)|$;

$$|K(\varphi)| = K_{(n+1)1}(\varphi)p_1 + \cdots + K_{(n+1)n}(\varphi)p_n - K_{(n+1)(n+1)}(\varphi) = -K_{(n+1)(n+1)}(\varphi) \neq 0.$$

Hence we obtain the implication (ii) $\Rightarrow$ (i). The above discussion proved the equivalence (i) $\Leftrightarrow$ (ii).

Next we prove the equivalence between (ii) and (iii). The implication (iii) $\Rightarrow$ (ii) follows directly from the expression of $K(\varphi)$. Hence it is sufficient to prove the implication (ii) $\Rightarrow$ (iii). We assume the condition of (ii), that is, $(K_{(n+1)1}(\varphi), \ldots, K_{(n+1)n}(\varphi)) = \mathbf{0}$ at w. We put $W := K_{(n+1)(n+1)}(\varphi) = \begin{pmatrix} (X_1)_{x_1} & \cdots & (X_1)_{x_n} \\ \vdots & & \vdots \\ (X_n)_{x_1} & \cdots & (X_n)_{x_n} \end{pmatrix}$. Let W_{ij} be the cofactor of the (i, j)th element of W and $\mathrm{adj}(W)$ be the adjugate matrix of W. In the same way as the discussion of the proof of the implication (ii) $\Rightarrow$ (i), we have $|W| \neq 0$ and $|\mathrm{adj}(W)| \neq 0$. By the assumption $K_{(n+1)1}(\varphi) = 0$, we have

$$0 = |K_{(n+1)1}(\varphi)| = \begin{vmatrix} (X_1)_{x_2} & \cdots & (X_n)_{x_2} \\ \vdots & & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} \\ (X_1)_z & \cdots & (X_n)_z \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} (X_2)_{x_2} & \cdots & (X_n)_{x_2} \\ \vdots & & \vdots \\ (X_2)_{x_n} & \cdots & (X_n)_{x_n} \end{vmatrix} (X_1)_z + \cdots + (-1)^{n+n} \begin{vmatrix} (X_1)_{x_2} & \cdots & (X_{n-1})_{x_2} \\ \vdots & & \vdots \\ (X_1)_{x_n} & \cdots & (X_{n-1})_{x_n} \end{vmatrix} (X_n)_z$$

$$= (-1)^{n-1}(W_{11}(X_1)_z + \cdots + W_{n1}(X_n)_z).$$

We also perform the same calculations for $K_{(n+1)2}(\varphi), \dots, K_{(n+1)n}(\varphi)$. Summarizing these calculations, we have the following simultaneous equations;

$$\begin{pmatrix} W_{11} & \cdots & W_{n1} \\ \vdots & & \vdots \\ W_{1n} & \cdots & W_{nn} \end{pmatrix} \begin{pmatrix} (X_1)_z \\ \vdots \\ (X_n)_z \end{pmatrix} = \operatorname{adj}(W) \begin{pmatrix} (X_1)_z \\ \vdots \\ (X_n)_z \end{pmatrix} = \mathbf{0}.$$

Then the condition $|\operatorname{adj}(W)| \neq 0$ derives the unique solution $\begin{pmatrix} (X_1)_z \\ \vdots \\ (X_n)_z \end{pmatrix} = \mathbf{0}$. Thus we obtain the implication (ii) $\Rightarrow$ (iii) and the equivalence (ii)$\Leftrightarrow$ (iii). □

By using the above proposition, we characterize the local behaviors of the domains of the first-order contact prolongations $\varphi^{(1)}$. Let $w_0 := ((x_i)_0, z_0)$ be a base point in $J^0(n,1)$. Roughly speaking, the shapes of the domains of the first-order contact prolongations can be classified into the following three types.

Type A: We assume that the condition (i)$'$ is true at the point w_0. Then the condition (i)$'$ is also true around w_0 by the equivalent open conditions (ii)$'$ and (iii)$'$.

EXAMPLE 2.3. We consider a diffeomorphism $\varphi : J^0(n,1) \to J^0(n,1)$ defined by $X_1 = z$, $X_i = x_i$ $(i = 2, \dots n)$ and $Z = x_1$. By the calculation of $|K(\varphi)|$, we have $|K(\varphi)| = -p_1$. Hence any point w_0 in $J^0(n,1)$ satisfies the condition (i)$'$.

Type B: We assume that the condition (i) is true at the point w_0. This case can be divided into the following two cases.

Type B-1: There exists a neighborhood U of w_0 which satisfies the following situation. The condition (i) holds at any point in U.

EXAMPLE 2.4. We consider a diffeomorphism $\varphi : J^0(n,1) \to J^0(n,1)$ defined by $X_i = x_{i+1}$, $X_n = x_1$ $(i = 1, \dots, n-1)$ and $Z = z$. Similarly, we have $|K(\varphi)| = (-1)^{(n+2)} \neq 0$. Hence any point w_0 in $J^0(n,1)$ satisfies the condition (i).

