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Relativity and the low-energy nd A_y puzzle

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We solve the Faddeev equation in an exactly Poincaré invariant formulation of the three-nucleon problem. The dynamical input is a relativistic nucleon-nucleon (NN) interaction that is exactly on-shell equivalent to the high-precision CD Bonn NN interaction. S-matrix cluster properties dictate how the two-body dynamics is embedded in the three-nucleon mass operator (rest Hamiltonian). We find that for neutron laboratory energies above ≈20 MeV relativistic effects on A_y are negligible. For energies below ≈20 MeV dynamical effects lower the nucleon analyzing power maximum slightly by ≈2% and Wigner rotations lower it further up to ≈10%, thereby increasing disagreement between data and theory. This indicates that three-nucleon forces (3NF) must provide an even larger increase of the A_y maximum than expected up to now.

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I. INTRODUCTION

High-precision nucleon-nucleon (NN) interactions such as AV18 [1], CDBonn [2], and Nijm I, II, and 93 [3] accurately describe the NN data set up to about 350 MeV. When these interactions are used to predict binding energies of three-nucleon ($3N$) systems they underestimate the experimental bindings of ^{3}H and ^{3}He by about 0.5–1 MeV [4,5]. This missing binding energy can be cured by introducing a three-nucleon force (3NF) into the nuclear Hamiltonian [5].

The study of elastic nucleon-deuteron (Nd) scattering and nucleon-induced deuteron breakup also revealed a number of cases where the nonrelativistic description based on pairwise interactions is insufficient to explain the data. Generally, the studied discrepancies between a theory based on NN interactions only and experiment become larger with increasing energy of the $3N$ system. Adding a 3NF that includes long-range 2π exchange to the pairwise interactions leads in some cases to a better description of the data. The parameters of such a 3NF must be separately adjusted to the experimental binding of ^{3}H and ^{3}He [6–8] for each NN interaction. The elastic Nd angular distribution in the region of its minimum and at backward angles is the best studied example [6,8]. The clear discrepancy in these angular regions at energies below ≈100 MeV nucleon laboratory energy between a theory based on NN interactions only and the cross-section data can be removed by adding modern 3NFs based on chiral effective field theory [9] to the nuclear Hamiltonian. At energies higher than ≈100 MeV current 3NFs [10,11] only partially improve the description of cross-section data, and the remaining discrepancies, which increase with energy, indicate the possibility of relativistic effects [12–14]. The need for a relativistic description of $3N$ scattering was also raised when precise measurements of the total cross section for neutron-deuteron (nd) scattering [15] were analyzed within the framework of nonrelativistic Faddeev calculations [16]. The NN interactions alone were insufficient to describe the data above ≈100 MeV.

In few-body models, off-shell effects, relativistic effects, and three-body force contributions cannot be cleanly separated. This is because different two-body interactions that give the same two-body S matrix are related by a unitary scattering equivalence [17]. To maintain this equivalence at the three-body level requires additional three-body interactions [18] in one of the Hamiltonians. Since relativistic two-body models are fit to the same data as the corresponding nonrelativistic models, there is a similar on-shell two-body scattering equivalence. Although the relativistic and nonrelativistic three-body predictions will be different one can in principle make up the difference (in a chosen frame) with a suitable three-body interaction. So, although it is possible to simulate "relativistic effects" with a three-body interaction, a Poincaré invariant treatment of the dynamics provides the most direct way to model the consequences of imposing Poincaré invariance and S-matrix cluster properties.

In this paper we investigate one particular representation of the Poincaré invariant three-body problem. We compare the predictions of relativistic and nonrelativistic three-body calculations where the input two-body interactions give the same two-body S-matrix, have the same internal two-body wave functions, and have a kinematic three-dimensional

Euclidean symmetry. The relativistic and nonrelativistic models differ in how these interactions appear in the three-body problem. In addition, the internal and single-particle variables are related by Galilean boosts in the nonrelativistic case and Lorentz boosts in the relativistic case. In the nonrelativistic case the two-body interactions for each pair are added to the center-of-mass kinetic energy operator. In the relativistic case, the nonlinear relation between the two- and three-body mass operators must be respected to obtain a scattering matrix that clusters into a product of the identity and the input two-body S matrix [19]. This nonlinear dependence of the three-body invariant mass operator on the two-body interaction has dynamical consequences for the three-body system, which complicates the structure of the Faddeev kernel.

In Ref. [20] we used a Poincaré invariant formulation of the $3N$ scattering problem. A technique for constructing the relativistic nucleon-nucleon interaction from a standard high-precision interaction was given in Ref. [21]. We used the same technique to construct the transition operators that appear in the kernel of the relativistic Faddeev equation. Application to a $3N$ bound state supported the relativistic effects previously found in Ref. [22].

Realistic NN interactions are fit by properly transforming experimental data to the center-of-momentum frame and fitting S-matrix elements computed by using the nonrelativistic Schrödinger equation to these data. Although the same data could be precisely fit by using S-matrix elements computed from a relativistic Schrödinger equation, this has not been done [23] with the same precision used to construct realistic interactions.

In Ref. [21], instead, an analytical scale transformation of momenta was used to relate NN interactions in the nonrelativistic and relativistic Schrödinger equations in such a way that the two-body scattering matrix elements are identified as $S_{\rm nr}(E_{\rm c.m.}) = S_r(E_{\rm c.m.}) = S_{\rm exp}(E_{\rm c.m.})$ as functions of the center-of-momentum energy [24]. In this work we use an alternative procedure [25] that generates a relativistic nucleon-nucleon interaction with the property that the relativistic and nonrelativistic two-body S matrices satisfy $S_r(\mathbf{k}^2) = S_{\rm nr}(\mathbf{k}^2) = S_{\rm exp}(\mathbf{k}^2)$, where $S_{\rm exp}(\mathbf{k}^2)$ is the experimental two-body S matrix, and $\mathbf{k}^2$ is the c.m. momentum of one of the particles.

When high-precision potentials are determined [26] by properly Lorentz transforming scattering data from the laboratory frame to the center-of-momentum frame the Lorentz invariant scalar product $p_{\rm target} \cdot p_{\rm beam} = mE_b = 2\mathbf{k}^2 + m^2$ is used to relate the laboratory beam energy E_b to the c.m. momentum $\mathbf{k}^2$. The potential is determined by comparing the transformed experimental scattering observables to the scattering observables computed by using the nonrelativistic Lippmann-Schwinger equation, identifying the $\mathbf{k}^2$ appearing in the Lippmann-Schwinger equation with the $\mathbf{k}^2$ computed from the invariant $p_{\rm target} \cdot p_{\rm beam}$. With this procedure the resulting interactions are constructed so that the S-matrix elements in the relativistic and nonrelativistic cases are identified as functions of $\mathbf{k}^2$ rather than c.m. energy. Though the difference in the two approaches leads to a small mismatch in the relativistic and nonrelativistic momentum, the interactions generated by the analytic scale transformation provide a useful first step to investigate the effects of the nonlinear relation between the three-body mass operator and two-body interactions.

In our initial studies [20,27,28] the interaction generated by the analytic scale transformation was used to study the changes in elastic nd scattering and breakup observables when the nonrelativistic form of the kinetic energy is replaced by the relativistic one and a proper treatment of the dynamics is included. We found that the elastic scattering cross section is only slightly influenced by relativity. Only at backward angles and higher energies are the elastic cross sections increased by relativity.

Because of the selectivity of the breakup reaction, however, regions of phase space were found at higher energies of the incoming nucleon where relativity leads to a characteristic pattern by which it changes the nonrelativistic breakup cross section. Namely, in this region of phase space fixing the angle of the first detected nucleon and changing the angle of the second nucleon provides variations of the nonrelativistic cross section by relativity, increasing or decreasing it by a factor up to ≈2. For spin observables the implemented relativistic features lead only to small effects.

Recently, an interesting estimate of "relativistic corrections" has been performed and its effect on low-energy nd analyzing power A_y has been estimated by using the plane-wave impulse approximation [29]. The estimate is based on a perturbative realization of the Poincaré Lie algebra to leading order in $1/c^2$ [30]. A large increase by ≈10% of the A_y maximum at laboratory energy $E_n = 3$ MeV has been found. The authors comment that their estimates are both exploratory and incomplete. In addition to the absence of final-state interaction, there are a number of other important differences from an exact formulation of this problem. Such a large effect, which would significantly reduce the discrepancy between theory and data in the region of the A_y maximum, calls for a relativistic study in an exactly Poincaré invariant treatment of the three-nucleon dynamics.

Our previous study [20], performed without inclusion of Wigner rotations, is too limited for spin observables. Therefore, to make definite conclusions for A_y we perform a complete dynamical calculation including the effects of Wigner rotations. We focus on that issue and do not include 3NFs.

The paper is organized as follows. In Sec. II we discuss the construction of our Poincaré invariant dynamical model. This includes a discussion of how high-precision interactions are used to construct the three-body mass operator (rest Hamiltonian). In Sec. III we discuss spin observables in Poincaré invariant quantum theory. In Sec. IV we discuss the formulation of the Faddeev equation to construct scattering observables for this three-nucleon mass operator. This includes an exact treatment of the Faddeev kernel, which avoids the approximations used in Refs. [20,27,28]. We also solve the relativistic $3N$ Faddeev equation with and without Wigner spin rotations and show and discuss results for the neutron analyzing power A_y. Section V contains a summary and conclusions.

