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Research Article

Some Similarity between Contractions and Kannan Mappings

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Contractions are always continuous and Kannan mappings are not necessarily continuous. This is a very big difference between both mappings. However, we know that relaxed both mappings are quite similar. In this paper, we discuss both mappings from a new point of view.

1. Introduction

Let (X, d) be a metric space and let T be a mapping on X. Then T is called a *contraction* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \le r d(x, y) \tag{1.1}$$

for all $x, y \in X$. T is called *Kannan* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty) \tag{1.2}$$

for all $x, y \in X$. We know that if X is complete, then every contraction and every Kannan mapping have a unique fixed point, see [1, 2]. We know that both conditions are independent, that is, there exist a contraction, which is not Kannan, and a Kannan mapping, which is not a contraction. Thus we cannot compare both conditions directly. So we compare both indirectly.

Fact 1

Banach fixed-point theorem, which is often called the Banach contraction principle, is very important because it is a very forceful tool in nonlinear analysis. We think that Kannan fixed-point theorem is also very important because Subrahmanyam [3] proved that Kannan theorem characterizes the metric completeness of underlying spaces, that is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point. On the other hand, Connell [4] gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. Thus the Banach theorem cannot characterize the metric completeness of X. Therefore, we consider that the notion of contractions is stronger from this point of view.

Fact 2

Using the notion of τ-distances, Suzuki [5] considered some weaker contractions and Kannan mappings and proved the following.

(i) If T is a contraction with respect to a τ-distance, then T is Kannan with respect to another τ-distance.

(ii) If T is Kannan with respect to a τ-distance, then T is a contraction with respect to another τ-distance.

That is, both conditions are completely the same.

Recently, Suzuki [6] proved the following theorem, see also [7].

Theorem 1.1 (see [6]). *Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases}1 & \text{if } 0\le r\le \frac{1}{2}(\sqrt{5}-1),\\ \dfrac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1)\le r\le \dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}}\le r<1.\end{cases} \tag{1.3}$$

Then for a metric space (X,d), the following are equivalent:

(i) X *is complete,*

(ii) *every mapping T on X, satisfying the following, has a fixed point: there exists $r\in[0,1)$ such that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$ for all $x,y\in X$.*

Remark 1.2. $\theta(r)$ is the best constant for every r.

The purpose of this paper is to prove a Kannan version of Theorem 1.1. Then we compare the theorem (Theorem 2.2) with Theorem 1.1 and attempt to judge which is stronger from our new point of view.

2. Kannan mappings

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In this section, we prove our main result. We begin with the following lemma.

Lemma 2.1. *Let (X, d) be a metric space and let T be a mapping on X. Let $x \in X$ satisfy $d(Tx, T^2x) \leq rd(x, Tx)$ for some $r \in [0, 1)$. Then for $y \in X$, either*

$$\frac{1}{1+r}d(x,Tx) \leq d(x,y) \quad \text{or} \quad \frac{1}{1+r}d(Tx,T^2x) \leq d(Tx,y) \tag{2.1}$$

holds.

Proof. We assume

$$\frac{1}{1+r}d(x,Tx) > d(x,y), \qquad \frac{1}{1+r}d(Tx,T^2x) > d(Tx,y). \tag{2.2}$$

Then we have

$$\begin{aligned} d(x,Tx) &\leq d(x,y) + d(y,Tx) \\ &< \frac{1}{1+r}\left(d(x,Tx) + d(Tx,T^2x)\right) \\ &\leq \frac{1}{1+r}\left(d(x,Tx) + rd(x,Tx)\right) = d(x,Tx). \end{aligned} \tag{2.3}$$

This is a contradiction. □

The following theorem is a Kannan version of Theorem 1.1.

Theorem 2.2. *Define a nonincreasing function φ from $[0, 1)$ into $(1/2, 1]$ by*

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}} \leq r < 1. \end{cases} \tag{2.4}$$

Let (X, d) be a complete metric space and let T be a mapping on X. Let $\alpha \in [0, 1/2)$ and put $r := \alpha/(1-\alpha) \in [0, 1)$. Assume that

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies } d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty) \tag{2.5}$$

for all $x, y \in X$, then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Proof. Since $\varphi(r) \leq 1$, $\varphi(r)d(x, Tx) \leq d(x, Tx)$ holds. From the assumption, we have