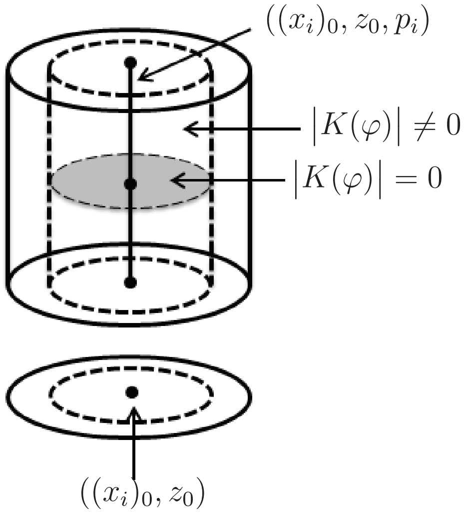

Figure 1. Type A

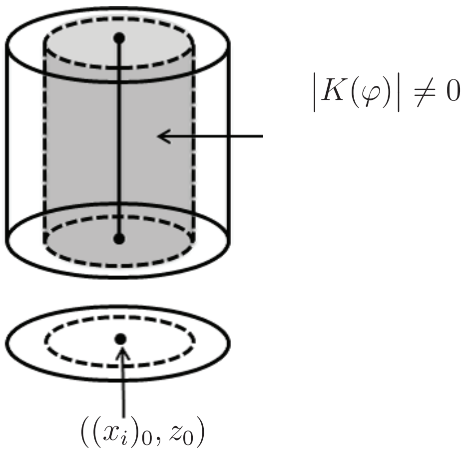

Figure 2. Type B-1

Type B-2: For any neighborhood U of w_0, there exists a point $w := (x_i, z)$ in U satisfying the condition (i)$'$.

Example 2.5. We consider a diffeomorphism $\varphi : J^0(n,1) \to J^0(n,1)$ defined by $X_1 = x_1 - \dfrac{1}{2}z^2$, $X_i = x_i$ $(i = 2, \dots, n)$ and $Z = z$. Similarly, we have $|K(\varphi)| = zp_1 - 1$. Hence any point w_0 in the hypersurface $\{z = 0\}$ of $J^0(n,1)$ satisfies the condition (i). On the other hand, any point w_0 in the open domain $\{z \neq 0\}$ of $J^0(n,1)$ satisfies the

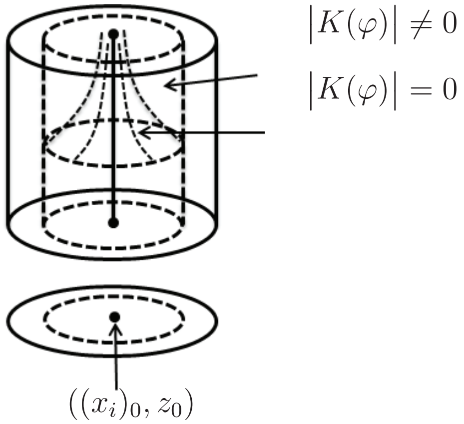

Figure 3. Type B-2

condition (i)$'$. Actually, it is sufficient to take a point v_0 in the fiber $(\pi_0^1)^{-1}(w_0)$ satisfying $p_1 = \dfrac{1}{z}$.

Through the discussion in this section, we note that the combination of Theorem 2.1 and Proposition 2.2 derives the following characterization.

COROLLARY 2.6. *Let $\varphi : J^0(n,1) \to J^0(n,1)$ be a diffeomorphism. The first-order contact prolongation $\varphi^{(1)}$ of φ can be defined at any point in $J^1(n,1)$ if and only if φ is a fiber-preserving diffeomorphism for the natural fibration $J^0(n,1) \to \mathbb{R}^n$ defined by $(x_1, \dots, x_n, z) \mapsto (x_1, \dots, x_n)$. Here a fiber-preserving diffeomorphism can be described by the form $\varphi(x_i, z) = (X_i(x_1, \dots, x_n), Z(x_1, \dots, x_n, z))$.*

3. Contact transformations of the 2-jet bundle

Let $(J(M,n), C^1)$ be a geometric 1-jet space which has the local canonical system $(J^1(n,1), C^1)$. We take a local contact form ϖ around any point $u \in J(M,n)$. An n-dimensional subspace v of the tangent space $T_u(J(M,n))$ is called an integral element of C^1 if $v \subset C^1(u)$ and $d\varpi|_v = 0$. Since $(J(M,n), C^1)$ is the $2n+1$-dimensional standard contact manifold, this integral element v is a Lagrangian subspace of the symplectic vector space $(C(u), d\varpi)$. We consider the Lagrangian Grassmannian bundle $L(J(M,n))$ over $J(M,n)$. Namely, $L(J(M,n))$ is defined by $L(J(M,n)) = \bigcup_{u \in J(M,n)} L_u$, where each fiber L_u is the Lagrangian Grassmannian manifold. Let $\Pi_1^2 : L(J(M,n)) \to J(M,n)$ be the bundle projection. Then the canonical differential system E of corank $n+1$ on $L(J(M,n))$ is defined by

$$E(v) = (\Pi_1^2)_*^{-1}(v) \subset T_v(L(J(M,n))) \xrightarrow{(\Pi_1^2)_*} T_u(J(M,n)),$$