II. POINCARé INVARIANT DYNAMICS

In quantum theory the principle of special relativity requires that the probabilities computed for equivalent experiments done in different inertial coordinate systems be identical. Since inertial coordinate systems are related by Poincaré transformations, it follows [31] that equivalent states in different inertial coordinate systems are related by a unitary representation, $U(\Lambda, a)$, of the Poincaré group. This emphasis on the invariance of experimental measurements in different inertial frames is different than the covariance requirements that are historically motivated by the way symmetries are realized in classical wave equations.

Since any representation of the Poincaré group can be decomposed into a direct integral of irreducible representations, one way to construct Poincaré invariant dynamics is to build it out of irreducible representations. The transformation properties of irreducible representations are well known and completely determined by group theoretical considerations. The dynamics is contained in the spectrum of the physical mass and spin operators, which determines the values and multiplicities of the Casimir invariants that appear in this decomposition.

Our construction begins with one-particle representations, which are irreducible representations. The particle's mass and spin fix the eigenvalues of the two Casimir invariants of the Poincaré group. For computations it is necessary to choose a basis for the irreducible representation space. This is done by choosing a maximal set of commuting Hermitian functions of the infinitesimal Poincaré generators. In addition to the mass and spin, it is possible to find four additional mutually commuting noninvariant functions of the generators. There is a second set of four operators that are conjugate to the noninvariant commuting observables. These operators change the eigenvalues and determine the spectrum of the commuting observables. All ten generators can be expressed as functions of these eight noninvariant operators and the two Casimir operators. The irreducible representation space, $\mathcal{H}$, is the space of square integrable functions of the eigenvalues of the four commuting operators [32].

Our choice of basis for irreducible representation spaces is the simultaneous eigenstates of the linear momentum $\mathbf{p}$ and the three-component of the canonical spin, j_{cz}, which is the observable corresponding to the spin measured in the particle's rest frame if the particle is transformed to its rest frame with a rotationless Lorentz transformation. In this basis the irreducible unitary representation of the Poincaré group is [32]

$$U(\Lambda, a)|(j, m)\mathbf{p}, \mu\rangle = \sum_{\mu'} |(j, m)\mathbf{p}', \mu'\rangle e^{ip'\cdot a}\sqrt{\frac{\omega(p')}{\omega(p)}} \times D^{j}_{\mu'\mu}[B^{-1}(\mathbf{p}'/m)\Lambda B(\mathbf{p}/m)], \quad (1)$$

where $p' = \Lambda p$, $\omega(p) = \sqrt{\mathbf{p}\cdot\mathbf{p} + m^2}$, and $B(\mathbf{p}/m)$ is the rotationless Lorentz transformation that takes a particle of mass m at rest to momentum $\mathbf{p}$. The quantity $R_w(\Lambda, p) := B^{-1}(\mathbf{p}'/m)\Lambda B(\mathbf{p}/m)$ is the standard rotationless-boost Wigner rotation. The representation given in Eq. (1) is unitary for states with a $\delta(\mathbf{p}-\mathbf{p}')$ normalization in the momentum. The important observation is that *all* mass $m > 0$ spin j irreducible representations of the Poincaré group in the $\{\mathbf{p}, j_{cz}\}$ basis have this form.

The two- or three-nucleon Hilbert space is the tensor product of two or three single-nucleon irreducible representation spaces: $\mathcal{H}\otimes\mathcal{H}$ or $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$. On each of these spaces

$$U_0(\Lambda, a) = U(\Lambda, a)\otimes U(\Lambda, a),$$
$$U_0(\Lambda, a) = U(\Lambda, a)\otimes U(\Lambda, a)\otimes U(\Lambda, a) \quad (2)$$

define kinematic representations of the Poincaré group. These representations are reducible and do not contain any dynamics. We build dynamical irreducible representations by adding suitable interactions to the mass Casimir operator of noninteracting irreducible representations. The first step needed to introduce interactions is then to decompose these noninteracting tensor product representations into a direct integral of irreducible representations. This is accomplished with Poincaré group Clebsch-Gordan coefficients in our chosen $\{\mathbf{p}, j_{cz}\}$ basis. The Poincaré group Clebsch-Gordan coefficients are the expansion coefficients of a linear combination of tensor product states that transform irreducibly. The desired noninteracting irreducible states are computed by (1) constructing rest eigenstates of the two-body system, (2) decomposing them into irreducible representations under SU(2) rotations, and (3) boosting the result to an arbitrary frame. The resulting Clebsch-Gordan coefficients in this basis are [19,32,33]

$$\begin{aligned}
&\langle \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2|(j, k)\mathbf{p}, \mu; l, s\rangle \\
&= \sum_{\mu_l\mu_s\mu_1'\mu_2'} \delta(\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2)\frac{\delta[k-k(\mathbf{p}_1, \mathbf{p}_2)]}{k^2}N^{-1}(p_1, p_2) \\
&\quad\times (l, \mu_l, s, \mu_s|j, \mu)(j_1, \mu_1', j_2, \mu_2'|s, \mu_s) \\
&\quad\times Y_{l\mu_l}[\hat{\mathbf{k}}(\mathbf{p}_1, \mathbf{p}_2)]D^{j_1}_{\mu_1'\mu_1}[R_w(B(\mathbf{p}/m_{120}), k_1)] \\
&\quad\times D^{j_2}_{\mu_2'\mu_2}[R_w(B(\mathbf{p}/m_{120}), k_2)] \qquad (3)\\
&= \int d\hat{\mathbf{k}} \sum_{\mu_l\mu_s\mu_1'\mu_2'} \delta(\mathbf{p}_1-\mathbf{p}_1(\mathbf{p}, \mathbf{k}))\delta(\mathbf{p}_2-\mathbf{p}_2(\mathbf{p}, \mathbf{k}))N(p_1, p_2) \\
&\quad\times (l, \mu_l, s, \mu_s|j, \mu)(j_1, \mu_1', j_2, \mu_2'|s, \mu_s)Y_{l\mu_l}(\hat{\mathbf{k}}) \\
&\quad\times D^{j_1}_{\mu_1'\mu_1}[R_w(B(\mathbf{p}/m_{120}), k_1)] \\
&\quad\times D^{j_2}_{\mu_2'\mu_2}[R_w(B(\mathbf{p}/m_{120}), k_2)]. \qquad (4)
\end{aligned}$$

In these expressions

$$\begin{aligned}
p^\mu &= p_1^\mu + p_2^\mu, \\
m_{120}^2 &= -p^2, \qquad (5)\\
k^\mu &= B^{-1}(\mathbf{p}/m_{120})^\mu{}_\nu \tfrac{1}{2}(p_1-p_2)^\nu,
\end{aligned}$$

$$N^{-2}(p_1, p_2) = \frac{\omega(k)\omega(k)[\omega(p_1)+\omega(p_2)]}{\omega(p_1)\omega(p_2)[\omega(k)+\omega(k)]}, \quad (6)$$

and the two-body invariant mass

$$m_{120} := 2\sqrt{\mathbf{k}^2+m^2} = 2\omega(k) \quad (7)$$

is replaced by the continuous variable $k := \sqrt{\mathbf{k}^2}$. The quantum numbers l and s are kinematically invariant quantities that distinguish multiple copies of representations with the same mass (k) and spin. For a two-nucleon system they have the same spectrum as the orbital and spin angular momentum operators in a partial-wave representation of the nonrelativistic basis.

An irreducible representation is also obtained by changing the order of the spin couplings in Eqs. (3) and (4), where the orbital angular momentum is first coupled to one of the spins, $j_2 + l = I$, and then the result is coupled to the second spin. $j_1 + I = j$. This representation is constructed by making the replacements

$$\sum_{s,\mu_s}(j_1,\mu_1',j_2,\mu_2'|s,\mu_s)(l,\mu_l,s,\mu_s|j,\mu) \to \sum_{I,\mu_I}(l,\mu_l',j_2,\mu_2|I,\mu_I)(I,\mu_I,j_1,\mu_1'|j,\mu) \quad (8)$$

in Eq. (3) or (4). In this representation the degeneracy parameters (l,s) are replaced by (l,I). When we construct three-body irreducible representations by successive pairwise coupling we use the coupling from Eq. (3) in the first Clebsch-Gordan coefficient and the coupling from Eq. (8) in the second Clebsch-Gordan coefficient. This allows us to identify the quantum numbers of the relativistic irreducible basis with the quantum numbers that we used in previous nonrelativistic calculations [34,35].