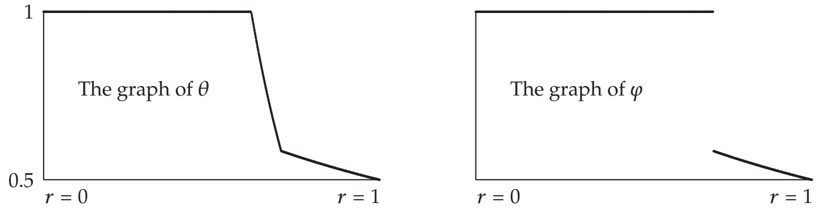
$$d(Tx,T^2x) \leq \alpha d(x,Tx) + \alpha d(Tx,T^2x), \tag{2.6}$$

and hence

$$d(Tx,T^2x) \leq rd(x,Tx) \tag{2.7}$$

for $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = T^n u$ for all $n \in \mathbb{N}$. From (2.7), we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} r^n d(u_0, u_1) < \infty. \tag{2.8}$$



(a) (b)

Figure 1

So $\{u_n\}$ is a Cauchy sequence in X and by the completeness of X, there exists a point z such that $u_n \to z$.

We next show

$$d(z, Tx) \le \alpha d(x, Tx), \quad \forall x \in X \text{ with } x \ne z. \tag{2.9}$$

Since $u_n \to z$, there exists $n_0 \in \mathbb{N}$ such that $d(u_n, z) \le (1/3)d(x, z)$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Then we have

$$\begin{aligned} \varphi(r)d(u_n, Tu_n) &\le d(u_n, Tu_n) = d(u_n, u_{n+1}) \\ &\le d(u_n, z) + d(u_{n+1}, z) \\ &\le \frac{2}{3}d(x, z) = d(x, z) - \frac{1}{3}d(x, z) \\ &\le d(x, z) - d(u_n, z) \le d(u_n, x), \end{aligned} \tag{2.10}$$

and hence

$$d(Tu_n, Tx) \le \alpha d(u_n, Tu_n) + \alpha d(x, Tx) \quad \text{for } n \in \mathbb{N} \text{ with } n \ge n_0. \tag{2.11}$$

Therefore, we obtain

$$\begin{aligned} d(z, Tx) &= \lim_{n\to\infty} d(u_{n+1}, Tx) = \lim_{n\to\infty} d(Tu_n, Tx) \\ &\le \lim_{n\to\infty} \left(\alpha d(u_n, Tu_n) + \alpha d(x, Tx)\right) \\ &= \alpha d(x, Tx) \end{aligned} \tag{2.12}$$

for $x \in X$ with $x \ne z$.

Let us prove that z is a fixed point of T. In the case where $0 \le r < 1/\sqrt{2}$, arguing by contradiction, we assume that $Tz \ne z$. Then we have, from (2.9),

$$d(z, T^2 z) \le \alpha d(Tz, T^2 z) \le \alpha r d(z, Tz), \tag{2.13}$$

and hence

$$\begin{aligned} d(z, Tz) &\le d(z, T^2 z) + d(Tz, T^2 z) \\ &\le \alpha r d(z, Tz) + r d(z, Tz) = \frac{r + 2r^2}{1 + r} d(z, Tz) \\ &< \frac{r + 1}{1 + r} d(z, Tz) = d(z, Tz). \end{aligned} \tag{2.14}$$

This is a contradiction. Therefore, we obtain $Tz = z$. In the case where $1/\sqrt{2} \leq r < 1$, from Lemma 2.1, either

$$\varphi(r)d(u_{2n}, u_{2n+1}) \leq d(u_{2n}, z) \quad \text{or} \quad \varphi(r)d(u_{2n+1}, u_{2n+2}) \leq d(u_{2n+1}, z) \tag{2.15}$$

holds for $n \in \mathbb{N}$. Thus there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\varphi(r)d(u_{n_j}, u_{n_j+1}) \leq d(u_{n_j}, z) \tag{2.16}$$

for $j \in \mathbb{N}$. From the assumption, we have

$$d(z, Tz) = \lim_{j\to\infty} d(u_{n_j+1}, Tz) \leq \lim_{j\to\infty} (\alpha d(u_{n_j}, u_{n_j+1}) + \alpha d(z, Tz)) = \alpha d(z, Tz). \tag{2.17}$$

Since $\alpha < 1/2$, we have $Tz = z$. Therefore, we have shown $Tz = z$ in both cases.

From (2.9), we obtain that the fixed point z is unique. □

Remark 2.3. Since $\theta(r) \leq \varphi(r)$ for every r, we can consider that Kannan is stronger from our new point of view. Though θ and φ are different, we remark that the graphs of θ and φ are quite similar.

The following theorem shows that $\varphi(r)$ is the best constant for every r.

Theorem 2.4. *Define a function φ as in Theorem 2.2. For every $\alpha \in [0, 1/2)$, putting $r = \alpha/(1-\alpha)$, there exist a complete metric space (X, d) and a mapping T on X such that T has no fixed points and*

$$\varphi(r)d(x, Tx) < d(x, y) \quad \text{implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \tag{2.18}$$

for all $x, y \in X$.

