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ON THE CALCULATION OF THE JAMES CONSTANT OF LORENTZ SEQUENCE SPACES

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Abstract. In [M. Kato and L. Maligranda, On James and Jordan-von Neumann constants of Lorentz sequence spaces, J. Math. Anal. Appl., 258(2001), 457–465], the James constant of the 2-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ is computed in the case where $2 \le q < \infty$. It is an open problem to compute it in the case where $1 \le q < 2$. In this paper, we completely determine the James constant of $d^{(2)}(\omega, q)$ in the case where $1 \le q < 2$.

1. Introduction and preliminaries

The notion of the James constant (or the non-square constant in the sense of James) of Banach spaces was introduced by Gao and Lau [4], and recently it has been studied by several authors (cf. [3, 4, 5, 6], etc.). The *James constant* $J(X)$ of a Banach space X is defined by

$$J(X) = \sup\Big\{ \min\big(||x+y||, ||x-y||\big) : x, y \in S_X \Big\},$$

where $S_X = \{x \in X : ||x|| = 1\}$. From [4], we have $\sqrt{2} \le J(X) \le 2$ for any Banach space X, and $J(X) = \sqrt{2}$ if X is a Hilbert space. Clearly, we have that $J(X) < 2$ if and only if X is uniformly non-square, that is, there exists a $\delta > 0$ such that $\|(x-y)/2\| \ge 1-\delta$, $x, y \in S_X$ imply $\|(x+y)/2\| \le 1-\delta$. They also calculated the James constant of L_p spaces, as follows: If $1 \le p \le \infty$ and $\dim L_p \ge 2$, then

$$J(L_p) = \max\{2^{1/p}, 2^{1/p'}\},$$

where $1/p + 1/p' = 1$. For some other results concerning the modulus of convexity, the James constant and the normal structure, we refer the reader to [3, 5].

For $0 < \omega < 1$ and $1 \le q < \infty$, the *2-dimensional Lorentz sequence space* $d^{(2)}(\omega, q)$ is $\mathbb{R}^2$ with the norm

$$\|x\|_{\omega,q} = (x_1^{*q} + \omega x_2^{*q})^{1/q}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where (x_1^*, x_2^*) is the nonincreasing rearrangement of $(|x_1|, |x_2|)$; that is, $x_1^* = \max\{|x_1|, |x_2|\}$ and $x_2^* = \min\{|x_1|, |x_2|\}$. Kato and Maligranda [6] considered the James constant of $d^{(2)}(\omega, q)$ and calculated it in the case where $q \ge 2$. That is, if $q \ge 2$, then

$$J\big(d^{(2)}(\omega, q)\big) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

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As in [6, Problem 1], it is an open problem to calculate it in the case where $1 \le q < 2$. In [7], the first and the second authors proved that, if $1 \le q < 2$ and $0 < \omega \le -1 + \sqrt{2}$, then

$$J\big(d^{(2)}(\omega, q)\big) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

Further, in [9], the third author, Yamano and Kato attempted to cover a part of the unknown case.

In this paper we completely determine the James constant of $d^{(2)}(\omega, q)$ in the case where $1 \le q < 2$.

To do it, we need some preliminaries. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $x, y \in \mathbb{R}$, and *normalized* if $\|(1, 0)\| = \|(0, 1)\| = 1$. Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the family of all continuous convex functions on $[0, 1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. As in [2, 8], AN_2 and Ψ_2 are in a one-one correspondence under the equation $\psi(t) = \|(1 - t, t)\|$ $(0 \le t \le 1)$. Let $\|\cdot\|_\psi$ be the absolute norm which corresponds to ψ, that is, for all $\psi \in \Psi_2$, let

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\dfrac{|y|}{|x| + |y|}\right) & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We recall that an absolute normalized norm $\|\cdot\|$ on $\mathbb{R}^2$ is symmetric in the sense that $\|(x, y)\| = \|(y, x)\|$ for all $x, y \in \mathbb{R}$ if and only if the corresponding function ψ is symmetric with respect to $t = 1/2$ (see [8]).

