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BOUNDS FOR THE K-GROUPS ASSOCIATED TO ABELIAN VARIETIES OVER A p-ADIC FIELD

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Abstract

For a product of curves $X = C_1 \times \cdots \times C_n$ over a p-adic field k, in [2] we proposed a conjecture that the kernel of the Albanese map for X is p-divisible when the base field is absolutely unramified and proved this under some assumptions. In this note, we report that when the Jacobian varieties of such curves $C_1, \dots, C_n$ all have good ordinary reduction, the Albanese kernel for the product $X = C_1 \times \cdots \times C_n$ is still p-divisible even if the base field is not unramified but its ramification is small enough.

1. Introduction

Let k be a finite extension of $\mathbf{Q}_p$, and let X be a smooth, projective, and geometrically connected variety over a field k. We consider the group $CH_0(X)$ of zero cycles on X modulo rational equivalence. There is a degree map $\deg : CH_0(X) \to \mathbf{Z}$ whose kernel is denoted by $F^1(X)$. Moreover, there is a generalization of the Abel-Jacobi map

$$\mathrm{alb}_X : F^1(X) \to \mathrm{Alb}_X(k)$$

called the **Albanese map** of X and its kernel is denoted by $F^2(X)$, where Alb_X is the dual abelian variety to the Picard variety of X. When X has a k-rational point, the degree map is surjective. In [2], we proposed the following conjecture:

CONJECTURE 1.1. *Suppose that $k/\mathbf{Q}_p$ is unramified. Let $X = C_1 \times \cdots \times C_n$ be the product of smooth projective curves $C_1, \dots, C_n$ over k with $C_i(k) \neq \emptyset$ for all i. We further assume that we are in one of the following two situations:*

(good) *The Jacobian variety J_i of C_i has good reduction, for $i = 1, \dots, n$.*

(mult) *The Jacobian variety J_i of C_i has split multiplicative reduction, for $i = 1, \dots, n$, that is, the curve C_i is a Mumford curve over k.*

Then, the kernel of the Albanese map $F^2(X)$ is p-divisible.

The case **(mult)** is settled ([2, Proposition 4.16]). For the case **(good)** also, there are partial results as follows: For elliptic curves $C_i = E_i$ over k which has good reduction, assuming that at most one of $E_1, \dots, E_n$ has good supersingular reduction, the conjecture above is proved ([2, Theorem 1.4, Corollary 1.5]). In this short note, we report that using computations in [5], the divisibility of the Albanese kernel holds over the base field with low ramification as follows:

THEOREM 1.2. *Let k be a finite extension of $\mathbf{Q}_p$ with ramification index $e_{k/\mathbf{Q}_p} < p-1$. Let $X = C_1 \times \cdots \times C_n$ be the product of smooth projective curves $C_1, \dots, C_n$ over k with $C_i(k) \neq \emptyset$. We further assume that we are in one of the following two situations:*

(ord) *The Jacobian variety J_i of C_i has good ordinary reduction, for $i = 1, \dots, n$.*

(mult) *The Jacobian variety J_i of C_i has split multiplicative reduction, for $i = 1, \dots, n$.*

Then, the Albanese kernel $F^2(X)$ is p-divisible.

For a product of curves $X = C_1 \times \cdots \times C_n$ with $C_i(k) \neq \emptyset$, the Albanese kernel $F^2(X)$ is related to the **Somekawa K-groups** associated to the Jacobian varieties as follows (for the definition of Somekawa K-groups, see [5]):

$$F^2(X) \xrightarrow{\simeq} \bigoplus_{2 \le v \le n} \bigoplus_{1 \le i_1 < i_2 < \cdots < i_v \le n} K(k; J_{i_1}, \dots, J_{i_r}), \tag{1}$$

where J_i is the Jacobian variety of C_i (*cf.* [2, (2.5)]). To show that $F^2(X)$ is p-divisible, from the above isomorphism, it is enough to show the all terms $K(k; J_{i_1}, \dots, J_{i_r})$ are p-divisible. In fact, it is known that $K(k; J_1, \dots, J_r)$ is divisible for $r \ge 3$ ([5, Remark 4.4.5]). Because of this, first we investigate the structure of the Somekawa K-group $K(k; A_1, A_2)$ attached to two abelian varieties A_1, A_2 in the next section, and then give a proof of the above theorem.