Three-particle irreducible representations for systems of noninteracting particles can be constructed by successive pairwise coupling of irreducible representations:

$$\begin{aligned}&\langle \mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2,\mathbf{p}_3,\mu_3|(J,q)\mathbf{P},\mu;\lambda,I,j_{23},k_{23},l_{23},s_{23}\rangle\\ &= \int d\mathbf{p}_{23}\sum_{\mu_{23}=-j_{23}}^{j_{23}}\langle \mathbf{p}_2,\mu_2,\mathbf{p}_3,\mu_3|(j_{23},k_{23})\mathbf{p}_{23},\mu_{23};l_{23},s_{23}\rangle\\ &\quad\times\langle \mathbf{p}_1,\mu_1,\mathbf{p}_{23},\mu_{23}|(J,q)\mathbf{P},\mu;\lambda,I\rangle. \end{aligned}\quad (9)$$

For three-nucleon scattering or bound-state problems it is sufficient and convenient to work in the three-body center-of-momentum frame. This simplifies the coefficients; the Wigner rotations in the second Clebsch-Gordan coefficient in Eq. (9) become the identity and the normalization factor $N(p_{23},p_1)\to 1$. Both of these factors are nontrivial in the first coefficient. The form of these coefficients for the case that particle 1 is a spectator (for details see Ref. [20]) using a shorthand notation is

$$\begin{aligned}&\langle \mathbf{p}_1,\mu_1',\mathbf{p}_2,\mu_2',\mathbf{p}_3,\mu_3'|(J,q)\mathbf{P}=\mathbf{0},\mu;\lambda,I,j_{23},k_{23},l_{23},s_{23}\rangle\\ &= \delta(\mathbf{0}-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\frac{1}{N(q_2,q_3)}\frac{\delta(q_1-q)}{q^2}\frac{\delta[k(\mathbf{q}_2,\mathbf{q}_3)-k]}{k^2}\\ &\quad\times\sum_{\mu_2\mu_3\mu_s}\sum_{\mu_l\mu_\lambda\mu_I}\left(\frac{1}{2},\mu_2,\frac{1}{2},\mu_3\middle|s,\mu_s\right)(l,\mu_l,s,\mu_s,|j,\mu_j)\\ &\quad\times\left(\lambda,\mu_\lambda,\frac{1}{2},\mu_1'\middle|I,\mu_I\right)(j,\mu_j,I,\mu_I|J,\mu)Y_{\lambda\mu_\lambda}(\hat{\mathbf{q}}_1)\\ &\quad\times Y_{l\mu_l}[\hat{\mathbf{k}}(\mathbf{q}_2,\mathbf{q}_3)]D^{\frac{1}{2}}_{\mu_2'\mu_2}[R_w(B(-\mathbf{q}_1/m_{023}),k_2(\mathbf{q}_2,\mathbf{q}_3))]\\ &\quad\times D^{\frac{1}{2}}_{\mu_3'\mu_3}[R_w(B(-\mathbf{q}_1/m_{023}),k_3(\mathbf{q}_2,\mathbf{q}_3))]. \end{aligned}\quad (10)$$

where

$$\begin{aligned}&[\mathbf{q_i},\omega(q_i)] = B^{-1}(\mathbf{P}/M)p_i, \quad i\in\{1,2,3\},\\ &[\mathbf{k}_i,\omega(k_i)] = B^{-1}(-\mathbf{q}_k/m_{ij})p_i,\end{aligned}\quad (11)$$

$$\sum_i \mathbf{q}_i = \mathbf{0}, \qquad M=\sum_{i=1}^{3}\sqrt{m^2+\mathbf{q}_i^2}. \quad (12)$$

The important property of the states

$$|(j,k)\mathbf{p},\mu;l,s\rangle \quad (13)$$

and

$$|(J,q)\mathbf{P},\mu;\lambda,I,j_{23},k_{23},l_{23},s_{23}\rangle \quad (14)$$

is that they transform irreducibly. The mass and spin are given by

$$m_{120} = 2\sqrt{m^2+\mathbf{k}^2}, \quad j, \quad (15)$$

$$M = \sqrt{4m^2+4\mathbf{k}^2+\mathbf{q}^2}+\sqrt{m^2+\mathbf{q}^2}, \quad J, \quad (16)$$

respectively.

To use these representations to construct dynamical representations an interaction is added to the two- or three-body invariant mass operator of the form

$$\begin{aligned}&\langle (j,k)\mathbf{p},\mu\cdots|v|\cdots\mathbf{p}',\mu'(j',k')\rangle\\ &=\delta_{\mu\mu'}\delta_{jj'}\delta(\mathbf{p}-\mathbf{p}')\langle k,\cdots\|v^j\|\cdots',k'\rangle\end{aligned}\quad (17)$$

for $N=2$ or

$$\begin{aligned}&\langle (J,q)\mathbf{P},\mu\cdots|V|\cdots\mathbf{P}',\mu'(J',q')\rangle\\ &=\delta_{\mu\mu'}\delta_{JJ'}\delta(\mathbf{P}-\mathbf{P}')\langle k,q,\cdots\|V^J\|\cdots',k',q'\rangle\end{aligned}\quad (18)$$

for $N=3$. Diagonalizing $m_{12}=m_{120}+v$ or $M=M_0+V$ in the noninteracting irreducible basis gives simultaneous eigenstates of m_{12}, $\mathbf{p}$, j^2, j_z for $N=2$ and of M, $\mathbf{P}$, J^2, J_z for $N=3$. In both the two- and three-body case these eigenstates, $|(j_{12},\lambda_{m_{12}})\mathbf{p}_{12},\mu_{12},\cdots\rangle$ and $|(J,\lambda_M)\mathbf{P},\mu,\cdots\rangle$, where $\lambda_{m_{12}}$ and λ_M are the eigenvalues of m_{12} and M, are complete on the two- and three-body Hilbert spaces, respectively.

The dynamical representation of the Poincaré group is defined by requiring that these eigenstates transform like Eq. (1) with the mass being replaced by the mass eigenvalues λ_M or $\lambda_{m_{12}}$. This representation is unitary and defines the dynamics of the system. With our choice of irreducible basis, $\{\mathbf{p},j_{cz}\}$, the resulting irreducible representations of the Poincaré group have a mass-independent representation of the three-dimensional Euclidean subgroup, which Dirac [36] called an "instant-form dynamics."

For the three-nucleon case there remains the problem of how to construct realistic NN interactions. For two-body interactions the relation

$$\begin{aligned}H_{12}^2-\mathbf{p}^2 &= m_{12}^2 = 4(\mathbf{k}^2+m^2)+4mv_{NN}\\ &= 4m(\underbrace{\mathbf{k}^2/m+v_{NN}}_{h_{\rm nr}=H_{\rm nr}-\frac{\mathbf{p}^2}{4m}}+m)\end{aligned}\quad (19)$$

implies that the square of the two-body mass operator has a simple relation to the nonrelativistic rest Hamiltonian with a "realistic" NN interaction [25], provided one identifies the spectrally equivalent relative momenta, $\mathbf{k}^2$. In the relativistic case, $k := B^{-1}(\mathbf{p}/m_{120})\frac{1}{2}(p_1 - p_2)$, one has

$$\mathbf{k} \equiv \mathbf{k}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{2} \times \left(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}\frac{\omega(p_2) - \omega(p_1)}{\omega(p_2) + \omega(p_1) + \sqrt{[\omega(p_2) + \omega(p_1)]^2 - \mathbf{p}^2}}\right), \tag{20}$$

whereas in the nonrelativistic case, $k = B_g^{-1}(\mathbf{p}/2m)\frac{1}{2}(p_1 - p_2)$, and

$$\mathbf{k} = \frac{1}{2}\left[\left(\mathbf{p}_1 - \frac{\mathbf{p}}{2m}m\right) - \left(\mathbf{p}_2 - \frac{\mathbf{p}}{2m}m\right)\right] = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2), \tag{21}$$

where $B(\mathbf{p}/m_{120})$ is a rotationless Lorentz boost and $B_g(\mathbf{p}/2m)$ is the corresponding Galilean boost.

The Kato-Birman invariance principle [37–39] implies that the Møller wave operators satisfy

$$\begin{aligned}\Omega_\pm(H, H_0)\colon &= s - \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t} \\ &= s - \lim_{t\to\pm\infty} e^{if(H)t}e^{-if(H_0)t} \\ &= \Omega_\pm(f(H), f(H_0)),\end{aligned} \tag{22}$$

where $f(x)$ is any piecewise differentiable function of bounded variation with positive derivative [39]. The functions $f(x) = x^2$ and $f(x) = x^{1/2}$ satisfy the conditions of the Kato-Birman theorem. Using Eq. (22) along with the kinematic Euclidean invariance of the Hamiltonians H_r and $H_{\rm nr}$ gives the following relation between the two-body scattering wave operators:

$$\begin{aligned}\Omega_\pm(H_r, H_{r0}) &= \Omega_\pm\left(H_r^2, H_{r0}^2\right) = \Omega_\pm\left(M_r^2, M_{r0}^2\right) \\ &= \Omega_\pm(M_r, M_{r0}).\end{aligned} \tag{23}$$

However, the identification from Eq. (19) along with the reparametrization $t \to t' = 4mt$ gives

$$\begin{aligned}\Omega_\pm\left(M_r^2, M_{r0}^2\right) &= s - \lim_{t\to\pm\infty} e^{ih_{\rm nr}4mt}e^{-h_{\rm nr0}4mt} \\ &= s - \lim_{t'\to\pm\infty} e^{ih_{\rm nr}t'}e^{-h_{\rm nr0}t'} = \Omega_{\pm\rm nr}(h_{\rm nr}, h_{\rm nr0}).\end{aligned} \tag{24}$$

Writing both wave operators as direct integrals over $\mathbf{k}^2 = mh_{0\rm nr} = (M_0^2 - 4m^2)/4$ leads to the identifications

$$\begin{aligned}\Omega_\pm(H_{\rm nr}, H_{0\rm nr}) = \Omega_\pm(h_{\rm nr}, h_{0\rm nr}) &= \int_\oplus \hat{\Omega}_\pm(\mathbf{k}^2)d\mathbf{k}^2 \\ &= \Omega_\pm(H_r, H_{0r}) = \Omega_\pm\left(M_{\rm nr}^2, M_{0\rm nr}^2\right)\end{aligned} \tag{25}$$

and

$$S(\mathbf{k}^2) = \Omega_{r+}^\dagger(\mathbf{k}^2)\Omega_{r-}(\mathbf{k}^2) = \Omega_{\rm nr+}^\dagger(\mathbf{k}^2)\Omega_{\rm nr-}(\mathbf{k}^2). \tag{26}$$

The identification of the relativistic and nonrelativistic wave operators as functions of $\mathbf{k}^2$ ensures that the relativistic two-body model is fit to the same two-body S-matrix (experimental data) as the nonrelativistic model provided the interactions are related by Eq. (19). The identification of the wave operators also implies the identity of the scattering wave functions as a function of $\mathbf{k}$. The identity of the bound-state wave functions is also due to Eq. (19).