where $\Pi_1^2(v) = u$ for $v \in L(J(M,n))$. The pair $(L(J(M,n)), E)$ is called the geometric 2-jet space. A diffeomorphism Φ of $L(J(M,n))$ onto itself is called a contact transformation of second order if it preserves the canonical differential system E, i.e., $\Phi_* E = E$. As in the case of the geometric 1-jet space, $(L(J(M,n)), E)$ has a local canonical system $(J^2(n,1), C^2)$. Namely, $J^2(n,1)$ is the coordinate space $J^2(n,1) := \{(x_i, z, p_i, p_{ij}) \mid 1 \le i \le j \le n\}$ and $C^2 := \{\varpi_0 = \varpi_i = 0\}$ is the canonical second-order contact structure described by the defining 1-forms $\varpi_0 := dz - \sum_{i=1}^{n} p_i\, dx_i$ and $\varpi_i := dp_i - \sum_{j=1}^{n} p_{ij}\, dx_j$, where $1 \le i \le n$, $p_{ij} = p_{ji}$ ([26]). This local canonical system $(J^2(n,1), C^2)$ is called the 2-jet space for n independent and one dependent variables. In the same way as above, if a (local) diffeomorphism $\phi : J^2(n,1) \to J^2(n,1)$ satisfies $\phi_* C^2 = C^2$, then ϕ is called a (local) contact transformation of second order. We also have the natural projections $\pi_1^2 : J^2(n,1) \to J^1(n,1)$ and $\pi_0^2 : J^2(n,1) \to J^0(n,1)$ defined by $\pi_1^2(x_i, z, p_i, p_{ij}) = (x_i, z, p_i)$ and $\pi_0^2 := \pi_0^1 \circ \pi_1^2$ respectively. The fibration $\pi_0^2 : J^2(n,1) \to J^1(n,1) \to J^0(n,1)$ gives a local structure of the (geometric) fibration $\Pi_0^2 := \Pi_0^1 \circ \Pi_1^2 : L(J(M,n)) \to J(M,n) \to M$. As in the case of Section 2, we mainly use the (local) fibration π_0^2 for the investigation of the contact prolongations.

3.1. Second-order contact prolongations of diffeomorphisms

The second-order contact prolongation $\varphi^{(2)} : J^2(n,1) \to J^2(n,1)$ of a (local) diffeomorphism $\varphi : J^0(n,1) \to J^0(n,1)$ is the unique (local) contact transformation $\varphi^{(2)} : J^2(n,1) \to J^2(n,1)$ satisfying $\pi_1^2 \circ \varphi^{(2)} = \varphi^{(1)} \circ \pi_1^2$, where $\varphi^{(1)} : J^1(n,1) \to J^1(n,1)$ is the first-order contact prolongation of φ. In this subsection, we calculate the explicit form of the second-order contact prolongation of a (local) diffeomorphism.

Now we prepare some fundamental properties for the determinant to be used in subsequent discussions. However, we omit the proofs.

Proposition 3.1. *Let m, n be natural numbers, h, l be non-negative integers, $1 \le k \le n$ and $1 \le i_1 < \cdots < i_k \le n$. We put*

$$\boldsymbol{a}_i := {}^t(a_{1i} \quad a_{2i} \quad \cdots \quad a_{ni}) \quad (1 \le i \le m), \qquad B := (\boldsymbol{b}_1 \quad \cdots \quad \boldsymbol{b}_l) := \begin{pmatrix} b_{11} & \cdots & b_{1l} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nl} \end{pmatrix},$$

$$\boldsymbol{a}_i^k := {}^t(a_{i_1 i} \quad a_{i_2 i} \quad \cdots \quad a_{i_k i}), \qquad \boldsymbol{b}_j^k := {}^t(b_{i_1 j} \quad b_{i_2 j} \quad \cdots \quad b_{i_k j}) \quad (1 \le j \le l),$$

$$B_k := (\boldsymbol{b}_1^k \quad \cdots \quad \boldsymbol{b}_l^k), \qquad C := \begin{pmatrix} c_{i_1 1} & \cdots & c_{i_1 h} \\ \vdots & & \vdots \\ c_{i_k 1} & \cdots & c_{i_k h} \end{pmatrix},$$

where $m + l = n + 1$, $1 + h + l = k$. *Then we have the equality;*

$$\begin{vmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m & B \\ |\boldsymbol{a}_1^k \quad B_k \quad C| & |\boldsymbol{a}_2^k \quad B_k \quad C| & \cdots & |\boldsymbol{a}_m^k \quad B_k \quad C| & \boldsymbol{0} \end{vmatrix} = 0.$$

PROPOSITION 3.2. *The exterior derivative of a determinant satisfies*

$$d\begin{vmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} = \begin{vmatrix} df_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ df_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} f_{11} & \cdots & f_{1n-1} & df_{1n} \\ \vdots & & \vdots & \vdots \\ f_{n1} & \cdots & f_{nn-1} & df_{nn} \end{vmatrix}.$$

We also prepare the useful explicit formula of the partial derivatives $(P_i)_{p_j}$ of P_i in a first-order contact prolongation $\varphi^{(1)} : (x_i, z, p_i) \mapsto (X_i, Z, P_i)$. By using the description (2), we calculate

$$(3) \quad (P_i)_{p_j} = \left(\frac{1}{|K(\varphi)|} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix} \right)_{p_j}$$

$$= \frac{1}{|K(\varphi)|^2} \left(\left. \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix} \right|_{p_j} \cdot |K(\varphi)| \right.$$

$$\left. - \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix} \cdot |K(\varphi)|_{p_j} \right).$$