On the contrary to the above theorem, when the base field k has ramified sufficiently, the Albanese kernel $F^2(X)$ may not be p-divisible. In fact, for a curve C over k and suppose that the Jacobian variety $J = \mathrm{Jac}_C$ has good ordinary reduction. Here, we assume that the base field k satisfies $J[p] \subset J(k)$. By the Weil pairing, this assumption implies $\mu_p \subset k$ and hence $e_{k/\mathbf{Q}_p} \ge p-1$ (*cf.* [3, Exercise A.7.8]). For the product $X = C \times C$, it is known that

$$F^2(X) \otimes \mathbf{Z}/p\mathbf{Z} \simeq K(k; J, J) \otimes \mathbf{Z}/p\mathbf{Z} \simeq (\mathbf{Z}/p\mathbf{Z})^{\oplus g^2},$$

where $g = \dim(J)$ ([4, Theorem 1.1]).

Notation. Throughout this note, we follow the notation used in [2]. In particular, for an abelian group G and $m \in \mathbf{Z}_{\ge 1}$, we write $G[m]$ and G/m for the kernel and cokernel of the multiplication by m on G respectively.

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2. Upper bounds for Somekawa K-groups

In this section, we use the following notation:

- k: a finite extension of $\mathbf{Q}_p$ with absolute ramification index $e_k = e_{k/\mathbf{Q}_p}$,
- $\mathbf{F}_k$: the residue field of k,
- k^{ur}: the maximal unramified extension of k, and
- $M^{\mathrm{ur}} := \max\{m \geq 0 \mid \mu_{p^m} \subset k^{\mathrm{ur}}\}$.

For an abelian variety A over k and for the Néron model $\mathscr{A}$ over $\mathcal{O}_k$ of A, we denote by $\mathscr{A}_s$ the special fiber of $\mathscr{A}$ and $\mathscr{A}_s^\circ$ the connected component of the zero element in $\mathscr{A}_s$. Recall that an abelian variety A has **split semi-ordinary reduction** (in the sense of [5]) if $\mathscr{A}_s^\circ$ is a semi-abelian variety over $\mathbf{F}_k$, the maximal abelian quotient $\bar{A}$ of $\mathscr{A}_s^\circ$ is ordinary and the maximal torus of $\mathscr{A}_s^\circ$ splits over $\mathbf{F}_k$. Namely, there is a short exact sequence

$$0 \to \mathbf{G}_m^{\oplus r} \to \mathscr{A}_s^\circ \to \bar{A} \to 0,$$

for some $r \geq 0$ and an ordinary abelian variety $\bar{A}$ over $\mathbf{F}_k$. To simplify the notation, we say that A has **split multiplicative reduction** if it has semi-ordinary reduction, and the connected component $\mathscr{A}_s^\circ$ is a split torus.

Let A_1 and A_2 be abelian varieties over k which have split semi-ordinary reduction. From [5, Theorem 4.5] the Somekawa K-group $K(k; A_1, A_2)$ attached to A_1 and A_2 is of the form

$$K(k; A_1, A_2) = K(k; A_1, A_2)_{\mathrm{fin}} \oplus K(k; A_1, A_2)_{\mathrm{div}} \tag{2}$$

for some finite group $K(k; A_1, A_2)_{\mathrm{fin}}$ and a divisible group $K(k; A_1, A_2)_{\mathrm{div}}$. From now on, we investigate the finite part $K(k; A_1, A_2)_{\mathrm{fin}}$. We denote by $K(k; A_1, A_2)_{\mathrm{fin}}[p^\infty]$ the p-torsion part $\varinjlim_{n \geq 1} K(k; A_1, A_2)[p^n]$ of the finite group $K(k; A_1, A_2)_{\mathrm{fin}}$.

As in Theorem 1.2 and Conjecture 1.1, we consider one of the following conditions:

(ord) The abelian variety A_i has good ordinary reduction for $i = 1, 2$.

(mult) The abelian variety A_i has split multiplicative reduction for $i = 1, 2$.