For a norm $\|\cdot\|$ on $\mathbb{R}^2$, we write $J\big(\|\cdot\|\big)$ for $J\big((\mathbb{R}^2, \|\cdot\|)\big)$. In [7], we characterized the James constant of $(\mathbb{R}^2, \|\cdot\|_\psi)$ in terms of ψ. That is,

Theorem 1 (Mitani and Saito [7]). *Let $\psi \in \Psi_2$. If ψ is symmetric with respect to $t = 1/2$, then*

$$J\big(\|\cdot\|_\psi\big) = \max_{0 \le t \le 1/2} \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right).$$

Note here that the norm $\|\cdot\|_{\omega,q}$ of $d^{(2)}(\omega, q)$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi_{\omega,q}(t) = \begin{cases} \big((1-t)^q + \omega t^q\big)^{1/q} & \text{if } 0 \le t \le 1/2, \\ \big(t^q + \omega(1-t)^q\big)^{1/q} & \text{if } 1/2 \le t \le 1. \end{cases}$$

Therefore we can give the James constant of $d^{(2)}(\omega, q)$ as follows:

Proposition 1. *For $0 < \omega < 1$ and $1 \le q < \infty$,*

$$J\big(d^{(2)}(\omega, q)\big)\big(= J\big(\|\cdot\|_{\psi_{\omega,q}}\big)\big) = \max_{0 \le t \le 1/2} \frac{2-2t}{\psi_{\omega,q}(t)}\psi_{\omega,q}\left(\frac{1}{2-2t}\right)$$

holds.

Our aim in this paper is the following:

Theorem. Let $1 \le q < 2$. Then we have

(i) If $0 < \omega \le (\sqrt{2}-1)^{2-q}$, then

$$J\big(d^{(2)}(\omega, q)\big) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

(ii) If $(\sqrt{2}-1)^{2-q} < \omega < 1$, then there exists a unique solution s_0 of the equation

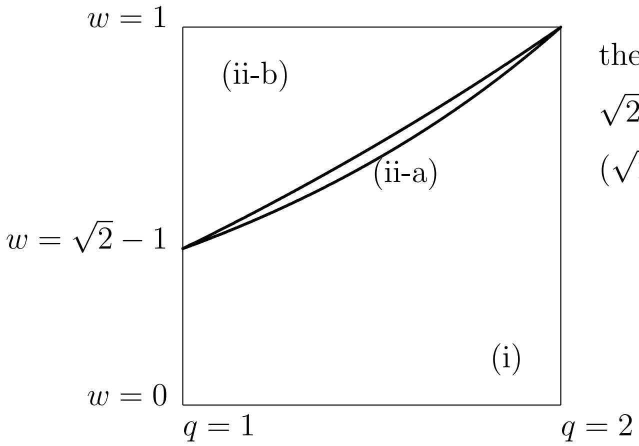
$$(1+s_0)^{q-1}(1-\omega s_0^{q-1}) = \omega(1-s_0)^{q-1}(1+\omega s_0^{q-1}), \quad 0 < s_0 < \omega^{1/(2-q)}.$$

(ii-a) If $(\sqrt{2}-1)^{2-q} < \omega \le \sqrt{2}^q - 1$, then

$$J\big(d^{(2)}(\omega, q)\big) = \max\left\{\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}, \; 2\left(\frac{1}{1+\omega}\right)^{1/q}\right\}.$$

(ii-b) If $\sqrt{2}^q - 1 < \omega < 1$, then

$$J\big(d^{(2)}(\omega, q)\big) = \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}.$$



the graphs of $\sqrt{2}^q - 1$ (above) and $(\sqrt{2}-1)^{2-q}$ (below)

2. Proof of Theorem

We define a function f from $[0, 1/2]$ into $\mathbb{R}$ by

$$f(t) \;=\; \tfrac{2-2t}{\psi_{\omega,q}(t)}\psi_{\omega,q}\left(\tfrac{1}{2-2t}\right) = \left(\tfrac{1+\omega(1-2t)^q}{(1-t)^q+\omega t^q}\right)^{1/q}$$

for t with $0 \le t \le 1/2$. We also put

$$g(s) = f\left(\frac{s}{1+s}\right) = \frac{\big((1+s)^q + \omega(1-s)^q\big)^{1/q}}{(1+\omega s^q)^{1/q}}$$

for s with $0 \le s \le 1$. We note that $J\big(d^{(2)}(\omega, q)\big) = \max\{g(s) : 0 \le s \le 1\}$, and we shall calculate the maximum of the function g. The derivative of g is

$$\begin{aligned} g'(s) \;&=\; \frac{\big((1+s)^q + \omega(1-s)^q\big)^{1/q-1}}{(1+\omega s^q)^{1/q+1}} \\ &\times\; \big\{(1+s)^{q-1}(1-\omega s^{q-1}) - \omega(1-s)^{q-1}(1+\omega s^{q-1})\big\}. \end{aligned}$$