Proof. In the case where $0 \leq r < 1/\sqrt{2}$, define a complete subset X of the Euclidean space $\mathbb{R}$ by $X = \{-1, 1\}$. We also define a mapping T on X by $Tx = -x$ for $x \in X$. Then T dose not have a fixed point and

$$\varphi(r)d(x, Tx) = 2 \geq d(x, y) \tag{2.19}$$

for all $x, y \in X$. In the case where $1/\sqrt{2} \leq r < 1$, define a complete subset X of the Euclidean space $\mathbb{R}$ by

$$X = \{0, 1\} \cup \{x_n : n \in \mathbb{N} \cup \{0\}\}, \tag{2.20}$$

where $x_n = (1-r)(-r)^n$ for all $n \in \mathbb{N} \cup \{0\}$. Define a mapping T on X by $T0 = 1$, $T1 = 1-r$, and $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then the following are obvious:

(i) $d(T0, T1) = r = \alpha d(0, T0) + \alpha d(1, T1)$,

(ii) $\varphi(r)d(0, T0) \geq \varphi(r)d(x_n, Tx_n) = d(0, x_n)$ for $n \in \mathbb{N} \cup \{0\}$.

Also, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(0, Tx_m) + d(0, Tx_n) = \alpha d(x_m, Tx_m) + \alpha d(x_n, Tx_n), \\ d(T1, Tx_n) - (\alpha d(1, T1) + \alpha d(x_n, Tx_n)) &\leq d(0, T1) + d(0, Tx_n) - (\alpha d(1, T1) + \alpha d(x_n, Tx_n)) \\ &= d(0, T1) - \alpha d(1, T1) = \frac{1-2r^2}{1+r} \leq 0 \end{aligned} \tag{2.21}$$

for $m, n \in \mathbb{N} \cup \{0\}$. □

3. Generalized Kannan mappings

It is a very natural question of whether or not another fixed-point theorem with θ exists. In this section, we give a positive answer to this problem.

Theorem 3.1. *Define a nonincreasing function θ as in Theorem 1.1. Let (X,d) be a complete metric space and let T be a mapping on X . Suppose that there exists $r \in [0,1)$ such that*

$$\theta(r)d(x,Tx) \le d(x,y) \quad \textit{implies } d(Tx,Ty) \le r \max\{d(x,Tx),d(y,Ty)\} \tag{3.1}$$

for all $x,y \in X$. Then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Proof. Since $\theta(r)d(x,Tx) \le d(x,Tx)$, we have, from the assumption,

$$d(Tx,T^2x) \le r \max\{d(x,Tx),d(Tx,T^2x)\} \tag{3.2}$$

and hence

$$d(Tx,T^2x) \le rd(x,Tx) \tag{3.3}$$

for $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = T^n u$ for all $n \in \mathbb{N}$. As in the proof of Theorem 2.2, we can prove that $\{u_n\}$ converges to some $z \in X$.

We next show

$$d(z,Tx) \le rd(x,Tx) \quad \text{for all } x \in X \text{ with } x \ne z. \tag{3.4}$$

Since $u_n \to z$, we have $\theta(r)d(u_n,Tu_n) \le d(u_n,x)$ for sufficiently large $n \in \mathbb{N}$. Hence we obtain, from the assumption,

$$\begin{aligned} d(z,Tx) &= \lim_{n\to\infty} d(u_{n+1},Tx) = \lim_{n\to\infty} d(Tu_n,Tx) \\ &\le \lim_{n\to\infty} r\max\{d(u_n,Tu_n),d(x,Tx)\} = rd(x,Tx) \end{aligned} \tag{3.5}$$

for $x \in X$ with $x \ne z$.

Let us prove that z is a fixed point of T. In the case where $0 \le r < 1/\sqrt{2}$, we note

$$\theta(r) \le \frac{1-r}{r^2}. \tag{3.6}$$

We will show, by induction,

$$d(T^n z,Tz) \le rd(z,Tz) \tag{3.7}$$

for $n \in \mathbb{N}$ with $n \ge 2$. When $n = 2$, (3.7) becomes (3.3), thus (3.7) holds. We assume $d(T^n z,Tz) \le rd(z,Tz)$ for some $n \in \mathbb{N}$ with $n \ge 2$. Since

$$d(z,Tz) \le d(z,T^n z) + d(T^n z,Tz) \le d(z,T^n z) + rd(z,Tz), \tag{3.8}$$

we have $d(z, Tz) \le (1/(1-r))d(z, T^n z)$, and hence

$$\theta(r)d(T^n z, T^{n+1} z) \le \frac{1-r}{r^2} d(T^n z, T^{n+1} z) \le \frac{1-r}{r^n} d(T^n z, T^{n+1} z) \\ \le (1-r)d(z, Tz) \le d(z, T^n z). \tag{3.9}$$