Theorem 2.1. *Assume one of the conditions* **(ord)** *or* **(mult)**. *Then, the p-torsion part $K(k; A_1, A_2)_{\mathrm{fin}}[p^\infty]$ is a quotient of $(\mathbf{Z}/p^{M^{\mathrm{ur}}})^{\oplus g_1 g_2}$, where $g_i = \dim(A_i)$.*

Proof. Recall that the Somekawa K-group $K(k; A_1, A_2)$ is a quotient of the Mackey product $(A_1 \otimes A_2)(k)$ for A_1 and A_2 regarding they are Mackey functors over k (for the definition and some properties of Mackey functors, see [5] or [2]). Hence, it is enough to show that there is a surjective homomorphism

$$(\mathbf{Z}/p^{M^{\mathrm{ur}}})^{\oplus g_1 g_2} \twoheadrightarrow (A_1 \otimes A_2)(k)/p^n$$

for any $n \geq 1$. Here, we divide the cases into **(ord)** and **(mult)**.

(The case **(ord)**) Consider the case **(ord)**. Recall that for a finite flat (commutative) group scheme $\mathscr{G}$ over $\mathcal{O}_k$, the correspondence $K \mapsto H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathscr{G})$ defines a Mackey functor ([5, Lemma 4.3.1]). Let $\mathscr{A}_i$ be the Néron model over $\mathcal{O}_k$ of A_i for each $i = 1, 2$. The connected-étale sequence

$$0 \to \mathscr{A}_i[p^n]^{\circ} \to \mathscr{A}_i[p^n] \to \mathscr{A}_i[p^n]^{\mathrm{et}} \to 0$$

(*cf.* [6, Section 1.4]) induces an exact sequence of Mackey functors

$$H^1_{\mathrm{fl}}(O_-, \mathscr{A}_i[p^n]^{\circ}) \to H^1_{\mathrm{fl}}(O_-, \mathscr{A}_i[p^n]) \to H^1_{\mathrm{fl}}(O_-, \mathscr{A}_i[p^n]^{\mathrm{et}}) \to 0 \tag{3}$$

([5, Lemma 4.3.3]). By [5, Lemma 4.3.3] again, we also have $H^1_{\mathrm{fl}}(O_-, \mathscr{A}_i[p^n]) \simeq A_i/p^n$ as Mackey functors. Put $\mathcal{H}_i^{\bullet} := H^1_{\mathrm{fl}}(O_-, \mathscr{A}_i[p^n]^{\bullet})$ for $\bullet \in \{\circ, \mathrm{et}\}$. By the right exactness of the Mackey products, we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
\mathcal{H}_1^{\circ} \otimes \mathcal{H}_2^{\circ} & \longrightarrow & \mathcal{H}_1^{\circ} \otimes A_2/p^n & \longrightarrow & \mathcal{H}_1^{\circ} \otimes \mathcal{H}_2^{\mathrm{et}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A_1/p^n \otimes \mathcal{H}_2^{\circ} & \longrightarrow & A_1/p^n \otimes A_2/p^n & \longrightarrow & A_1/p^n \otimes \mathcal{H}_2^{\mathrm{et}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}_1^{\mathrm{et}} \otimes \mathcal{H}_2^{\circ} & \longrightarrow & \mathcal{H}_1^{\mathrm{et}} \otimes A_2/p^n & \longrightarrow & \mathcal{H}_1^{\mathrm{et}} \otimes \mathcal{H}_2^{\mathrm{et}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array} \tag{4}$$

For a finite unramified extension K/k, the norm $H_i^*(K) \to H_i^*(k)$ is surjective ([5, Lemma 4.3.1]). Hence, the norm map for K/k on $\mathcal{H}_1^{\circ} \otimes \mathcal{H}_2^{\circ}$ and $\mathcal{H}_1^{\circ} \otimes \mathcal{H}_2^{\mathrm{et}}$ are surjective. By the diagram chase, same holds on $\mathcal{H}_1^{\circ} \otimes A_2/p^n$. In the same way, the norm map for K/k on $\mathcal{H}_1^{\mathrm{et}} \otimes \mathcal{H}_2^{\circ}$ and $\mathcal{H}_1^{\mathrm{et}} \otimes \mathcal{H}_2^{\mathrm{et}}$ are surjective, so is $\mathcal{H}_1^{\mathrm{et}} \otimes A_2/p^n$. From the vertical and the middle short exact sequence in the above diagram (4), the norm map $(A_1/p^n \otimes A_2/p^n)(K) \to (A_1/p^n \otimes A_2/p^n)(k)$ is surjective.