In our calculations we use the interaction v defined by $v := m_{12} - m_{120}$, which we construct [40] from the NN interaction in Eq. (19) by iterating

$$\{m_{120}, v\} = 4mv_{NN} - v^2 \tag{27}$$

in the irreducible plane-wave basis. Because m_{12} and m_{12}^2 have the same eigenvectors, the $\mathbf{k}$ dependence of the wave functions constructed from m_{12} are also identical to the corresponding nonrelativistic wave functions.

Equation (27) in the irreducible plane-wave basis has the form

$$\begin{aligned}&\langle k, l, s|v^j|k', l', s'\rangle \\ &= 2m\frac{\langle k, l, s|v_{NN}^j|k', l', s'\rangle}{\omega(k) + \omega(k')} - \sum_{l''s''}\int k''^2dk'' \\ &\quad\times \frac{\langle k, l, s|v^j|k'', l'', s''\rangle\langle k'', l'', s''|v^j|k', l', s'\rangle}{2\omega(k) + 2\omega(k')}.\end{aligned} \tag{28}$$

The iteration converges quickly for realistic interactions [40]. Although a mathematical proof of convergence of the iteration of Eq. (28) is lacking, the results of the iterations are easily tested because the resulting $m_{120} + v$ must have the same eigenfunctions as the nonrelativistic two-body Hamiltonian.

We applied this approach using the CD Bonn potential as the nonrelativistic interaction $v_{NN}(k, k')$. In our previous studies [20,27,28] we used the momentum transformation of Ref. [24] and in addition restricted our calculation to leading order terms in the $\mathbf{p}/\omega$ and v/ω expansions only:

$$V(\mathbf{k}, \mathbf{k}'; \mathbf{q}) = v(\mathbf{k}, \mathbf{k}')\left(1 - \frac{\mathbf{q}^2}{8\omega(k)\omega(k')}\right). \tag{29}$$

We checked that in most cases this simple approximation leads to practically the same results as the exact approach applied in the present study.

Once the two-body mass operator is constructed, the three-body mass operator for the interacting (ij) pair is the well-defined nonlinear function of the two-body mass:

$$M_{(ij)(k)} = \sqrt{(m_{ij0} + v_{ij})^2 + \mathbf{q}_k^2} + \sqrt{m^2 + \mathbf{q}_k^2}, \tag{30}$$

where v_{ij} is embedded in the three-body Hilbert space so it commutes with $\mathbf{q}_k$. If this is interpreted as the rest energy operator, the interacting pair and spectator energies are additive in the rest frame. This implies that the S-matrix clusters properly in the rest frame while the invariance of S-matrix in all frames ensures that this property extends to all inertial coordinate systems.

Pairwise interactions in the three-body system are defined by

$$\begin{aligned}V_{(ij)(k)} &= M_{(ij)(k)} - M_0 \\ &= \sqrt{(m_{ij0} + v_{ij})^2 + \mathbf{q}^2} - \sqrt{m_{ij0}^2 + \mathbf{q}^2}.\end{aligned} \tag{31}$$

For any pair of particles these interactions commute with kinematic momentum and spin and are independent of the momentum and magnetic quantum numbers. This ensures that the sum of the interactions has the general form of Eq. (18).

A generalization of the method used in Eqs. (27) and (28) can be used to construct $V_{(ij)(k)}$ by iterating

$$\left\{\sqrt{m_{ij0}^2+\mathbf{q}^2}, V_{(ij)(k)}\right\} = v^2 + \{m_{ij0}, v\} - V_{(ij)(k)}^2. \quad (32)$$

Specifically,

$$\begin{aligned}&\langle k,\cdots|V_{(ij)(k)}(\mathbf{q}^2)|k',\cdots\rangle\\ &= \frac{1}{\sqrt{m_{ij0}^2(k)+\mathbf{q}^2}+\sqrt{m_{ij0}^2(k')+\mathbf{q}^2}}\\ &\quad\times\Bigg[\langle k,\cdots|v^2|k',\cdots\rangle + 2(\omega(k)+\omega(k'))\langle k|v|k'\rangle\\ &\quad-\sum_{l''s''}\int\langle k,\cdots|V_{(ij)(k)}(\mathbf{q}^2)|k'',\cdots\rangle k''^2dk''\langle k'',\cdots|\\ &\quad\times V_{(ij)(k)}(\mathbf{q}^2)|k',\cdots\rangle\Bigg]. \quad (33)\end{aligned}$$

This iteration also converges and is used to construct interactions V_{ij} for each pair of particles.

The three-body mass (rest energy) operator is

$$M = M_0 + V_{12} + V_{23} + V_{31}, \quad (34)$$

where M_0 is the three-body kinematic invariant mass given by

$$M_0 = \sqrt{m_{120}^2+\mathbf{q}^2} + \sqrt{m^2+\mathbf{q}^2}. \quad (35)$$

Our relativistic Faddeev equation is based on this mass operator [Eq. (34)] with two-body interactions constructed from the CD Bonn interaction using Eqs. (27), (28), (32), and (33).

Finally, just as in the nonrelativistic case, there is a natural order of coupling of the irreducible representations for computing each pairwise interaction. The change of basis relating different orders of coupling is needed for the implementation of the Faddeev equation as an integral equation. The required basis change only changes the invariant degeneracy quantum numbers associated with each order of coupling:

$$\begin{aligned}&\langle (jm)\mathbf{P},\mu(ab)(c)|(j'm')\mathbf{P}',\mu'(de)(f)\rangle\\ &\quad=\delta(\mathbf{P}-\mathbf{P}')\delta_{jj'}\delta_{\mu\mu'}R^{jm}[(ab)(c);(de)(f)]. \quad (36)\end{aligned}$$

The invariants $R^{jm}[(ab)(c);(de)(f)]$ are Racah coefficients for the Poincaré group. They are constructed by using four Poincaré Clebsch-Gordan coefficients. We compute this quantity using the Balian-Brezin method [41], where the variables associated with one order of coupling are expressed in terms of the variables associated with another order of coupling. The invariant coefficient, $R^{jm}[(ab)(c);(de)(f)]$, can be computed by evaluating the expression at zero momentum, averaging over the magnetic quantum numbers, and evaluating the resulting expression at any kinematically allowed set of momenta [42,43]. These Racah coefficients contain Wigner rotations and Jacobians that do not appear in the nonrelativistic permutation operators. Explicit expressions are given in Appendix B.

III. SPIN OBSERVABLES

In a Poincaré invariant quantum theory or relativistic quantum field theory the spin of a particle can be defined as the angular momentum that is measured in the particle's rest frame. For any nonzero momentum $\mathbf{p}$, different Lorentz transformations can be used to transform a particle at rest to a frame where the particle has momentum $\mathbf{p}$. Because the commutator of two rotationless boost generators,

$$[K_j, K_k] = -i\epsilon_{jkl}J_l, \quad (37)$$

is a rotation generator, the spin of the particle with momentum $\mathbf{p}$ depends on the choice of Lorentz transformation that transforms the particle's momentum from zero to $\mathbf{p}$. To have an unambiguous definition of the spin it is necessary to choose a standard set of $\mathbf{p}$-dependent Lorentz boosts, $B(\mathbf{p}/m)^\mu{}_\nu$, that transform a particle of mass m at rest to momentum $\mathbf{p}$. Then the spin of the particle can be unambiguously defined as the value of the spin measured in the particles rest frame if it is transformed to the rest frame by using the standard Lorentz transformation. With this definition, if two spins are equal in one frame they are equal in all frames.

The choice of standard boost is not unique because if $R(\mathbf{p}/m)$ is any $\mathbf{p}$-dependent rotation and

$$B'(\mathbf{p}/m) := B(\mathbf{p}/m)R(\mathbf{p}/m) \quad (38)$$

then both $B^{-1}(\mathbf{p}/m)^\mu{}_\nu$ and $B^{-1\prime}(\mathbf{p}/m)^\mu{}_\nu$ both transform $\mathbf{p}$ to zero.

Each choice of boost leads to a different spin operator, corresponding to a different prescription for measuring the spin in an arbitrary frame. The rotation

$$R(\mathbf{p}/m) = B^{-1}(\mathbf{p}/m)B'(\mathbf{p}/m) \quad (39)$$

that relates different boosts is called a generalized Melosh rotation [32,44].

The spin operator $\mathbf{j}_x$ associated with a boost $B_x(\mathbf{p}/m)$ is defined as the following function of the Poincaré generators:

$$(0,\mathbf{j}_x) = B_x^{-1}(\mathbf{p}/m)^\mu{}_\nu W^\mu/m, \quad (40)$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}p_\alpha M_{\beta\gamma}$ is the Pauli-Lubanski vector, $M_{\beta\gamma}$ is the relativistic angular momentum tensor, and $B_x(\mathbf{p}/m)$ is the boost matrix with the parameters $\mathbf{p}/m$ and m replaced by the mass and momentum operators.