Therefore, by the assumption, we have

$$d(T^{n+1} z, Tz) \le r \max\{d(T^n z, T^{n+1} z), d(z, Tz)\} = rd(z, Tz). \tag{3.10}$$

By induction, (3.7) holds for $n \in \mathbb{N}$ with $n \ge 2$. Arguing, by contradiction, we assume $Tz \ne z$. Then from (3.7), $T^n z \ne z$ holds for all $n \in \mathbb{N}$. Then by (3.4), we have

$$d(T^{n+1} z, z) \le rd(T^n z, T^{n+1} z) \le r^{n+1} d(z, Tz). \tag{3.11}$$

This implies $T^n z \to z$, which contradicts (3.7). Therefore, we obtain $Tz = z$. In the case where $1/\sqrt{2} \le r < 1$, as in the proof of Theorem 2.2, we can show that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\varphi(r)d(u_{n_j}, u_{n_j+1}) \le d(u_{n_j}, z)$ for $j \in \mathbb{N}$. From the assumption, we have

$$d(z, Tz) = \lim_{j\to\infty} d(u_{n_j+1}, Tz) \le \lim_{j\to\infty} r \max\{d(u_{n_j}, u_{n_j+1}), d(z, Tz)\} = rd(z, Tz). \tag{3.12}$$

Since $r < 1$, the above inequality implies that $Tz = z$. Therefore, we have shown that $Tz = z$ in both cases.

From (3.4), we obtain that the fixed point z is unique. □

Remark 3.2. When the second author was proving Theorem 1.1, he did not feel that $\theta(r)$ was natural. However, since the above proof is easier to understand how $\theta(r)$ works, the authors can faintly feel that $\theta(r)$ is natural.

The following theorem shows that $\theta(r)$ is the best constant for every r.

Theorem 3.3. *Define a function θ as in Theorem 1.1. Then for any $r \in [0, 1)$, there exist a complete metric space (X, d) and a mapping T on X such that T has no fixed points and*

$$\theta(r)d(x, Tx) < d(x, y) \quad \text{implies } d(Tx, Ty) \le r \max\{d(x, Tx), d(y, Ty)\} \tag{3.13}$$

for all $x, y \in X$.

Proof. We have already shown the conclusion in the case where $0 \le r \le (1/2)(\sqrt{5}-1)$ or $1/\sqrt{2} \le r < 1$ because $\varphi(r) = \theta(r)$ holds. So let us consider the case where $(1/2)(\sqrt{5}-1) < r < 1/\sqrt{2}$. Define a complete subset X of the Euclidean space $\mathbb{R}$ by $X = \{x_n : n \in \mathbb{N}\}$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 1 - r$, and $x_n = (1 - r - r^2)(-r)^{n-3}$ for $n \ge 3$. Define a mapping T on X by $Tx_n = x_{n+1}$ for $n \in \mathbb{N}$. Then the following are obvious:

(i) $d(Tx_0, Tx_1) = r = rd(x_0, Tx_0) = r \max\{d(x_0, Tx_0), d(x_1, Tx_1)\}$,

(ii) $\theta(r)d(x_0, Tx_0) \ge \theta(r)d(x_2, Tx_2) = 1 - r = d(x_0, x_2)$,

(iii) $\theta(r)d(x_0, Tx_0) \ge \theta(r)d(x_n, Tx_n) = ((1 - r^2)/r^2)d(x_0, x_n) \ge d(x_0, x_n)$ for $n \ge 3$,

(iv) $d(Tx_1, Tx_2) = r^2 = rd(x_1, Tx_1)$.

Since

$$x_3 < x_5 < x_7 < \cdots < x_0 < \cdots < x_8 < x_6 < x_4 < x_2 < x_1, \tag{3.14}$$

we have the following:

(i) $d(Tx_1, Tx_n) < d(x_2, x_3) = r^2 = rd(x_1, Tx_1)$ for $n \geq 3$,

(ii) $d(Tx_2, Tx_n) - rd(x_2, Tx_2) \leq d(x_3, x_4) - r^3 = 2r^2 - 1 \leq 0$ for $n \geq 3$,

(iii) $d(Tx_m, Tx_n) \leq d(Tx_m, Tx_{m+1}) = rd(x_m, Tx_m)$ for $3 \leq m < n$.

This completes the proof. □

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