By replacing a sufficiently large unramified extension field K of k with k (and M^{ur} does not vary), we have

$$\mathscr{A}_i[p^n]^{\mathrm{et}} \simeq (\mathbf{Z}/p^n)^{\oplus g_i}, \qquad \text{and} \qquad \mathscr{A}_i[p^n]^{\circ} \simeq (\mu_{p^n})^{\oplus g_i}.$$

From [5, Lemma 4.3.3], we have $\mathcal{H}_i^{\circ} \simeq (U/p^n)^{\oplus g_i}$ and $\mathcal{H}_i^{\mathrm{et}} \simeq (\mathcal{Z}/p^n)^{\oplus g_i}$, where $\mathcal{Z}$ and U are Mackey functors defined by $K \mapsto \mathbf{Z}$ and $K \mapsto U_K = \mathcal{O}_K^{\times}$ respectively. By [5, Lemma 4.2.2] and the Mackey product $\otimes$ commutes with $\oplus$, we have

$$\mathcal{H}_1^{\mathrm{et}} \otimes \mathcal{H}_2^{\mathrm{et}} \simeq (\mathcal{Z}/p^n)^{\oplus g_1} \otimes (\mathcal{Z}/p^n)^{\oplus g_2} \simeq (\mathcal{Z}/p^n \otimes \mathcal{Z}/p^n)^{\oplus g_1 g_2} = 0,$$

$$\mathcal{H}_1^{\circ} \otimes \mathcal{H}_2^{\mathrm{et}} \simeq (U/p^n)^{\oplus g_1} \otimes (\mathcal{Z}/p^n)^{\oplus g_2} \simeq (U/p^n \otimes \mathcal{Z}/p^n)^{\oplus g_1 g_2} = 0, \qquad \text{and}$$

$$\mathcal{H}_1^{\mathrm{et}} \otimes \mathcal{H}_2^{\circ} = 0 \qquad \text{by the same computations above.}$$

Applying these results to the diagram (4), we obtain a surjective homomorphism

$$\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ \twoheadrightarrow A_1/p^n \otimes A_2/p^n. \tag{5}$$

By [5, Lemma 4.3.3, Lemma 4.2.2] again, the former group $\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ$ has a representation

$$\begin{aligned}(\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)(k) &\simeq ((U/p^n)^{\oplus g_1} \otimes (U/p^n)^{\oplus g_2})(k) \\ &\simeq (U/p^n \otimes U/p^n)(k)^{\oplus g_1 g_2} \qquad \text{(because } \otimes \text{ commutes with } \oplus) \\ &\simeq \bigoplus_{i=1}^{g_1 g_2} \mathbf{Z}/p^{n_i}\end{aligned}$$

for some $n_i \leq M^{\mathrm{ur}}$. By (5), we have the required surjective homomorphism

$$(\mathbf{Z}/p^{M^{\mathrm{ur}}})^{\oplus g_1 g_2} \twoheadrightarrow (\mathcal{H}_1^\circ \otimes \mathcal{H}_2^\circ)(k) \twoheadrightarrow (A_1/p^n \otimes A_2/p^n)(k) = (A_1 \otimes A_2)(k)/p^n$$

for any n. The assertion follows from this.