Here each term of the above can be transformed as follows. First of all, we transform

$$\left. \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix} \right|_{p_j}$$

$$= \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & Z_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} & 0 \\ (X_1)_{x_j} & \cdots & (X_{i-1})_{x_j} & Z_{x_j} & (X_{i+1})_{x_j} & \cdots & (X_n)_{x_j} & 1 \\ (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & Z_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & 0 \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & 0 \end{vmatrix} \quad (\because \ Prop.\ 3.2)$$

$$= (-1)^{j+n+1} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & Z_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} \\ (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & Z_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z \end{vmatrix}$$

$$= (-1)^{i+j+n} \begin{vmatrix} Z_{x_1} & (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ Z_{x_{j-1}} & (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} \\ Z_{x_{j+1}} & (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ Z_{x_n} & (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} \\ Z_z & (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z \end{vmatrix}.$$

We denote the determinant in the last description by A_1. Next we perform the following transformation different from the above.

$$\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & Z_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & Z_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & Z_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}$$

$$= (-1)^{n-i} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & Z_z & -1 \end{vmatrix}.$$

Finally we expand the term $|K(\varphi)|_{p_j}$;

$$|K(\varphi)|_{p_j} = \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{vmatrix}_{p_j} \underset{Prop.\ 3.2}{=} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & 0 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} & 0 \\ (X_1)_{x_j} & \cdots & (X_n)_{x_j} & 1 \\ (X_1)_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} & 0 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & 0 \\ (X_1)_z & \cdots & (X_n)_z & 0 \end{vmatrix}$$

$$= (-1)^{j+n+1} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} \\ \vdots & & \vdots \\ (X_1)_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} \\ (X_1)_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} \\ \vdots & & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} \\ (X_1)_z & \cdots & (X_n)_z \end{vmatrix}$$

$$= (-1)^{i+j+n} \begin{vmatrix} (X_i)_{x_1} & (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ (X_i)_{x_{j-1}} & (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} \\ (X_i)_{x_{j+1}} & (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ (X_i)_{x_n} & (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} \\ (X_i)_z & (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z \end{vmatrix}.$$

We also denote the determinant in the last description by A_2. Thus we have the following form of $(P_i)_{p_j}$ by substituting these expansions into (3).

$$(4) \qquad (P_i)_{p_j} = \frac{1}{|K(\varphi)|^2} \left((-1)^{i+j+n} A_1 \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{vmatrix} \right.$$

$$
+(-1)^{j+1}\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & Z_z & -1 \end{vmatrix} A_2 \Bigg)
$$

$$
= \frac{(-1)^{i+j+n+1}}{|K(\varphi)|^2}\left(-A_1 \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{vmatrix} \right.
$$

$$
\left. +(-1)^{i+n}\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & Z_z & -1 \end{vmatrix} A_2 \right).
$$

Now, by using Proposition 3.1, we also have the following condition;

$$
0 = \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_i)_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_i)_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_i)_z & (X_{i+1})_z & \cdots & (X_n)_z & Z_z & -1 \\ 0 & \cdots & 0 & A_2 & 0 & \cdots & 0 & A_1 & \begin{vmatrix} p_1 & (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_{j-1} & (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} \\ p_{j+1} & (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_n & (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} \\ -1 & (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z \end{vmatrix} \end{vmatrix}
$$

$$
= (-1)^{n+2+i}\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & Z_z & -1 \end{vmatrix} A_2 - A_1 \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{vmatrix}
$$

$$
+ |J\varphi| \cdot \begin{vmatrix} p_1 & (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_{j-1} & (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} \\ p_{j+1} & (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_n & (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} \\ -1 & (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z \end{vmatrix}
$$

$$
= (-1)^{i+n} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & Z_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & Z_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & Z_z & -1 \end{vmatrix} A_2 - A_1 \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{vmatrix}
$$

$$
+ (-1)^{n-1}|J\varphi| \cdot \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} & p_{j-1} \\ (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} & p_{j+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}.
$$

By substituting this condition into (4), we have the expansion;

$$
(P_i)_{p_j} = \frac{(-1)^{i+j+n+1}}{|K(\varphi)|^2}(-1)^n|J\varphi| \cdot \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{i-1})_{x_1} & (X_{i+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_{j-1}} & \cdots & (X_{i-1})_{x_{j-1}} & (X_{i+1})_{x_{j-1}} & \cdots & (X_n)_{x_{j-1}} & p_{j-1} \\ (X_1)_{x_{j+1}} & \cdots & (X_{i-1})_{x_{j+1}} & (X_{i+1})_{x_{j+1}} & \cdots & (X_n)_{x_{j+1}} & p_{j+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{i-1})_{x_n} & (X_{i+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{i-1})_z & (X_{i+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}.
$$

Let $K_{ij}(\varphi)$ be the cofactor of the (i,j)th element of $K(\varphi)$. Consequently, we obtain the following simple formula of $(P_i)_{p_j}$.