Although this quantity has the appearance of a four-vector it is not because of the operator dependence of the arguments of the boost. Instead, under Lorentz transformations the spin transforms like

$$U^\dagger(\Lambda,0)\mathbf{j}_x U(\Lambda,0) = R_{wx}(\Lambda,p)\mathbf{j}_x, \quad (41)$$

where

$$R_{wx}(\Lambda,p) := B_x^{-1}(\Lambda\mathbf{p}/m)\Lambda B_x(\mathbf{p}/m) \quad (42)$$

is the Wigner rotation associated with the boost $B_x(\mathbf{p}/m)$. It is a consequence of the Poincaré commutation relations that the components of any of these spin observables satisfy the SU(2) commutation relations

$$[j_{xl}, j_{xm}]_- = i\epsilon_{lmn}j_{xn} \quad (43)$$

for any x. The operator $\mathbf{j}^2$ is independent of the choice of boost because the generalized Melosh rotations leave the scalar product of two vectors unchanged. The spin defined with the textbook rotationless or canonical boost is called the canonical spin.

It is natural to ask how these different types of spins are measured in the laboratory. Spins of isolated elementary or composite particles are measured in the laboratory through their response to classical electromagnetic fields. In the one-photon exchange approximation the photon couples to matrix elements of a covariant current operator. Imposing Poincaré covariance, current conservation, and discrete symmetries allows one to express all current matrix elements in terms of an independent set of matrix elements, which have a one-to-one correspondence with invariant form factors. All conventional form factors can be expressed in terms of Breit frame matrix elements with canonical spin and a quantization axis parallel to the Breit frame momentum transfer.

In the SL(2, $\mathbb{C}$) representation canonical boosts are represented by positive Hermitian matrices. They have the general form

$$B(\mathbf{p}/m) = \exp(\boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2), \tag{44}$$

where $\boldsymbol{\rho}$ is the rapidity of the Lorentz transformation and $\boldsymbol{\sigma}$ are the Pauli matrices. In this paper all of our spins are canonical spins. The SO(1, 3) representation of canonical boosts are the standard rotationless boosts.

Given a definition of the form factors in terms of independent current matrix elements in a given basis, it is also possible to express them in terms of current matrix elements in any other standard frame by using any other basis [45]. For example, the expression in terms of Breit frame canonical spin matrix elements can be replaced by a different independent set of laboratory frame helicity spin matrix elements. In quantum field theory the choice of boost is built into conventions used to define the Dirac spinors. The relation of the invariant form factors to current matrix elements with different choices of spin determines the relationship between different spin observables and experiment.

The spin degrees of freedom of the asymptotic incoming or outgoing particles are most conveniently expressed in terms of traces of density matrices, which is a reflection of the fact that realistic initial and/or final states are generally not pure states. Scattering spin observables in cross sections are formally defined by [34]

$$\langle O \rangle = \frac{\mathrm{Tr}(S_f T S_i T^\dagger)}{\mathrm{Tr}(T T^\dagger)}, \tag{45}$$

where T is the invariant scattering amplitude for the reaction under consideration. The connection between the invariant scattering amplitude defined in the particle data book [46] and the transition amplitudes constructed by solving our formulation of the relativistic Faddeev equations is given in Ref. [47].

The quantities S have the form

$$S = \sum s_a S_a, \tag{46}$$

where the index runs over all N_i or N_f initial or final sets of magnetic quantum numbers, S_a are a basis for $N_i \times N_i$ or $N_f \times N_f$ matrices that are orthonormal with respect to the trace norm, and s_a are constant coefficients [34].

If the initial and final asymptotic states are represented in a canonical spin basis, then the magnetic quantum numbers that appear in the invariant amplitudes T transform with Wigner rotations under Lorentz transformations. The result is that the spin observable $\langle O \rangle$ will not be invariant unless the matrices S_a or coefficients s_a are defined to transform in a manner that leaves the observable invariant.

Any spin observable can be made Lorentz invariant, by defining the invariant observable as its value in a given frame if it is transformed to the frame with a specific Lorentz boost. This can be used to get an invariant definition of the vector or tensor polarizations.

In this paper invariant spin observables are defined to be the values of the observable in the laboratory frame (rest frame of the target). To evaluate the corresponding spin observable it is only necessary to evaluate the expression

$$\langle O \rangle = \frac{\mathrm{Tr}(S_f T S_i T^\dagger)}{\mathrm{Tr}(T T^\dagger)} \tag{47}$$

for the values of the invariant amplitudes with laboratory kinematics.

This observable is equal its value evaluated in other frames by using the formula

$$\langle O \rangle = \frac{Tr(S_f D^\dagger M D' S_i D'^\dagger M^\dagger D)}{\mathrm{Tr}(M M^\dagger)}, \tag{48}$$

where the invariant amplitudes are evaluated in the other frame and D and D' are products of Wigner D functions of the Wigner rotations associated with the boost from the laboratory frame to the other frame.

Our specific interest in this paper is the observable A_y with polarized incoming nucleon. The convention used to define A_y is the Madison convention, where the laboratory frame scattering plane is in the xz plane. The observable A_y is defined as

$$\langle A_y \rangle = \frac{\mathrm{Tr}[T(\sigma_y \times I_d) T^\dagger]}{\mathrm{Tr}(T T^\dagger)}. \tag{49}$$

Because the Wigner rotation for canonical boost along the direction of a particle's momentum is the identity, σ_y is unchanged and so A_y can also be evaluated in the c.m. frame without making any compensating Wigner rotations.

IV. FADDEEV EQUATION

The nucleon-deuteron scattering with neutron and protons interacting through a NN interaction v_{NN} alone is described in terms of a breakup operator T satisfying the Faddeev-type integral equation [34,48]

$$T|\phi\rangle = tP|\phi\rangle + tPG_0T|\phi\rangle. \tag{50}$$

The two-nucleon ($2N$) t matrix results from solving the Lippmann-Schwinger equation with the interaction v_{NN}. The permutation operator $P = P_{12}P_{23} + P_{13}P_{23}$ is given in terms of the transposition P_{ij} that interchanges nucleons i and j. The incoming state $|\phi\rangle = |\mathbf{q}_0\rangle|\phi_d\rangle$ describes the free nucleon-deuteron motion with relative momentum $\mathbf{q}_0$ and the deuteron state vector $|\phi_d\rangle$. Finally, G_0 is the resolvent of the three-body center-of-mass kinetic energy.

The elastic nd scattering transition operator U is given in terms of T by [34,48]

$$U = PG_0^{-1} + PT. \tag{51}$$

This is our standard nonrelativistic formulation, which is equivalent to the nonrelativistic $3N$ Schrödinger equation plus boundary conditions. The formal structure of these equations in the relativistic case remains the same but the ingredients change. As explained in Ref. [22] the relativistic $3N$ rest Hamiltonian (mass operator) has the same form as the nonrelativistic one; only the momentum dependence of the kinetic energy changes and the relation of the pair interactions in the three-body problem to the pair interactions in the two-body problem changes. Consequently, all the formal steps leading to Eqs. (50) and (51) remain the same.

The free relativistic invariant mass of three identical nucleons in their c.m. system has the form given by Eq. (35) whereas the free two-body mass operator has the form of Eq. (7).

As introduced in Ref. [19] the pair forces in the relativistic $3N$ $2+1$ mass operator are given by Eq. (31), where $V = V(\mathbf{q}^2)$ reduces to the interaction v for $\mathbf{q} = 0$.

The transition matrix that appears in the kernel of the Faddeev equation (50) is obtained by solving the Lippmann-Schwinger equation, which must be solved as a function of $\mathbf{q}^2$:

$$t(\mathbf{k},\mathbf{k}';\mathbf{q}^2) = V(\mathbf{k},\mathbf{k}';\mathbf{q}^2) + \int d^3k'' \times \frac{V(\mathbf{k},\mathbf{k}'';\mathbf{q}^2)t(\mathbf{k}'',\mathbf{k}';\mathbf{q}^2)}{\sqrt{(2\omega(\mathbf{k}')^2+\mathbf{q}^2} - \sqrt{(2\omega(\mathbf{k}'')^2+\mathbf{q}^2} + i\epsilon}. \tag{52}$$

The input two-body interactions are computed by solving Eqs. (32) and (33).

The new relativistic ingredients in Eqs. (50) and (51) will therefore be the t operator [Eq. (52)] (expressed in partial waves) and the resolvent of the $3N$ invariant mass,

$$G_0 = \frac{1}{E + i\epsilon - M_0}, \tag{53}$$

where M_0 is given by Eq. (35), E, the total $3N$ c.m. energy expressed in terms of the initial neutron momentum $\mathbf{q}_0$ relative to the deuteron, is

$$E = \sqrt{(M_d)^2 + \mathbf{q}_0^2} + \sqrt{m^2 + \mathbf{q}_0^2}, \tag{54}$$

and M_d is the deuteron rest mass.

Currently, Eq. (50) in its nonrelativistic form is numerically solved for any NN interaction by using a momentum space partial-wave decomposition. Details are presented in Ref. [48]. This turns Eq. (50) into a coupled set of two-dimensional integral equations. As shown in Ref. [20], in the relativistic case we can keep the same formal structure, though the permutation operators are replaced by the corresponding Racah coefficients [Eq. (36)] for the Poincaré group. These coefficients include both Jacobians and Wigner rotations that do not appear in the nonrelativistic permutation operators [34,35]. These coefficients are computed in Appendix B by using methods that we have applied to compute the nonrelativistic permutation operators.

In the nonrelativistic case the partial-wave projected momentum space basis is

$$\left|pq(ls)j\left(\lambda\tfrac{1}{2}\right)IJ\left(t\tfrac{1}{2}\right)T\right\rangle, \tag{55}$$

where p and q are the magnitudes of standard Jacobi momenta (see Refs. [34,35]), obtained by transforming single-particle momenta to the rest frame of a two- or three-body system using Galilean boosts, the $(ls)j$ two-body quantum numbers have obvious meaning, $(\lambda 1/2)I$ refer to the third nucleon (described by the momentum q), J is the total $3N$ angular momentum, and the rest are isospin quantum numbers. In the relativistic case this basis is replaced by the irreducible plane-wave states defined in Eq. (10).