(The case **(mult)**) Next, we consider the case **(mult)**. From the assumption on A_i for $i = 1, 2$, there exists a split torus $T_i \simeq \mathbf{G}_m^{\oplus g_i}$ and a free abelian subgroup $L_i \subset T_i(k)$ such that

$$T_i(k')/L_i \simeq A_i(k') \tag{6}$$

for any finite extension k'/k. The quotient map $T_i \twoheadrightarrow A_i$ induces a surjection $(T_1 \otimes T_2)(k)/p^n \twoheadrightarrow (A_1 \otimes A_2)(k)/p^n$. This gives

$$((\mathbf{G}_m \otimes \mathbf{G}_m)(k)/p^n)^{\oplus g_1 g_2} \twoheadrightarrow (T_1 \otimes T_2)(k)/p^n \twoheadrightarrow K(k; A_1, A_2)/p^n$$

for any $n \geq 1$ (cf. [7, Remark 4.2 (2)]). Since we have a surjective homomorphism $\mathbf{Z}/p^{M^{\mathrm{ur}}} \twoheadrightarrow (\mathbf{G}_m \otimes \mathbf{G}_m)(k)/p^n$ ([5, Lemma 4.2.2]), we obtain surjective homomorphisms

$$(\mathbf{Z}/p^{M^{\mathrm{ur}}})^{\oplus g_1 g_2} \twoheadrightarrow ((\mathbf{G}_m \otimes \mathbf{G}_m)(k)/p^n)^{\oplus g_1 g_2} \twoheadrightarrow K(k; A_1, A_2)/p^n. \qquad \square$$

Proof of Theorem 1.2. As we referred in Section 1, to show the divisibility of $F^2(X)$ for the product $X = C_1 \times \cdots \times C_n$, by using the decomposition (1) it is enough to show that $K(k; J_1, \ldots, J_r)$ is divisible for any $r \geq 2$ under the condition **(ord)** or **(mult)** for the Jacobian variety J_i of curves C_i $(i = 1, \ldots, r)$. For the case $r \geq 3$, this follows from [5, Remark 4.4.5].

Now, we consider the case $r = 2$. Since we are assuming $e_k < p - 1$, we have $\mu_p \not\subset k$. If the extension $k(\mu_p)/k$ is unramified, then $e_{k(\mu_p)} = e_k < p - 1$. This contradicts with $e_{k(\mu_p)/\mathbf{Q}_p} = (p-1)e_{k(\mu_p)/\mathbf{Q}_p(\mu_p)} \geq p - 1$. As a result, we have $\mu_p \not\subset k^{\mathrm{ur}}$ and hence $M^{\mathrm{ur}} = 0$. From Theorem 2.1, the finite p-part is $K(k; J_1, J_2)_{\mathrm{fin}}[p^\infty] = 0$. By considering the decomposition (2), the groups $K(k; J_1, J_2) \simeq F^2(X)$ are p-divisible. $\square$

REMARK 2.2. If both of A_1 and A_2 have good reduction, then it is known that $K(k; A_1, A_2)$ is l-divisible for any prime $l \neq p$ ([5, Theorem 3.5]). Therefore, in the case **(ord)**, we have $K(k; A_1, A_2)_{\mathrm{fin}} = K(k; A_1, A_2)_{\mathrm{fin}}[p^\infty]$ and the K-group $K(k; A_1, A_2)$ becomes *divisible* when $M^{\mathrm{ur}} = 0$.

Closing this note, we explain some reasons why we have to restrict to the same reduction type of A_1 and A_2 as in the conditions **(ord)** and **(mult)**. Consider the following situation: Suppose A_1 has good ordinary reduction, and A_2 has split multiplicative reduction. From the exact sequence (3), we have an exact sequence of Mackey functors:

$$\mathcal{H}_1^\circ \to A_1/p^n \to \mathcal{H}_1^{\mathrm{et}} \to 0.$$

As in (6), there exists a split torus $T_2 \simeq \mathbf{G}_m^{\oplus g_2}$ and a subgroup $L_2 \subset T_2(k)$ such that $T_2(k')/L_2 \simeq A_2(k')$ for any finite extension k'/k. In particular, there is a surjection $T_2/p^n \twoheadrightarrow A_2/p^n$ of Mackey functors. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
\mathcal{H}_1^\circ \otimes T_2/p^n & \longrightarrow & \mathcal{H}_1^\circ \otimes A_2/p^n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
A_1/p^n \otimes T_2/p^n & \longrightarrow & A_1/p^n \otimes A_2/p^n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{H}_1^{\mathrm{et}} \otimes T_2/p^n & \longrightarrow & \mathcal{H}_1^{\mathrm{et}} \otimes A_2/p^n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & & .
\end{array}$$