Lemma 3.3. *Let $\varphi^{(1)} : (x_i, z, p_i) \mapsto (X_i, Z, P_i)$ be a first-order contact prolongation. Then the partial derivatives $(P_i)_{p_j}$ of P_i are given by*

$$
(P_i)_{p_j} = -\frac{|J\varphi|}{|K(\varphi)|^2} K_{ij}(\varphi) \qquad (i, j = 1, \dots, n).
$$

From now on, let us proceed to the description of the second-order prolongation $\varphi^{(2)} : J^2(n,1) \to J^2(n,1)$ of a diffeomorphism $\varphi : J^0(n,1) \to J^0(n,1)$. Let $\varphi^{(2)} : (x_i, z, p_i, p_{ij}) \mapsto (X_i(x_i, z), Z(x_i, z),\ P_i = P_i(x_i, z, p_i),\ P_{ij} = P_{ij}(x_i, z, p_i, p_{ij}))$ denote the second-order contact prolongation of a diffeomorphism φ. In the same way as the discussion of the first-order prolongation $\varphi^{(1)}$, we examine the pullback equation for generator 1-forms of C^2 by $\varphi^{(2)}$;

$$(5)\qquad \begin{pmatrix} dZ - P_1\, dX_1 - \cdots - P_n\, dX_n \\ dP_1 - P_{11}\, dX_1 - \cdots - P_{1n}\, dX_n \\ \vdots \\ dP_n - P_{n1}\, dX_1 - \cdots - P_{nn}\, dX_n \end{pmatrix}$$

$$= \begin{pmatrix} f & 0 & \cdots & 0 \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{pmatrix} \begin{pmatrix} dz - p_1\, dx_1 - \cdots - p_n\, dx_n \\ dp_1 - p_{11}\, dx_1 - \cdots - p_{1n}\, dx_n \\ \vdots \\ dp_n - p_{n1}\, dx_1 - \cdots - p_{nn}\, dx_n \end{pmatrix},$$

where the above functions $f(x_i, z, p_i)$, $f_i(x_i, z, p_i, p_{ij})$ and $f_{ij}(x_i, z, p_i, p_{ij})$ satisfy;

$$\begin{vmatrix} f & 0 & \cdots & 0 \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{vmatrix} \neq 0.$$

In the previous section, we already discussed the first equation in (5). Hence we investigate the remaining equations except for this first equation. We fix the $(i+1)$th equation for $i = 1, \ldots, n$. The left-hand side can be expanded as

$$\begin{aligned} &dP_i - P_{i1}\, dX_1 - \cdots - P_{in}\, dX_n \\ &\quad = \left(\sum_{k=1}^{n} (P_i)_{x_k} dx_k + (P_i)_z dz + \sum_{k=1}^{n} (P_i)_{p_k} dp_k \right) \\ &\qquad - P_{i1} \left(\sum_{k=1}^{n} (X_1)_{x_k} dx_k + (X_1)_z dz \right) - \cdots - P_{in} \left(\sum_{k=1}^{n} (X_n)_{x_k} dx_k + (X_n)_z dz \right) \\ &\quad = \sum_{k=1}^{n} ((P_i)_{x_k} - P_{i1}(X_1)_{x_k} - \cdots - P_{in}(X_n)_{x_k}) dx_k \\ &\qquad + ((P_i)_z - P_{i1}(X_1)_z - \cdots - P_{in}(X_n)_z) dz + \sum_{k=1}^{n} (P_i)_{p_k} dp_k. \end{aligned}$$

On the other hand, the right-hand side can be expanded as

$$\begin{aligned} &f_i \left(dz - \sum_{k=1}^{n} p_k\, dx_k \right) + f_{i1} \left(dp_1 - \sum_{k=1}^{n} p_{1k}\, dx_k \right) + \cdots + f_{in} \left(dp_n - \sum_{k=1}^{n} p_{nk}\, dx_k \right) \\ &\quad = \sum_{k=1}^{n} (-f_i p_k - f_{i1} p_{1k} - \cdots - f_{in} p_{nk}) dx_k + f_i\, dz + \sum_{k=1}^{n} f_{ik}\, dp_k. \end{aligned}$$

Thus we have the following simultaneous equations by comparing both expansions.

$$\begin{cases} (P_i)_{x_1} - P_{i1}(X_1)_{x_1} - \cdots - P_{in}(X_n)_{x_1} = -f_i p_1 - f_{i1} p_{11} - \cdots - f_{in} p_{n1}, \\ \vdots \\ (P_i)_{x_n} - P_{i1}(X_1)_{x_n} - \cdots - P_{in}(X_n)_{x_n} = -f_i p_n - f_{i1} p_{1n} - \cdots - f_{in} p_{nn}, \\ (P_i)_z - P_{i1}(X_1)_z - \cdots - P_{in}(X_n)_z = f_i, \\ (P_i)_{p_1} = f_{i1}, \\ \vdots \\ (P_i)_{p_n} = f_{in}. \end{cases}$$

We rewrite the above system of equations in matrix form;

$$(6) \qquad \begin{pmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \end{pmatrix} \begin{pmatrix} P_{i1} \\ \vdots \\ P_{in} \\ -f_i \end{pmatrix} = \begin{pmatrix} (P_i)_{x_1} + \sum_{k=1}^{n} (P_i)_{p_k} p_{k1} \\ \vdots \\ (P_i)_{x_n} + \sum_{k=1}^{n} (P_i)_{p_k} p_{kn} \\ (P_i)_z \end{pmatrix}.$$