We put $\alpha = q-1$ and define a function g_1 from $[0,1]$ into $\mathbb{R}$ by

$$g_1(s) = (1+s)^\alpha(1-\omega s^\alpha) - \omega(1-s)^\alpha(1+\omega s^\alpha)$$

for s with $0 \le s \le 1$. We also define

$$g_2(s) = \log\big((1+s)^\alpha(1-\omega s^\alpha)\big) - \log\big(\omega(1-s)^\alpha(1+\omega s^\alpha)\big)$$

for s with $0 \le s \le 1$. Note that for any s, $g_2(s) \ge 0$ if and only if $g'(s) \ge 0$. Since

$$g_2(s) = \alpha\log(1+s) + \log(1-\omega s^\alpha) - \log\omega - \alpha\log(1-s) - \log(1+\omega s^\alpha),$$

we have $\lim_{s\to+0} g_2(s) = -\log\omega > 0$ and $\lim_{s\to 1-0} g_2(s) = +\infty$. The derivative of g_2 is

$$g_2'(s) \;=\; \tfrac{2\alpha(1+\omega s^{\alpha+1})(1-\omega s^{\alpha-1})}{(1-s)(1+s)(1+\omega s^\alpha)(1-\omega s^\alpha)}.$$

Hence the function g_2 has the minimum at $s = \omega^{1/(1-\alpha)}$ and

$$g_2(\omega^{1/(1-\alpha)}) = (1-\alpha)\log\left(\frac{1-\omega^{1/(1-\alpha)}}{\omega^{1/(1-\alpha)}\big(1+\omega^{1/(1-\alpha)}\big)}\right).$$

Since $(1-u)/u(1+u) \ge 1 (u > 0) \Leftrightarrow 0 < u \le -1+\sqrt{2}$, it is easy to see that

$$g'(\omega^{1/(1-\alpha)}) \ge 0 \Leftrightarrow g_2(\omega^{1/(1-\alpha)}) \ge 0 \Leftrightarrow 0 < \omega \le (-1+\sqrt{2})^{2-q}.$$

Hence if $0 < \omega \le (-1+\sqrt{2})^{2-q}$ then we have $g'(s) \ge 0$ for all s, and so g is a non-decreasing function. Therefore we obtain

$$J\big(d^{(2)}(\omega,q)\big) = \max\{g(s) : 0 \le s \le 1\} = g(1) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

Let us consider the case $(-1+\sqrt{2})^{2-q} < \omega < 1$. Since $g'(\omega^{1/(1-\alpha)}) < 0$, by the following table, we can take s_0, s_1 such that $g'(s_0) = g'(s_1) = 0$ and $0 < s_0 < \omega^{1/(1-\alpha)} < s_1 < 1$.

s	0		s_0		$\omega^{1/(1-\alpha)}$		s_1		1
$g_2'(s)$		$-$	$-$	$-$	0	$+$	$+$	$+$	
$g_2(s)$	$+$	$\searrow$	0	$\searrow$	$-$	$\nearrow$	0	$\nearrow$	∞
$g'(s)$		$+$	0	$-$	$-$	$-$	0	$+$	
$g(s)$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$	

Since s_0 is a relative maximum of the function g, we have

$$J\big(d^{(2)}(\omega,q)\big) = \max\big\{g(s_0), g(1)\big\}.$$

Since $(1+s_0)^{q-1}(1-\omega s_0^{q-1}) = \omega(1-s_0)^{q-1}(1+\omega s_0^{q-1})$ by $g'(s_0) = 0$, we have

$$\begin{aligned}
&\big((1+s_0)^q + \omega(1-s_0)^q\big)(1+\omega s_0^{q-1}) \\
=\;& (1+s_0)^q(1+\omega s_0^{q-1}) + \omega(1-s_0)^{q-1}(1+\omega s_0^{q-1})(1-s_0) \\
=\;& (1+s_0)^q(1+\omega s_0^{q-1}) + (1+s_0)^{q-1}(1-\omega s_0^{q-1})(1-s_0) \\
=\;& (1+s_0)^{q-1}\big\{(1+s_0)(1+\omega s_0^{q-1}) + (1-\omega s_0^{q-1})(1-s_0)\big\} \\
=\;& 2(1+s_0)^{q-1}(1+\omega s_0^q).
\end{aligned}$$

Then we have

$$g(s_0) = \left(\frac{(1+s_0)^q + \omega(1-s_0)^q}{1+\omega s_0^q}\right)^{1/q} = \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}.$$

Therefore we obtain

$$J\big(d^{(2)}(\omega,q)\big)=\max\left\{\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q},2\left(\frac{1}{1+\omega}\right)^{1/q}\right\}.$$

It is easy to prove that $\omega>\sqrt{2}^q-1$ if and only if $\sqrt{2}>2\left(\frac{1}{1+\omega}\right)^{1/q}$. Since $\sqrt{2}\le J(X)\le 2$ for any Banach space X, we have

$$J\big(d^{(2)}(\omega,q)\big)=\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}$$

in the case where $\omega>\sqrt{2}^q-1$. This completes the proof.

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