The basis states [Eq. (10)] are used for the evaluation of the partial-wave representation of the permutation operator P with Wigner rotations of spin states for nucleons 2 and 3 included. In the relativistic case we adopt the following shorthand notation for the irreducible three-body states, which also includes isospin quantum numbers coupled in the same order:

$$\begin{aligned}|k,q,\alpha\rangle &:= \left|kq(ls)j\left(\lambda,\tfrac{1}{2}\right)IJ\left(t\tfrac{1}{2}\right)T\right\rangle \\ &= \left|(J,q)\mathbf{P},\mu;\lambda,I,j_{23},k_{23},l_{23},s_{23}\right\rangle\left|\left(t\tfrac{1}{2}\right)T\right\rangle.\end{aligned} \tag{56}$$

Equipped with that, projecting Eq. (50) onto the basis states $|k,q,\alpha\rangle$ one encounters as in the nonrelativistic notation [35]

$$\begin{aligned}{}_1\langle kq\alpha|P|k'q'\alpha'\rangle_1 &= {}_1\langle kq\alpha|k'q'\alpha'\rangle_2 + {}_1\langle kq\alpha|k'q'\alpha'\rangle_3 \\ &= 2\,{}_1\langle kq\alpha|k'q'\alpha'\rangle_2.\end{aligned} \tag{57}$$

This is evaluated by inserting the complete basis of states $|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2,\mathbf{p}_3,\mu_3\rangle$ and using Eq. (10). It can be expressed in a form that closely resembles the one appearing in the nonrelativistic regime [34,35]:

$$\begin{aligned}{}_1\langle kq\alpha|P|k'q'\alpha'\rangle_1 = \int_{-1}^{1} dx \frac{\delta(k-\pi_1)}{k^2}\frac{\delta(k'-\pi_2)}{k'^2} \\ \times \frac{1}{N_1(q,q',x)}\frac{1}{N_2(q,q',x)} G^{BB}_{\alpha\alpha'}(q,q',x),\end{aligned} \tag{58}$$

where all ingredients are given in Appendix B.

Because of the short-range nature of the NN interaction it can be considered negligible beyond a certain value $j_{\max}$ of the total angular momentum in the two-nucleon subsystem. Generally, with increasing energy $j_{\max}$ will also increase. For $j > j_{\max}$ we set the t matrix to zero, which yields a finite number of coupled channels for each total angular momentum J and total parity $\pi = (-)^{l+\lambda}$ of the $3N$ system. To achieve converged results at our energies we used all partial-wave

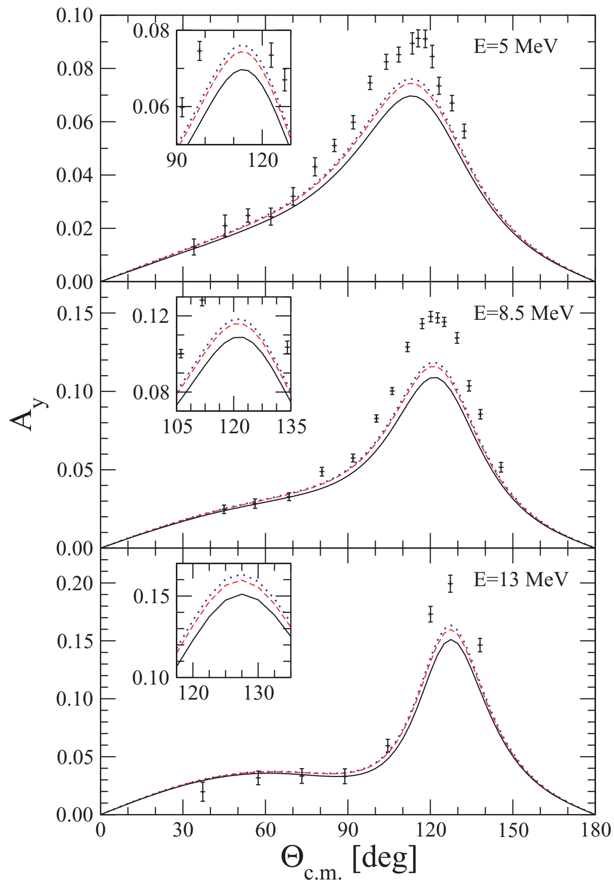

FIG. 1. (Color online) The nucleon analyzing power A_y for nd elastic scattering at various laboratory energies $E_n^{\rm lab}$ of the incoming neutron. The dotted line is the result of the nonrelativistic Faddeev calculation with the CD Bonn potential. The relativistic predictions without and with Wigner spin rotations are shown by the dashed and solid lines, respectively. The nd data at 5 and 8.5 MeV are from Ref. [49] and at 13 MeV are from Ref. [50].

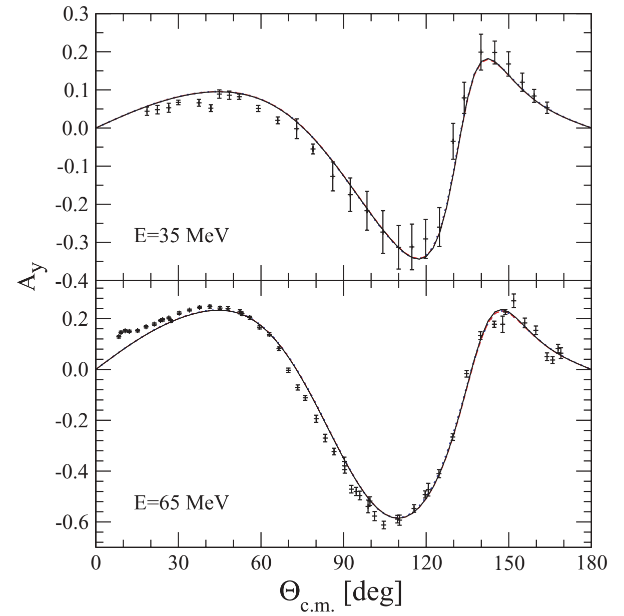

FIG. 2. (Color online) The nucleon analyzing power A_y for nd elastic scattering at $E_n^{\rm lab} = 35$ and 65 MeV. The dotted line is the result of the nonrelativistic Faddeev calculation with the CD Bonn potential. The relativistic predictions without and with Wigner spin rotations are shown by the dashed and solid lines, respectively. All theoretical predictions are practically overlapping. The pd data at 35 MeV are from Ref. [51] and at 65 MeV are from Ref. [52].

states with total angular momenta of the $2N$ subsystem up to $j_{\rm max} = 5$ and took into account all total angular momenta of the $3N$ system up to $J = 25/2$. This leads to a system of up to 143 coupled integral equations in two continuous variables for a given J and parity.

V. RESULTS AND DISCUSSION

The subject of the present study is to investigate the influence of relativity on the nd elastic scattering nucleon analyzing power A_y at low energies. We define the invariant observable A_y to be the value measured in the laboratory frame (target at rest).

To this aim we solved Faddeev equations at a number of the incoming neutron laboratory energies $E_n = 5, 8.5$, and 13 MeV. To check the energy dependence of the effect we added two additional energies $E_n = 35$ and 65 MeV. To see the importance of specific relativistic features we solved the equation in the relativistic case with and without Wigner rotations. This allowed us to see which effects, dynamical corrections (induced by the momentum dependence of the two-body force together with kinematical relativistic effects coming from the use of the Poincaré Jacobi variables) or Wigner rotations, dominate for A_y.

Figures 1 and 2 illustrate the results. When only dynamical effects are taken into account (i.e., Wigner rotations are neglected) then at low energies of the incoming neutron the relativistic and nonrelativistic predictions are practically the same with the exception of the angular region close to the maximum of A_y, where the relativistic prediction is ≈2% below the nonrelativistic one (see Fig. 1). This small effect disappears at higher energies (see Fig. 2). Including Wigner rotations significantly lowers the values of A_y in a large region of angles around the maximum. The changes in the maximum are up to ≈10%. Again, when the energy of the incoming neutron increases nonrelativistic and relativistic predictions are practically identical.

The large changes of A_y occur in a region of energies where this observable is extremely sensitive to changes in $^3P_0, ^3P_1$, and $^3P_2 - {}^3F_2$ NN force components [53]. At energies where this sensitivity dies out the relativistic effects for A_y also become negligible. This allows us to conclude that the large effects seen for A_y at low energies are due to amplification of changes of the ^3P_j contributions from relativity by a large sensitivity of A_y to P waves.

In a recent study [29] the changes of A_y from relativity by 10% at $E_n^{\rm lab} = 3$ MeV have been reported. Very probably the opposite sign of the effect found in that study can be attributed to the impulse approximation used when calculating A_y.

VI. SUMMARY AND OUTLOOK

We numerically solved the $3N$ Faddeev equation for *nd* scattering including relativistic kinematics, dynamical relativistic effects, and Wigner rotations at the neutron laboratory energies $E_n^{\rm lab} = 5, 8.5, 13, 35,$ and 65 MeV. As dynamical input we took the nonrelativistic NN potential CD Bonn and generated in the $2N$ c.m. system an exactly on-shell equivalent relativistic interaction v, by solving numerically the nonlinear quadratic equation relating matrix elements of the nonrelativistic and relativistic potentials. We checked that the approximate procedure using an analytical scale transformation of momenta applied in our previous studies provides practically the same results as the present exact approach. In addition a similar nonlinear equation [Eq. (32)] was used to generate the momentum-dependent two-body interaction embedded in the three-particle Hilbert space.