By Cramer's rule, we obtain the following expression of the unique solution of (6);

$$P_{ij} = \frac{1}{|K(\varphi)|} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & (P_i)_{x_1} + \sum_{k=1}^{n} (P_i)_{p_k} p_{k1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & (P_i)_{x_n} + \sum_{k=1}^{n} (P_i)_{p_k} p_{kn} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix},$$

for all j $(1 \leq j \leq n)$, and

$$f_i = -\frac{1}{|K(\varphi)|} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & (P_i)_{x_1} + \sum_{k=1}^{n} (P_i)_{p_k} p_{k1} \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & (P_i)_{x_n} + \sum_{k=1}^{n} (P_i)_{p_k} p_{kn} \\ (X_1)_z & \cdots & (X_n)_z & (P_i)_z \end{vmatrix}.$$

We substitute the description of $(P_i)_{p_j}$ in Lemma 3.3 into the description of P_{ij};

$$P_{ij} = \frac{1}{|K(\varphi)|}\left(\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & (P_i)_{x_1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & (P_i)_{x_n} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}\right.$$

$$\left. + \sum_{k=1}^{n}\left(-\frac{|J\varphi|}{|K(\varphi)|^2}K_{ik}(\varphi)\right)\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & p_{k1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & p_{kn} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{j-1})_z & 0 & (X_{j+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}\right).$$

Moreover, by using the cofactors of $K(\varphi)$, we have

$$\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & p_{k1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & p_{kn} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{j-1})_z & 0 & (X_{j+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix} = \sum_{l=1}^{n} K_{jl}(\varphi)p_{kl}.$$

Thus we obtain the clear description of P_{ij};

$$P_{ij} = \frac{1}{|K(\varphi)|}\left(|A_{ij}| - \frac{|J\varphi|}{|K(\varphi)|^2}\sum_{l=1}^{n}\sum_{k=1}^{n} K_{ik}(\varphi)K_{jl}(\varphi)p_{kl}\right), \qquad \text{where}$$

$$A_{ij} := {}^t\begin{pmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & (P_i)_{x_1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & (P_i)_{x_n} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 \end{pmatrix}.$$

We remark that the equality $|A_{ij}| = |A_{ji}|$ follows from the equality $P_{ij} = P_{ji}$. The following statement is a summary of the above discussions.

THEOREM 3.4. *Let $\varphi^{(1)} : (x_i, z, p_i, p_{ij}) \mapsto (X_i, Z, P_i, P_{ij})$ be the second-order contact prolongation of a diffeomorphism $\varphi : (x_i, z) \mapsto (X_i, Z)$. We take any point v of $J^2(n,1)$. The second-order contact prolongation $\varphi^{(2)}$ can be defined at v if and only if $|K(\varphi)| \neq 0$ at v. Under this condition $|K(\varphi)| \neq 0$, we have the description of P_{ij};*

$$P_{ij} = \frac{1}{|K(\varphi)|}\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & (P_i)_{x_1} + \sum_{k=1}^{n}(P_i)_{p_k}p_{k1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & (P_i)_{x_n} + \sum_{k=1}^{n}(P_i)_{p_k}p_{kn} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 \end{vmatrix}$$

$$= \frac{1}{|K(\varphi)|}\left(|A_{ij}| - \frac{|J\varphi|}{|K(\varphi)|^2}\sum_{l=1}^{n}\sum_{k=1}^{n}K_{ik}(\varphi)K_{jl}(\varphi)p_{kl}\right), \qquad \textit{where } |A_{ij}| = |A_{ji}|.$$

Remark 3.5. We note that the condition $|K(\varphi)| \neq 0$ appears in both Theorem 2.1 and Theorem 3.4. Namely, the open domain $D := \{|K(\varphi)| \neq 0\} \subset J^1(n,1)$ is the domain of the first-order contact prolongations $\varphi^{(1)}$, and the its total space $\hat{D} := (\pi_1^2)^{-1}(D)$ is also the domain of the second-order contact prolongations $\varphi^{(2)}$. Consequently, we obtain the following commutative diagram for contact prolongations under the condition $|K(\varphi)| \neq 0$;

$$\begin{array}{ccc} J^2(n,1) \supset \hat{D} & \xrightarrow{\varphi^{(2)}} & J^2(n,1) \\ {\scriptstyle \pi_1^2}\downarrow & \circlearrowright & \downarrow{\scriptstyle \pi_1^2} \\ J^1(n,1) \supset D & \xrightarrow{\varphi^{(1)}} & J^1(n,1) \\ {\scriptstyle \pi_0^1}\downarrow & \circlearrowright & \downarrow{\scriptstyle \pi_0^1} \\ J^0(n,1) & \xrightarrow{\varphi} & J^0(n,1). \end{array}$$

3.2. General contact transformations of second order

Let us start from the following famous lifting theorem for the contact transformations on 2-jet bundles (c.f. Theorem 3.2 in [24]).