As we have $\mathcal{H}_1^\bullet \otimes T_2/p^n \simeq (\mathcal{H}_1^\bullet \otimes \mathbf{G}_m/p^n)^{\oplus g_2}$, we consider the short exact sequence

$$\mathcal{H}_1^\circ \otimes \mathbf{G}_m/p^n \to A_1/p^n \otimes \mathbf{G}_m/p^n \to \mathcal{H}_1^{\mathrm{et}} \otimes \mathbf{G}_m/p^n \to 0.$$

After replacing a finite unramified extension K/k with k, by [5, Lemma 4.2.2] we have

$$(\mathcal{H}_1^\circ \otimes \mathbf{G}_m/p^n)(k) \simeq (U/p^n \otimes \mathbf{G}_m/p^n)(k)^{\oplus g_1} \simeq (\mathbf{Z}/p^{n_1})^{\oplus g_1},$$

$$(\mathcal{H}_1^{\mathrm{et}} \otimes \mathbf{G}_m/p^n)(k) \simeq (\mathcal{Z}/p^n \otimes \mathbf{G}_m/p^n)(k)^{\oplus g_1} \simeq (\mathbf{Z}/p^n)^{\oplus g_1}$$

for some $n_1 \leq M^{\mathrm{ur}}$. For this reason, the term $\mathcal{H}^{\mathrm{et}} \otimes \mathbf{G}_m/p^n$ is not bounded. In fact, using the formal group low $\hat{A}_1$ of A_1, there is a short exact sequence

$$(\hat{A}_1 \otimes \mathbf{G}_m)(k)/p^n \to (A \otimes \mathbf{G}_m)(k)/p^n \to \bar{A}_1(\mathbf{F}_k)/p^n \to 0, \tag{7}$$

where $\bar{A}_1$ is the reduction of A_1, and also a surjective homomorphism $(\mathbf{Z}/p^{M^{\mathrm{ur}}})^{\oplus g_1} \twoheadrightarrow (\hat{A}_1 \otimes \mathbf{G}_m)(k)/p^n$ for any $n \geq 1$ (*cf.* [1, Theorem 3.5]). Therefore, one can present an upper bound of the order of the p-torsion part of the Somekawa K-group as follows:

LEMMA 2.3. *We assume that A_1 has good ordinary reduction, and A_2 has split multiplicative reduction. Then, we have an inequality*

$$\mathrm{ord}_p(\#K(k; A_1, A_2)_{\mathrm{fin}}[p^\infty]) \leq g_2(g_1 M^{\mathrm{ur}} + \mathrm{ord}_p(\#\bar{A}_1(\mathbf{F}_k))), \tag{8}$$

where ord_p *is the order function normalized as* $\mathrm{ord}_p(p) = 1$.

PROOF. As we noted above, there are surjective homomorphisms

$$((A_1 \otimes \mathbf{G}_m)(k)/p^n)^{\oplus g_2} \twoheadrightarrow (A_1 \otimes A_2)(k)/p^n \twoheadrightarrow K(k; A_1, A_2)/p^n.$$

The product $(A_1 \otimes \mathbf{G}_m)(k)$ has also the following decomposition:

$$(A_1 \otimes \mathbf{G}_m)(k) = (A_1 \otimes \mathbf{G}_m)(k)_{\mathrm{fin}} \oplus (A_1 \otimes \mathbf{G}_m)(k)_{\mathrm{div}}$$

for some finite group $(A_1 \otimes \mathbf{G}_m)(k)_{\mathrm{fin}}$ and a divisible group $(A_1 \otimes \mathbf{G}_m)(k)_{\mathrm{div}}$ ([1, Lemma 3.1]). By taking the limit of the exact sequence (7), we have

$$\mathrm{ord}_p(\#(A_1 \otimes \mathbf{G}_m)(k)_{\mathrm{fin}}[p^\infty]) \leq g_1 M^{\mathrm{ur}} + \mathrm{ord}_p(\#\bar{A}_1(\mathbf{F}_k)).$$

Thus, we obtain the required inequality (8). □

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