We found that at low energies the effects of Wigner rotations are most important for the analyzing power. They lower the maximum of A_y by up to ≈10%. The dynamical relativistic effects are of minor importance for A_y and provide small changes of A_y in a region close to its maximum. They lower A_y by only ≈2%. The relativistic effects disappear at higher energies.

Wigner rotations are negligible for the cross section and all other spin observables in elastic Nd scattering with the exception of four low-energy spin correlations $C_{z,yz}, C_{x,xy}, C_{y,xx-yy}$, and $C_{y,yy}$ and four low-energy spin transfer coefficients: those from deuteron to deuteron, $K_z^{y'z'}$ and $K_x^{x'y'}$, and those from deuteron to neutron, $K_{yz}^{z'}$ and $K_{xy}^{x'}$. Very probably this can be traced back as in the case of A_y to their sensitivity to ^3P waves.

These results shed new light on the low-energy analyzing power puzzle. It is known that the existing discrepancies between A_y data and theoretical predictions based only on NN potentials cannot be removed when the current three-nucleon force, mostly of 2π-exchange character [10,11], is included in the nuclear Hamiltonian. This indicated that additional $3N$ forces should be added to the 2π-exchange-type forces. Such forces provided by χPT in NNLO and NNNLO orders are expected to provide the solution for the A_y puzzle [9,54]. It seems that in view of the present result they must increase the maximum of A_y more than expected up to now. However, A_y is a very sensitive observable and our approach, using the irreducible $\{\mathbf{p}, j_{cz}\}$ basis, is only one of many possible basis choices that lead to Poincaré invariant dynamical theories that are two-body scattering equivalent, which may give different three-body predictions.

ACKNOWLEDGMENTS

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APPENDIX A: SPINORS

In our calculations SL(2, $\mathbb{C}$) matrices are used to represent Wigner rotations and Lorentz transformations. In this way spin algebra is reduced to working with 2×2 complex matrices, and there is no need for representations in terms of Euler angles. The relevant relations are given in this Appendix.

The coordinates of a four-vector p^μ can be labeled by the 2×2 Hermitian matrix P:

$$P := p^\mu \sigma_\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \tag{A1}$$

where σ_μ are the identity and the three Pauli matrices. The components of p^μ can be extracted from the matrix P by using

$$p^\mu = \tfrac{1}{2}\,\mathrm{Tr}\,(\sigma_\mu P) = \tfrac{1}{2}\,\mathrm{Tr}\,(P\sigma_\mu). \tag{A2}$$

Because

$$\det(P) = (p^0)^2 - (\vec{p}\,)^2 = -\eta_{\mu\nu} p^\mu p^\nu = m^2 \tag{A3}$$

and

$$P = P^\dagger \tag{A4}$$

for real p^μ, it follows that any linear transformation that preserves the Hermiticity and determinant of P is a real Lorentz transformation. It is easy to show that if A is a complex 2×2 matrix with $\det(A) = 1$ then the transformation

$$P \to P' = APA^\dagger \tag{A5}$$

has both of these properties:

$$\det(P') = \det(P) \quad \text{and} \quad P = P^\dagger \to P' = P'^\dagger. \tag{A6}$$

The most general 2×2 matrix with determinant 1 can be written as

$$A = \pm e^{\frac{i}{2}\boldsymbol{\sigma}\cdot\mathbf{z}}, \tag{A7}$$

where $\mathbf{z} = \boldsymbol{\theta} - i\boldsymbol{\rho}$. If $\boldsymbol{\rho} = \mathbf{0}$ then $A = U(\boldsymbol{\theta})$ is unitary and corresponds to an SU(2) rotation through an angle $|\boldsymbol{\theta}|$ about the $\hat{\boldsymbol{\theta}}$ axis. If $\boldsymbol{\theta} = \mathbf{0}$ then A is a positive Hermitian matrix that corresponds to a rotationless (canonical) Lorentz boost with rapidity $|\boldsymbol{\rho}|$ in the direction $\hat{\boldsymbol{\rho}}$.

The rotation $U(\boldsymbol{\theta})$ is given by

$$A \to U(\boldsymbol{\theta}) = \sigma_0 \cos(\theta/2) + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}\sin(\theta/2). \tag{A8}$$

The axis of rotation can be extracted by using

$$\hat{\boldsymbol{\theta}} = -i\frac{\mathrm{Tr}[\boldsymbol{\sigma} U(\boldsymbol{\theta})]}{|\mathrm{Tr}[\boldsymbol{\sigma} U(\boldsymbol{\theta})]|} \tag{A9}$$

and the angle of rotation can be extracted from

$$\theta = 2\tan^{-1}\left(\frac{\mathrm{Tr}[-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}U(\boldsymbol{\theta})]}{\mathrm{Tr}[U(\boldsymbol{\theta})]}\right). \tag{A10}$$

The rotationless boost that transforms a particle of mass m at rest to total momentum $\mathbf{p}$ can be labeled by the final four-velocity $\mathbf{q} := \mathbf{p}/m$:

$$A \to B(\mathbf{q}) = \sigma_0 \cosh(\rho/2) + \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \sinh(\rho/2), \quad \text{(A11)}$$

where $\boldsymbol{\rho} = \hat{\mathbf{p}}\rho$ is the rapidity of the Lorentz boost, which is related to $\mathbf{q}$ by

$$\cosh(\rho/2) = \sqrt{\frac{q^0+1}{2}} = \sqrt{\frac{p^0+m}{2m}} \quad \text{(A12)}$$

and

$$\sinh(\rho/2) = \frac{|\mathbf{q}|}{\sqrt{2(q^0+1)}} = \frac{|\mathbf{p}|}{\sqrt{2m(p^0+m)}}. \quad \text{(A13)}$$

$B(\mathbf{q})$ satisfies

$$B(\mathbf{q})\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} B^\dagger(\mathbf{q}) = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix} = P, \quad \text{(A14)}$$

where

$$p^0 = \omega(p) = \sqrt{m^2 + \mathbf{p}\cdot\mathbf{p}}. \quad \text{(A15)}$$

This rotationless boost is called the canonical boost. The inverse transformation is obtained by reversing the sign of $\mathbf{q}$ or $\mathbf{p}$.

The Wigner D functions are homogeneous polynomials of degree $2j$ in the coefficients of the SL(2, $\mathbb{C}$) matrices A_{ij}:

$$D^j_{\mu\nu}(A) := \sum_{\alpha=0}^{2j} \frac{[(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!]^{1/2}}{(j+\mu-\alpha)!\alpha!(\alpha-\mu+\nu)!(j-\nu-\alpha)!} \times A_{11}^{j+\mu-\alpha} A_{12}^{\alpha} A_{21}^{\alpha-\mu+\nu} A_{22}^{j-\nu-\alpha}. \quad \text{(A16)}$$

These are representations of both SL(2, $\mathbb{C}$) and SU(2). The $j = 1/2$ representation is just the matrix A.

APPENDIX B: PERMUTATION OPERATOR

Using Eq. (10) twice for the bra state ${}_1\langle k, q, \alpha|$ and the ket state $|k', q', \alpha'\rangle_2$ one gets for the matrix element of the permutation operator in our partial wave basis

$$\begin{aligned}
&{}_1\langle k, q, \alpha|P|k', q,' \alpha'\rangle_1 = 2\,{}_1\langle k, q, \alpha|k', q', \alpha'\rangle_2 \\
&= 2 \sum_{m_1 m_2 m_3} \sum_{\mu_2\mu_3\mu_s} \sum_{\mu_l\mu_\lambda\mu_I\mu} \sum_{\mu_2'\mu_3'\mu_{s'}} \sum_{\mu_{l'}\mu_{\lambda'}\mu_{I'}\mu'} \\
&\times \left(\lambda\mu_\lambda \tfrac{1}{2}, m_1, \middle| I, \mu_I\right)(j, \mu, I, \mu_I|J, M) \\
&\times \left(\tfrac{1}{2}, \mu_2, \tfrac{1}{2}, \mu_3 \middle| s, \mu_s\right)(l, \mu_l, s, \mu_s|j, \mu) \\
&\times \left(\lambda', \mu_{\lambda'}, \tfrac{1}{2}, m_2, \middle| I', \mu_{I'}\right)(j', \mu', I', \mu_{I'}|J, M) \\
&\times \left(\tfrac{1}{2}, \mu_2', \tfrac{1}{2}, \mu_3' \middle| s', \mu_{s'}\right)(l', \mu_{l'}, s', \mu_{s'}|j', \mu') \\
&\times \int d\hat{\mathbf{q}}\, d\hat{\mathbf{q}}' \frac{1}{N(\mathbf{q}', -\mathbf{q}-\mathbf{q}')} \frac{1}{N(-\mathbf{q}-\mathbf{q}', \mathbf{q})} \\
&\times \frac{\delta(k - |\mathbf{k}(\mathbf{q}', -\mathbf{q}-\mathbf{q}')|)}{\mathbf{k}^2} \frac{\delta(k' - |\mathbf{k}(-\mathbf{q}-\mathbf{q}', \mathbf{q})|)}{\mathbf{k}'^2} \\
&\times Y^*_{\lambda\mu_\lambda}(\hat{\mathbf{q}}) Y^*_{l\mu_l}(\hat{\mathbf{k}}(\mathbf{q}', -\mathbf{q}-\mathbf{q}')) Y_{\lambda'\mu_{\lambda'}}(\hat{\mathbf{q}}') Y_{l'\mu_{l'}}(\hat{\mathbf{k}}(-\mathbf{q}-\mathbf{q}', \mathbf{q})) \\
&\times D^{\frac{1}{2}*}_{m_2\mu_2}[R_w(B(-\mathbf{q}/2\omega_m(\mathbf{k})), (\mathbf{k}(\mathbf{q}', -\mathbf{q}-\mathbf{q}'), \omega_m(\mathbf{k}))] \\
&\times D^{\frac{1}{2}*}_{m_3\mu_3}[R_w(B(-\mathbf{q}/2\omega_m(\mathbf{k})), (-\mathbf{k}(\mathbf{q}', -\mathbf{q}-\mathbf{q}'), \omega_m(\mathbf{k}))] \\
&\times D^{\frac{1}{2}}_{m_3\mu_2'}[R_w(B(-\mathbf{q}'/2\omega_m(\mathbf{k})), (\mathbf{k}(-\mathbf{q}-\mathbf{q}', \mathbf{q}), \omega_m(\mathbf{k}))] \\
&\times D^{\frac{1}{2}}_{m_1\mu_3'}[R_w(B(-\mathbf{q}'/2\omega_m(\mathbf{k})), (-\mathbf{k}(-\mathbf{q}-\mathbf{q}', \mathbf{q}), \omega_m(\mathbf{k}))] \\
&\times {}_1\left\langle \left(t\tfrac{1}{2}\right) T \middle| \left(t'\tfrac{1}{2}\right) T \right\rangle_2,
\end{aligned} \quad \text{(B1)}$$