Theorem 3.6 (Bäcklund [1], Yamaguchi [24]). *Let $(J(M,n),C)$ be a geometric 1-jet space (i.e. standard contact manifold) and $(L(J(M,n)),E)$ be the geometric 2-jet space. A first-order contact transformation $\phi : J(M,n) \to J(M,n)$ induces a unique second-order contact transformation $\hat{\phi} : L(J(M,n)) \to L(J(M,n))$ such that $\hat{\phi}(v) = \phi_*(v)$ for $v \in L(J(M,n))$. Conversely a second-order contact transformation $\psi : L(J(M,n)) \to L(J(M,n))$ induces an unique contact transformation $\phi : J(M,n) \to J(M,n)$ such that $\psi = \phi_*$.*

Theorem 3.6 was provided by A. V. Bäcklund in its local form ([1]). After that Yamaguchi proved the statement in the above global form ([24]). This theorem tells us that every (local) contact transformation of second order can be realized as a prolonged first-order (local) contact transformation. Incidentally, in our notation, the second-order prolongation $\hat{\psi} : J^2(\mathbb{R}^n, \mathbb{R}) \to J^2(\mathbb{R}^n, \mathbb{R})$ of a (local) first-order contact transformation $\psi : J^1(\mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$ is also defined as the unique (local) contact transformation satisfying $\pi_1^2 \circ \hat{\psi} = \psi \circ \pi_1^2$ from the same argument in the above ([20]). Based on Theorem 3.6, we give the explicit form of the general second-order contact transformation as a second-order prolongation $\hat{\psi}$.

Theorem 3.7. *Let $\hat{\psi} : (x_i, z, p_i, p_{ij}) \mapsto (X_i, Z, P_i, P_{ij})$ be the second-order prolongation of a first-order contact transformation $\psi : (x_i, z, p_i) \mapsto (X_i, Z, P_i)$. We take any point v of $J^2(n, 1)$. Moreover we put*

$$L(\psi) := \begin{pmatrix} (X_1)_{x_1} & \cdots & (X_1)_{x_n} & (X_1)_z & (X_1)_{p_1} & \cdots & (X_1)_{p_n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (X_n)_{x_1} & \cdots & (X_n)_{x_n} & (X_n)_z & (X_n)_{p_1} & \cdots & (X_n)_{p_n} \\ p_1 & \cdots & p_n & -1 & & & \\ p_{11} & \cdots & p_{1n} & & \ddots & & \mathbf{0} \\ \vdots & & \vdots & \mathbf{0} & & \ddots & \\ p_{n1} & \cdots & p_{nn} & & & & -1 \end{pmatrix}.$$

The second-order prolongation $\hat{\psi}$ can be defined at v if and only if $|L(\psi)| \neq 0$ at v. Under this condition $|L(\psi)| \neq 0$, we have the description of P_{ij};

$$P_{ij} \underset{(\#1)}{=} \frac{1}{|L(\psi)|} \begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & (P_i)_{x_1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 & p_{11} & \cdots & p_{n1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & (P_i)_{x_n} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n & p_{1n} & \cdots & p_{nn} \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 & & & \\ (X_1)_{p_1} & \cdots & (X_{j-1})_{p_1} & (P_i)_{p_1} & (X_{j+1})_{p_1} & \cdots & (X_n)_{p_1} & & \ddots & & \mathbf{0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \mathbf{0} & & \ddots & \\ (X_1)_{p_n} & \cdots & (X_{j-1})_{p_n} & (P_i)_{p_n} & (X_{j+1})_{p_n} & \cdots & (X_n)_{p_n} & & & & -1 \end{vmatrix}$$

$$\underset{(\#2)}{=} \frac{\begin{vmatrix} \frac{dX_1}{dx_1} & \cdots & \frac{dX_{j-1}}{dx_1} & \frac{dP_i}{dx_1} & \frac{dX_{j+1}}{dx_1} & \cdots & \frac{dX_n}{dx_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{dX_1}{dx_n} & \cdots & \frac{dX_{j-1}}{dx_n} & \frac{dP_i}{dx_n} & \frac{dX_{j+1}}{dx_n} & \cdots & \frac{dX_n}{dx_n} \end{vmatrix}}{\left| \left(\frac{dX_\alpha}{dx_\beta} \right)_{1 \le \alpha, \beta \le n} \right|},$$

where $\dfrac{d}{dx_i} := \dfrac{\partial}{\partial x_i} + p_i \dfrac{\partial}{\partial z} + \sum_{j=1}^{n} p_{ji} \dfrac{\partial}{\partial p_j}.$

PROOF. The equality (#1) follows from the same argument as in Theorem 3.4, so we omit the proof. In the following, we prove the equality (#2). Two determinants on the right-hand side of the equality (#1) can be rewritten as follows;