with

$$\begin{aligned}
\mathbf{k}(\mathbf{q}', -\mathbf{q}-\mathbf{q}') &\equiv \mathbf{q}' + \tfrac{1}{2}\mathbf{q}[1 + y_1(q, q', x)], \\
\mathbf{k}(-\mathbf{q}-\mathbf{q}', \mathbf{q}) &\equiv -\mathbf{q} - \tfrac{1}{2}\mathbf{q}'[1 + y_2(q, q', x)].
\end{aligned} \quad \text{(B2)}$$

In Eq. (B2)

$$y_1(q, q', x) = \frac{E_{\mathbf{q}'} - E_{\mathbf{q}+\mathbf{q}'}}{E_{\mathbf{q}'} + E_{\mathbf{q}+\mathbf{q}'} + \sqrt{(E_{\mathbf{q}'} + E_{\mathbf{q}+\mathbf{q}'})^2 - \mathbf{q}^2}}, \quad \text{(B3)}$$

with $x = \hat{q}\cdot\hat{q}'$, $y_2(q, q', x) = y_1(q', q, x)$, and $E_{\mathbf{q}} \equiv \omega_m(\mathbf{q})$.

Proceeding as in Refs. [41–43] one gets the following expression for the matrix element of the permutation operator P:

$$\begin{aligned}
&{}_1\langle k, q\alpha|P|k', q'\alpha'\rangle_1 \\
&= \int_{-1}^{1} dx \frac{\delta(k - \pi_1)}{k^2} \frac{\delta(k' - \pi_2)}{k'^2} \\
&\times \frac{1}{N_1(q, q', x)} \frac{1}{N_2(q, q', x)} G^{BB}_{\alpha\alpha'}(q, q', x),
\end{aligned} \quad \text{(B4)}$$

with

$$\begin{aligned}
\pi_1 &= \sqrt{q'^2 + \tfrac{1}{4}q^2(1+y_1)^2 + qq'x(1+y_1)}, \\
\pi_2 &= \sqrt{q^2 + \tfrac{1}{4}q'^2(1+y_2)^2 + qq'x(1+y_2)},
\end{aligned} \quad \text{(B5)}$$

$$\begin{aligned}
N_1(q, q', x) &\equiv N(\mathbf{q}', -\mathbf{q}-\mathbf{q}'), \\
N_2(q, q', x) &\equiv N(-\mathbf{q}-\mathbf{q}', \mathbf{q}),
\end{aligned}$$

and

$$\begin{aligned}
&G^{BB}_{\alpha\alpha'}(qq'x) \\
&= \frac{4\pi^{3/2}}{2J+1}(-1)^{t'}\delta_{TT'}\delta_{M_T M_{T'}}\sqrt{\hat{t}\hat{t}'}\begin{Bmatrix} 1/2 & 1/2 & t \\ 1/2 & T & t' \end{Bmatrix} \\
&\times \sqrt{\hat{\lambda}} \sum_{\mu_2\mu_3}\left(\tfrac{1}{2}\mu_2\ \tfrac{1}{2}\mu_3 \middle| s\mu_2+\mu_3\right) \\
&\times \sum_{\mu_2'}\left(\sum_{m_3} D^{1/2\,*}_{m_3\mu_3}[R_w(B(-\mathbf{q}/2\omega_m(\mathbf{k})), (-\mathbf{k}, \omega_m(\mathbf{k}))]\right. \\
&\times \left. D^{1/2}_{m_3\mu_2'}[R_w(B(\mathbf{q}'/2\omega_m(\mathbf{k}')), (\mathbf{k}', \omega_m(\mathbf{k}'))]\right) \\
&\times \sum_{\mu_3'}\left(\tfrac{1}{2}, \mu_2', \tfrac{1}{2}, \mu_3' \middle| s', \mu_2'+\mu_3'\right) \\
&\times \sum_{m_1}\left(\lambda 0 \tfrac{1}{2}, m_1, \middle| I, m_1\right) D^{1/2}_{m_1\mu_3'} \\
&\times [R_w(B(-\mathbf{q}'/2\omega_m(\mathbf{k}')), (-\mathbf{k}', \omega_m(\mathbf{k}'))]
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\mu} (l, \mu - \mu_2 - \mu_3, s, \mu_2 + \mu_3 | j, \mu) \\
&\times (-)^{\mu-\mu_2-\mu_3} Y_{l\,-(\mu-\mu_2-\mu_3)}(\hat{\mathbf{p}}) \\
&\times (j, \mu, I, m_1 | J, \mu + m_1) \\
&\times \sum_{\mu'} (l', \mu' - \mu'_2 - \mu'_3, s', \mu'_2 + \mu'_3 | j', \mu') \\
&\times Y_{l'\mu'-\mu'_2-\mu'_3}(\hat{\mathbf{p}}') \\
&\times (j', \mu', I', \mu + m_1 - \mu' | J, \mu + m_1) \\
&\times \sum_{m_2} \bigg(\lambda', \mu + m_1 - \mu' \\
&- m_2, \frac{1}{2}, m_2 \bigg| I', \mu + m_1 - \mu' \bigg) \\
&\times D^{1/2\;*}_{m_2\mu_2}[R_w(B(-\mathbf{q}/2\omega_m(\mathbf{k})), \\
&\times (\mathbf{k}, \omega_m(\mathbf{k}))] Y_{\lambda'\mu+m_1-\mu'-m_2}(\hat{\mathbf{q}}'). \qquad \text{(B6)}
\end{aligned}
$$

We use standard notation $\hat{l} \equiv 2l + 1$. It is assumed that the z-axis is along $\mathbf{q}$ and the momentum $\mathbf{q}'$ lies in the $x - z$ plane, which leads to the following components of the $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{k}$, and $\mathbf{k}'$ vectors:

$$
\begin{aligned}
\mathbf{q} &= [0, 0, q], \\
\mathbf{q}' &= [q'\sqrt{1-x^2}, 0, q'x], \\
\mathbf{k} &= [q'\sqrt{1-x^2}, 0, q'x + \tfrac{1}{2}q(1 + y_1(q, q', x))], \\
\mathbf{k}' &= \big[-\tfrac{1}{2}q'(1 + y_2(q, q', x))\sqrt{1-x^2}, 0, \\
&\quad - q - \tfrac{1}{2}q'(1 + y_2(q, q', x))x\big]. \qquad \text{(B7)}
\end{aligned}
$$

Though the direct evaluation of Euler angles as arguments of the Wigner D functions could be used as in Ref. [20], here we use the SL(2, $\mathbb{C}$) representations of Lorentz transfomations discussed in Appendix A. This leads to

$$
\begin{aligned}
&D^{1/2}[R_w(B(-\mathbf{q}/M_0), (\mathbf{k}, m))] \\
&\quad = B(\mathbf{p}_2/m) B(-\mathbf{q}/M_0) B(\mathbf{k}/m) \\
&\quad = \sqrt{\frac{E_0 + M_0}{2M_0}} \sqrt{\frac{\omega(k) + m}{\omega(p_2) + m}} \\
&\qquad - \frac{\mathbf{k} \cdot \mathbf{q}}{\sqrt{2M_0(E_0 + M_0)(\omega(k) + m)(\omega(p_2) + m)}} \\
&\qquad + i\mathbf{k} \times \mathbf{q} \cdot \boldsymbol{\sigma} \frac{1}{\sqrt{2M_0(E_0 + M_0)(\omega(k) + m)(\omega(p_2) + m)}}, \qquad \text{(B8)}
\end{aligned}
$$

where

$$E_0 = \sqrt{M_0^2 + q^2}, \qquad \text{(B9)}$$

$$M_0 = 2\omega(k), \qquad \text{(B10)}$$

$$\mathbf{p}_2(\mathbf{k}, -\mathbf{q}) = \mathbf{k} - \mathbf{q} \left(\frac{\omega(k)}{M_0} - \mathbf{k} \cdot \mathbf{q} \frac{1}{M_0(E_0 + M_0)} \right). \qquad \text{(B11)}$$

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