$$\begin{vmatrix} (X_1)_{x_1} & \cdots & (X_{j-1})_{x_1} & (P_i)_{x_1} & (X_{j+1})_{x_1} & \cdots & (X_n)_{x_1} & p_1 & p_{11} & \cdots & p_{n1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (X_1)_{x_n} & \cdots & (X_{j-1})_{x_n} & (P_i)_{x_n} & (X_{j+1})_{x_n} & \cdots & (X_n)_{x_n} & p_n & p_{1n} & \cdots & p_{nn} \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 & & & \\ (X_1)_{p_1} & \cdots & (X_{j-1})_{p_1} & (P_i)_{p_1} & (X_{j+1})_{p_1} & \cdots & (X_n)_{p_1} & & \ddots & & \mathbf{0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \mathbf{0} & & \ddots & \\ (X_1)_{p_n} & \cdots & (X_{j-1})_{p_n} & (P_i)_{p_n} & (X_{j+1})_{p_n} & \cdots & (X_n)_{p_n} & & & & -1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{dX_1}{dx_1} & \cdots & \frac{dX_{j-1}}{dx_1} & \frac{dP_i}{dx_1} & \frac{dX_{j+1}}{dx_1} & \cdots & \frac{dX_n}{dx_1} & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{dX_1}{dx_n} & \cdots & \frac{dX_{j-1}}{dx_n} & \frac{dP_i}{dx_n} & \frac{dX_{j+1}}{dx_n} & \cdots & \frac{dX_n}{dx_n} & 0 & 0 & \cdots & 0 \\ (X_1)_z & \cdots & (X_{j-1})_z & (P_i)_z & (X_{j+1})_z & \cdots & (X_n)_z & -1 & & & \\ (X_1)_{p_1} & \cdots & (X_{j-1})_{p_1} & (P_i)_{p_1} & (X_{j+1})_{p_1} & \cdots & (X_n)_{p_1} & & \ddots & & \mathbf{0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \mathbf{0} & & \ddots & \\ (X_1)_{p_n} & \cdots & (X_{j-1})_{p_n} & (P_i)_{p_n} & (X_{j+1})_{p_n} & \cdots & (X_n)_{p_n} & & & & -1 \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} \frac{dX_1}{dx_1} & \cdots & \frac{dX_{j-1}}{dx_1} & \frac{dP_i}{dx_1} & \frac{dX_{j+1}}{dx_1} & \cdots & \frac{dX_n}{dx_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{dX_1}{dx_n} & \cdots & \frac{dX_{j-1}}{dx_n} & \frac{dP_i}{dx_n} & \frac{dX_{j+1}}{dx_n} & \cdots & \frac{dX_n}{dx_n} \end{vmatrix}, \qquad \text{and}$$

$$|L(\psi)| = (-1)^{n+1} \begin{vmatrix} \frac{dX_1}{dx_1} & \cdots & \frac{dX_n}{dx_1} \\ \vdots & & \vdots \\ \frac{dX_1}{dx_n} & \cdots & \frac{dX_n}{dx_n} \end{vmatrix}.$$

We obtain the equality (#2) by substituting these descriptions into the right-hand side of (#1). □

REMARK 3.8. Since ψ is a first-order contact transformation, we remark that the following condition for P_i is in hiding within the above expression of P_{ij};

$$\begin{pmatrix} (X_1)_{x_1} & \cdots & (X_n)_{x_1} & p_1 \\ \vdots & & \vdots & \vdots \\ (X_1)_{x_n} & \cdots & (X_n)_{x_n} & p_n \\ (X_1)_z & \cdots & (X_n)_z & -1 \\ (X_1)_{p_1} & \cdots & (X_n)_{p_1} & 0 \\ \vdots & & \vdots & \vdots \\ (X_1)_{p_n} & \cdots & (X_n)_{p_n} & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ \vdots \\ P_n \\ -f \end{pmatrix} = \begin{pmatrix} Z_{x_1} \\ \vdots \\ Z_{x_n} \\ Z_z \\ Z_{p_1} \\ \vdots \\ Z_{p_n} \end{pmatrix}.$$

Here if a contact transformation ψ satisfies the condition $\begin{pmatrix} (X_1)_{p_1} & \cdots & (X_n)_{p_1} \\ \vdots & & \vdots \\ (X_1)_{p_n} & \cdots & (X_n)_{p_n} \end{pmatrix} = \mathbf{0}$, then ψ becomes a first-order prolongation $\varphi^{(1)}$ of some diffeomorphism φ.

A comparison between Theorem 3.7 and Theorem 3.4 enables us to understand elementary the marked differences between the pseudo group of (prolonged) first-order contact transformations and the pseudo group of (prolonged) diffeomorphisms (point transformations). A typical example of such a difference is the following Legendre transformation of general type.

Example 3.9. Let I be a nonempty subset of the set $\{1,\dots,n\}$. The Legendre transformation of general type $\psi : (x_i, z, p_i) \mapsto (X_i, Z, P_i)$ is defined by $X_i := p_i$, $X_j := x_j$, $Z := z - \sum_{i\in I} p_i x_i$, $P_i := -x_i$, $P_j := p_j$, where $i \in I$, $j \in \{1,\dots,n\}\setminus I$. In the case of $I = \{1,\dots,n\}$ ψ is the usual full Legendre transformation, otherwise it is the partial Legendre transformation ([22]). By using Theorem 3.7, we can calculate the second-order prolongation of these Legendre transformations. For example, the second-order prolongation of the full Legendre transformation is given by the description $P_{ij} = -\dfrac{q_{ij}}{|(p_{\alpha\beta})_{1\le\alpha,\beta\le n}|}$, where q_{ij} denotes the cofactor of the (i,j)th element of the matrix $(p_{\alpha\beta})_{1\le\alpha,\beta\le n}$.

Remark 3.10. In [4], the author made reference to Klein's conjecture ([7]): The group of contact Cremona (i.e. birational) transformations of the projective plane is generated by the subgroups of the prolonged point transformations and the (full) Legendre transformations. In this connection, the author proved the following statement. Any polynomial contact automorphism of the 3-dimensional affine space can be realized as a composition of some number of the prolonged polynomial point automorphisms and the (full) Legendre transformations. In our notation the 3-dimensional affine space is $J^1(1,1)$ and the contact transformation $T : J^1(1,1) \to J^1(1,1)$ is said to be a contact polynomial automorphism if T and its inverse T^{-1} are automorphisms consisting of polynomials.

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